A parametric q-supercongruence and two Dwork-type q-supercongruences

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Abstract. Employing the creative microscoping method and the Chinese remainder theorem for polynomials, we give a parametric q-supercongruences and two Dwork-type q-supercongruences. As a corollary we deduce that, for primes $p\equiv 1 \pmod 4$ and $r\geqslant 1$,

$$\sum_{k=0}^{p^r-1}(6k+1)\frac{(\frac12)_k(\frac14)_k^2}{k!^3}4^k\equiv p\sum_{k=0}^{p^{r-1}-1}(6k+1)\frac{(\frac12)_k(\frac14)_k^2}{k!^3}4^k \pmod{p^{3r}},$$

and a similar result for primes $p\equiv 3 \pmod 4$ and $r\geqslant 2$, $\sum_{k=0}^{p^r-1}(6k+1)\frac{(\frac12)_k(\frac14)_k^2 4^k}{k!^3}\equiv p^2\sum_{k=0}^{p^{r-2}-1}(6k+1)\frac{(\frac12)_k(\frac14)_k^2 4^k}{k!^3} \pmod{p^{3r-2}}$, where $(x)_0=1$ and $(x)_k=x(x+1)\cdots(x+k-1)$ for $k\geqslant 1$.

Keywords: q-congruences; creative microscoping; Gasper and Rahman's quadratic summation; Chinese remainder theorem for polynomials; Dwork-type supercongruences

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1. Introduction

More than one century ago, Ramanujan [19] listed seventeen mysterious infinite expansions for $1/\pi$, such as

$$\sum_{k=0}^{\infty}(6k+1)\frac{(\frac12)_k^3}{k!^3 4^k}=\frac{4}{\pi},$$

where $(x)_k=x(x+1)\cdots(x+k-1)$ dentoes the Pochhammer symbol. In 1997, motivated by Ramanujan's work, Van Hamme [22] observed thirteen beautiful p-adic analogues of Ramanujan series or Ramanujan-type series. For instance, he conjectured that, for primes $p>3$,

$$\sum_{k=0}^{(p-1)/2}(6k+1)\frac{(\frac12)_k^3}{k!^3 4^k}\equiv(-1)^{(p-1)/2}p \pmod{p^4}. \tag{1.1}$$

All of Van Hamme's supercongruences have been proved by different authors (see [18,21]). The supercongruence (1.1) was first confirmed by Long [16] through hypergeometric identities.

It should be mentioned that there are many Ramanujan-type supercongruences in the literature. For example, by making use of the Wilf–Zeilberger method, Guillera and Zudilin [4] established three supercongruences, such as

$$\sum_{k=0}^{p-1}\frac{(\frac12)_k^3}{k!^3}(3k+1)4^k\equiv p \pmod{p^3}\quad \text{for odd primes } p. \tag{1.2}$$

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A further generalization of (1.2) was conjectured by Sun [20, Conjecture 5.1(ii)] and was confirmed by Wang and Hu [23].

In 2019, the first author and Zudilin [12] introduced the so-called "creative microscoping" method to consturct q-analogues of supercongruences modulo p^3 for odd primes p. The first author [6] then utilized this method along with the Chinese remainder theorem for coprime polynomials to give q-analogues of (1.1) and (1.2) modulo p^4 as follows: for positive odd integers n, modulo $[n]\Phi_n(q)^3$,

$$\sum_{k=0}^{(n-1)/2}[6k+1]\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^4;q^4)_k^3}q^{k^2}\equiv(-q)^{(1-n)/2}[n]\left\{1+\frac{(n^2-1)(1-q)^2}{24}[n]^2\right\},$$

$$\sum_{k=0}^{n-1}[3k+1]\frac{(q;q^2)_k^3q^{-\binom{k+1}{2}}}{(q;q)_k^2(q^2;q^2)_k}\equiv q^{(1-n)/2}[n]\left\{1+\frac{(n^2-1)(1-q)^2}{24}[n]^2\right\}. \tag{1.3}$$

Throughout the paper, $[n]=(1-q^n)/(1-q)$ is the *q-integer*, and $(x;q)_n=(1-x)(1-xq)\cdots(1-xq^{n-1})$ $(n\geqslant 0)$ stands for the *q-shifted factorial*. For simplicity, we will also use the abbreivated notation: $(a_1,\ldots,a_m;q)_n=(a_1;q)_n\ldots(a_m;q)_n$. Moreover, $\Phi_n(q)$ represents the n-th *cyclotomic polynomial*, which can be factorized as

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}}(q-\zeta^k),$$

where ζ is an n-th primitive root of unity.

Recently, employing the creative microscoping method and Gasper and Rahman's quadratic summation, the present authors [10] estalbished the following q-supercongruence: for positive integers n subject to $n\equiv 1 \pmod 4$, modulo $[n]\Phi_n(q)^3$,

$$\sum_{k=0}^{(n-1)/2}[6k+1]\frac{(q;q^4)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{-k^2}\equiv q^{(1-n)/2}[n]\left\{1+\frac{(n^2-1)(1-q)^2}{24}[n]^2\right\}, \tag{1.4}$$

which generalizes a result of He and Wang [14, Theorem 2.4]. More supercongruences and q-supercongruences can be found in [9–11, 13, 15, 24–26].

Our first goal is to establish the following common generalization of (1.3) and (1.4).

Theorem 1.1. *Let d and r be positive integers with r odd and $\gcd(d,r)=1$. Let n be a positive integer such that $n\equiv r \pmod{2d}$ and $(n-1)/2\leqslant(n-r)/d\leqslant n-1$. Then*

$$\begin{aligned}\sum_{k=0}^{(n-r)/d}&[3dk+r]\frac{(q^r;q^{2d})_k^2(q^d;q^{2d})_k}{(q^d;q^d)_k^2(q^{2d};q^{2d})_k}q^{k(d-dk-2r)/2}\\ &\equiv q^{(r-n)/2}[n]\left\{1+\frac{(n^2-1)(1-q)^2}{24}[n]^2\right\}\pmod{[n]\Phi_n(q)^3}.\end{aligned} \tag{1.5}$$

It is easy to see that, if $d=r=1$, then the q-supercongruence (1.5) reduces to (1.3), while if $d=2$ and $r=1$, then (1.5) reduces to (1.4). Moreover, the first author [8] has already proved (1.5) modulo $[n]\Phi_n(q)^2$.

The $d=1,2$ cases of Theorem 1.1 can be stated as the following two corollaries.

Corollary 1.2. *Let n and r be positive odd integers with $n \geqslant 2r-1$. Then*

$$\sum_{k=0}^{n-r}[3k+r]\frac{(q^r;q^2)_k^2(q;q^2)_k}{(q;q)_k^2(q^2;q^2)_k}q^{k(1-k-2r)/2}$$
$$\equiv q^{(r-n)/2}[n]\left\{1+\frac{(n^2-1)(1-q)^2}{24}[n]^2\right\} \pmod{[n]\Phi_n(q)^3}.$$

Corollary 1.3. *Let n and r be positive odd integers with $n \equiv r \pmod 4$ and $r \leqslant 1$. Then*

$$\sum_{k=0}^{(n-r)/2}[6k+r]\frac{(q^r;q^4)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{k(1-k-r)}$$
$$\equiv q^{(r-n)/2}[n]\left\{1+\frac{(n^2-1)(1-q)^2}{24}[n]^2\right\} \pmod{[n]\Phi_n(q)^3}.$$

Letting n be a prime power and then taking the limits as $q \to 1$ in (1.5), we deduce the following conclusion.

Corollary 1.4. *Let d and r be positive integers with r odd and $\gcd(d,r)=1$. Let $p>3$ be a prime and s a positive integer such that $p^s \equiv r \pmod{2d}$ and $(p^s-1)/2 \leqslant (p^s-r)/d \leqslant p^s-1$. Then*

$$\sum_{k=0}^{(p^s-r)/d}(3dk+r)\frac{(\frac{1}{2})_k(\frac{r}{2d})_k^2}{k!^3}4^k \equiv p^s \pmod{p^{s+3}}.$$

Swisher [21, (J.3)] predicted that the supercongruence (1.1) has the following generalization: for primes $p>3$ and positive r,

$$\sum_{k=0}^{(p^r-1)/2}(6k+1)\frac{(\frac{1}{2})_k^3}{k!^34^k} \equiv (-1)^{(p-1)/2}p\sum_{k=0}^{(p^{r-1}-1)/2}(6k+1)\frac{(\frac{1}{2})_k^3}{k!^34^k} \pmod{p^{4r}}. \tag{1.6}$$

Swisher's prediction can be regarded as a particular instance of Dwork-type supercongruences [2,17]. The first author [5] succeeded in proving that (1.6) is true modulo p^{3r} by establishing a proper q-analogue. The present authors [10] proved the Dwork-type supercongruence: for any prime $p \equiv 1 \pmod 4$ and positive r,

$$\sum_{k=0}^{(p^r-1)/2}(6k+1)\frac{(\frac{1}{2})_k(\frac{1}{4})_k^2}{k!^3}4^k \equiv p\sum_{k=0}^{(p^{r-1}-1)/2}(6k+1)\frac{(\frac{1}{2})_k(\frac{1}{4})_k^2}{k!^3}4^k \pmod{p^{3r}}. \tag{1.7}$$

Our second goal is to prove the following Dwork-type supercongruences, which are companions of (1.7): for any prime $p \equiv 1 \pmod 4$ and positive r,

$$\sum_{k=0}^{p^r-1}(6k+1)\frac{(\frac{1}{2})_k(\frac{1}{4})_k^2}{k!^3}4^k \equiv p\sum_{k=0}^{p^{r-1}-1}(6k+1)\frac{(\frac{1}{2})_k(\frac{1}{4})_k^2}{k!^3}4^k \pmod{p^{3r}}, \tag{1.8}$$

and for any prime $p \equiv 3 \pmod 4$ and $r \geqslant 2$,

$$\sum_{k=0}^{p^r-1}(6k+1)\frac{(\frac{1}{2})_k(\frac{1}{4})_k^2}{k!^3}4^k \equiv p^2\sum_{k=0}^{p^{r-2}-1}(6k+1)\frac{(\frac{1}{2})_k(\frac{1}{4})_k^2}{k!^3}4^k \pmod{p^{3r-2}}. \tag{1.9}$$

We shall prove (1.8) and (1.9) by establishing the following Dwork-type q-supercongruences.

Theorem 1.5. *Let $n \equiv 1 \pmod 4$ be an integer with $n > 1$ and let $r \geqslant 1$. Then, modulo $[n^r]\prod_{j=1}^{r}\Phi_{n^j}(q)^2$,*

$$\sum_{k=0}^{n^r-1}[6k+1]\frac{(q;q^4)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{-k^2}$$
$$\equiv q^{(1-n)/2}[n]\sum_{k=0}^{n^{r-1}-1}[6k+1]_{q^n}\frac{(q^n;q^{4n})_k^2(q^{2n};q^{4n})_k}{(q^{2n};q^{2n})_k^2(q^{4n};q^{4n})_k}q^{-nk^2}. \tag{1.10}$$

Theorem 1.6. *Let $n \equiv 3 \pmod 4$ be a positive integer and let $r \geqslant 2$. Then, modulo $[n^r]\prod_{j=2}^{r}\Phi_{n^j}(q)^2$,*

$$\sum_{k=0}^{n^r-1}[6k+1]\frac{(q;q^4)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{-k^2}$$
$$\equiv q^{(1-n^2)/2}[n^2]\sum_{k=0}^{n^{r-2}-1}[6k+1]_{q^{n^2}}\frac{(q^{n^2};q^{4n^2})_k^2(q^{2n^2};q^{4n^2})_k}{(q^{2n^2};q^{2n^2})_k^2(q^{4n^2};q^{4n^2})_k}q^{-n^2k^2}. \tag{1.11}$$

Letting $n = p$ be a prime and taking $q \to 1$ in (1.10) and (1.11), we obtain (1.8) and (1.9), respectively.

The paper is organized as follows. We prove Theorems 1.1, 1.5, and 1.6 in Sections 2–4, respectively. Our principal tools are the creative microscoping method, Gasper and Rahman's quadratic summation, and the Chinese remainder theorem for coprime polynomials. Finally, in Section 5, we put forward two conjectures on Dwork-type supercongruences for further study.

2. Proof of Theorem 1.1

Note that q-series identities play an important role in the study of q-supercongruences. Here we need to use Gasper and Rahman's quadratic summation (see [3, (3.8.12)]), which can be written as

$$\sum_{k=0}^{\infty}\frac{1-aq^{3k}}{1-a}\frac{(a,b,q/b;q)_k(d,f,a^2q/df;q^2)_kq^k}{(aq/d,aq/f,df/a;q)_k(q^2,aq^2/b,abq;q^2)_k}$$
$$+\frac{(aq,f/a,b,q/b;q)_\infty(d,a^2q/df,fq^2/d,df^2q/a^2;q^2)_\infty}{(a/f,fq/a,aq/d,df/a;q)_\infty(aq^2/b,abq,fq/ab,bf/a;q^2)_\infty}$$
$$\times\sum_{k=0}^{\infty}\frac{(f,bf/a,fq/ab;q^2)_kq^{2k}}{(q^2,fq^2/d,df^2q/a^2;q^2)_k}$$
$$=\frac{(aq,f/a;q)_\infty(aq^2/bd,abq/d,bdf/a,dfq/ab;q^2)_\infty}{(aq/d,df/a;q)_\infty(aq^2/b,abq,bf/a,fq/ab;q^2)_\infty}. \tag{2.1}$$

It was Wei [24] who first made use of (2.1) to produce some q-supercongruences. Later the first author [7] derived more q-supercongruences modulo $\Phi_n(q)^2$ or $\Phi_n(q)^3$ from Gasper and Rahman's summation (2.1).

We now use (2.1) to establish the following identity.

Lemma 2.1. *Let* $n\equiv r \pmod{2d}$ *be a positive integer with* $0\leqslant (n-r)/d\leqslant n-1$. *Then*

$$\sum_{k=0}^{(n-r)/d}\frac{1-q^{3dk+r-n}}{1-q^{r-n}}\frac{(q^{r-n};q^d)_k(ab^dq^r,q^r/a,q^{d-2n}/b^d;q^{2d})_k}{(q^{2d};q^{2d})_k(q^{d-n}/ab^d,aq^{d-n},b^dq^{r+n};q^d)_k}q^{k(d-dk-2r)/2+nk}=0. \tag{2.2}$$

Proof. Putting $b=q^{-2m}$ in (2.1), we obtain

$$\begin{aligned}&\sum_{k=0}^{2m}\frac{1-aq^{3k}}{1-a}\frac{(a,q^{-2m},q^{1+2m};q)_k(d,f,a^2q/df;q^2)_kq^k}{(aq/d,aq/f,df/a;q)_k(q^2,aq^{2+2m},aq^{1-2m};q^2)_k}\\&=\frac{(aq,aq^2,f/a,fq/a,aq^{2+2m}/d,aq^{1-2m}/d,dfq^{-2m}/a,dfq^{1+2m}/a;q^2)_\infty}{(aq/d,aq^2/d,df/a,dfq/a,aq^{2+2m},aq^{1-2m},fq^{-2m}/a,fq^{1+2m}/a;q^2)_\infty}\\&=\frac{(aq^2,dq/a,fq/a,aq^2/df;q^2)_{2m}}{(q/a,aq^2/d,aq^2/f,dfq/a;q^2)_{2m}},\end{aligned} \tag{2.3}$$

which was already noticed by He and Wang [14] but was not stated explicitly by them.

Letting $q\mapsto q^d$, and then taking $a=q^{r-n}$, $d=ab^dq^r$, $f=q^r/a$, and $m\to\infty$ in (2.3), we see that the left-hand side of (2.2) is equal to

$$\frac{(q^{r-n+2d},ab^dq^{d+n},q^{d+n}/a,q^{-r-n+2d}/b^d;q^{2d})_\infty}{(q^{d-r+n},q^{-n+2d}/ab^d,aq^{-n+2d},b^dq^{r+d+n};q^{2d})_\infty}=0.$$

This is because the factor $(q^{r-n+2d};q^{2d})_\infty=0$ appears in the numerator. □

We require the following parametric version of (1.4).

Lemma 2.2. *Let* $n\equiv r \pmod{2d}$ *be a positive integer with* $0\leqslant (n-r)/d\leqslant n-1$. *Let* a *and* b *be indeterminates. Then, modulo* $\Phi_n(q)(1-ab^dq^n)(a-q^n)$,

$$\begin{aligned}&\sum_{k=0}^{(n-r)/d}[3dk+r]\frac{(q^r;q^d)_k(ab^dq^r,q^r/a,q^d/b^d;q^{2d})_k}{(q^{2d};q^{2d})_k(q^d/ab^d,aq^d,b^dq^r;q^d)_k}q^{k(d-dk-2r)/2}\\&\equiv[r]\frac{(q^{r+d};q^d)_{(n-r)/d}}{(b^dq^r;q^d)_{(n-r)/d}}b^{(n-r)/2}q^{(r-n)/2}.\end{aligned} \tag{2.4}$$

Proof. In view of $q^n\equiv 1 \pmod{\Phi_n(q)}$, the identity (2.2) leads to

$$\sum_{k=0}^{(n-r)/d}[3dk+r]\frac{(q^r;q^d)_k(ab^dq^r,q^r/a,q^d/b^d;q^{2d})_k}{(q^{2d};q^{2d})_k(q^d/ab^d,aq^d,b^dq^r;q^d)_k}q^{k(d-dk-2r)/2}\equiv 0 \pmod{\Phi_n(q)}.$$

Since $(q^{r+d};q^d)_{(n-r)/d}$ has the factor $1-q^n$, while $(b^dq^r;q^d)_{(n-r)/d}$ is coprime with $\Phi_n(q)$, the right-hand side of (2.4) is congruent to 0 as well. This means that the q-congruence (2.4) is true modulo $\Phi_n(q)$.

For $a=q^n$ or $a=q^{-n}/b^d$, the left-hand side of (2.4) equals

$$\sum_{k=0}^{(n-r)/d}\frac{1-q^{3dk+r}}{1-q}\frac{(q^r;q^d)_k(q^{r-n},b^dq^{r+n},q^d/b^d;q^{2d})_k}{(q^{2d};q^{2d})_k(q^{d+n},q^{d-n}/b^d,b^dq^r;q^d)_k}q^{k(d-dk-2r)/2}. \tag{2.5}$$

Plugging the parameter substitutions $q \mapsto q^d$, $a = q^r$, $d = q^{r-n}$, and $f = b^d q^{r+n}$ into (2.3), we see that (2.5) can be written as

$$\begin{aligned}
[r]\frac{(q^{r+2d}, q^{d-n}, b^d q^{d+n}, q^{2d-r}/b^d; q^{2d})_\infty}{(q^{d-r}, q^{2d+n}, q^{2d-n}/b^d, b^d q^{r+d}; q^{2d})_\infty} &= [r]\frac{(q^{r+2d}, q^{d-n}; q^{2d})_{(n-r)/(2d)}}{(q^{2d-n}/b^d, b^d q^{r+d}; q^{2d})_{(n-r)/(2d)}} \\
&= [r]\frac{(q^{r+d}; q^d)_{(n-r)/d}}{(b^d q^r; q^d)_{(n-r)/d}} b^{(n-r)/2} q^{(r-n)/2}.
\end{aligned}$$

This indicates that (2.4) is true modulo $1 - ab^2q^n$ and $a - q^n$. Since $\Phi_n(q)$, $1 - ab^2q^n$, and $a - q^n$ are pairwise coprime polynomials in q, we get the q-congruence (2.4). □

We also require the following q-congruence.

Lemma 2.3. *Let $n \equiv r \pmod{2d}$ be a positive integer with $0 \leqslant (n-r)/d \leqslant n-1$. Let a and b be indeterminates. Then, modulo $b - q^n$,*

$$\begin{aligned}
&\sum_{k=0}^{(n-1)/2} [3dk + r]\frac{(q^r; q^d)_k (ab^d q^r, q^r/a, q^d/b^d; q^{2d})_k}{(q^{2d}; q^{2d})_k (q^d/ab^d, aq^d, b^d q^r; q^d)_k} q^{k(d-dk-2r)/2} \\
&\quad \equiv [r]\frac{(q^{r+2d}, q^{2d-r}/b^d; q^{2d})_{(n-1)/2}}{(aq^{2d}, q^{2d}/ab^d; q^{2d})_{(n-1)/2}}. \qquad (2.6)
\end{aligned}$$

Proof. Letting $q \mapsto q^d$, and taking $a = q^r$, $d = q^r/a$, and $f = aq^{nd+r}$ in (2.3), we get

$$\begin{aligned}
&\frac{1}{[r]}\sum_{k=0}^{(n-1)/2} \frac{1 - q^{3dk+r}}{1-q}\,\frac{(q^r; q^d)_k (q^r/a, aq^{nd+r}, q^{d-nd}; q^{2d})_k}{(q^{2d}; q^{2d})_k (aq^d, q^{d-nd}/a, q^{nd+r}; q^d)_k} q^{k(d-dk-2r)/2} \\
&\quad = \frac{(q^{r+2d}, q^d/a, aq^{nd+d}, q^{2d-nd-r}; q^{2d})_\infty}{(q^{d-r}, aq^{2d}, q^{2d-nd}/a, q^{nd+r+d}; q^{2d})_\infty} \\
&\quad = \frac{(q^{r+2d}, q^{2d-nd-r}; q^{2d})_{(n-1)/2}}{(aq^{2d}, q^{2d-nd}/a; q^{2d})_{(n-1)/2}}.
\end{aligned}$$

This means that the two sides of (2.6) are equal for $b = q^n$, and therefore the q-congruence (2.6) is true. □

Finally, we need the following result similar to [6, Lemma 2.1].

Lemma 2.4. *Let d and n be positive integers with n odd. Let a be an indeterminate. Then*

$$(aq^{2d}, q^{2d}/a; q^{2d})_{(n-1)/2} \equiv (-1)^{(n-1)/2}\frac{(1-a^n)q^{-d(n-1)^2/4}}{(1-a)a^{(n-1)/2}} \pmod{\Phi_n(q)}, \qquad (2.7)$$

$$(q^d; q^d)_{n-1} \equiv n \pmod{\Phi_n(q)}. \qquad (2.8)$$

Proof. Since $q^n \equiv 1 \pmod{\Phi_n(q)}$, we have

$$(q^{2d}/a; q^{2d})_{(n-1)/2} = (1 - q^{2d}/a)(1 - q^{4d}/a)\cdots(1 - q^{dn-d}/a)$$

$$
\begin{aligned}
&\equiv (1-q^{2d-dn}/a)(1-q^{4d-dn}/a)\cdots(1-q^{-d}/a)\\
&=(-1)^{(n-1)/2}(aq^d,q^{2d})_{(n-1)/2}\frac{q^{-d(n-1)^2/4}}{a^{(n-1)/2}} \pmod{\Phi_n(q)}.
\end{aligned}
$$

Therefore, the left-hand side of (2.7) is congruent to

$$
(-1)^{(n-1)/2}(aq^d;q^{2d})_{n-1}\frac{q^{-d(n-1)^2/4}}{a^{(n-1)/2}} \pmod{\Phi_n(q)}.
$$

The proof of (2.7) then follows from the fact

$$
(aq^d;q^d)_{n-1}=\sum_{k=0}^{n-1}(-1)^k q^{d\binom{k+1}{2}}\begin{bmatrix}n-1\\ k\end{bmatrix}_{q^d} a^k \equiv \sum_{k=0}^{n-1} a^k \pmod{\Phi_n(q)}, \tag{2.9}
$$

where ${m \brack k}_q=(q;q)_m/((q;q)_k(q;q)_{m-k})$ denotes the q-binomial coefficient and we have used the q-binomial theorem [1, p. 36].

Putting $a=1$ in (2.9), we immediately get (2.8). □

We are now able to present a proof of Theorem 1.1.

Proof of Theorem 1.1. Obviously, $\Phi_n(q)(1-ab^dq^n)(a-q^n)$ and $b-q^n$ are coprime polynomials in q. Noticing the following two q-congruences,

$$
\frac{(b-q^n)(ab^dq^n+ab^{d+1}-a^2b^d-1)}{(a-b)(1-ab^{d+1})}\equiv 1 \pmod{(1-ab^dq^n)(a-q^n)}, \tag{2.10}
$$

$$
\frac{(1-ab^dq^n)(a-q^n)}{(a-b)(1-ab^{d+1})}\equiv 1 \pmod{b-q^n}. \tag{2.11}
$$

and using the Chinese reminder theorem for coprime polynomials, we acquire the the q-congruence from (2.4) and (2.6): modulo $\Phi_n(q)(1-ab^dq^n)(a-q^n)(b-q^n)$,

$$
\begin{aligned}
&\sum_{k=0}^{(n-r)/d}[3dk+r]\frac{(q^r;q^d)_k(ab^dq^r,q^r/a,q^d/b^d;q^{2d})_k}{(q^{2d};q^{2d})_k(q^d/ab^d,aq^d,b^dq^r;q^d)_k}q^{k(d-dk-2r)/2}\\
&\quad\equiv [r]\frac{(q^{r+d};q^d)_{(n-r)/d}}{(b^dq^r;q^d)_{(n-r)/d}}b^{(n-r)/2}q^{(r-n)/2}\frac{(b-q^n)(ab^dq^n+ab^{d+1}-a^2b^d-1)}{(a-b)(1-ab^{d+1})}\\
&\qquad+[r]\frac{(q^{r+2d},q^{2d-r}/b^d;q^{2d})_{(n-1)/2}}{(aq^{2d},q^{2d}/ab^d;q^{2d})_{(n-1)/2}}\frac{(1-ab^dq^n)(a-q^n)}{(a-b)(1-ab^{d+1})}.
\end{aligned}\tag{2.12}
$$

Moreover, we have

$$
\begin{aligned}
[r](q^{r+2d};q^{2d})_{(n-1)/2}&=[r](q^{r+2d};q^{2d})_{(n-r)/(2d)}(q^{n+2d};q^{2d})_{(n-1)/2-(n-r)/(2d)}\\
&\equiv (q^r;q^{2d})_{(n-r)/(2d)}[n](q^{2d};q^{2d})_{(n-1)/2-(n-r)/(2d)} \pmod{\Phi_n(q)^2},\\
(q^{2d-r};q^{2d})_{(n-1)/2}&=(-1)^{(n-1)/2}(q^{d(1-n)+r};q^{2d})_{(n-1)/2}q^{(n^2-1)d/4-(n-1)r/2}.
\end{aligned}
$$

It follows that

$$\begin{aligned}[r](q^{r+2d},q^{2d-r};q^{2d})_{(n-1)/2} &\equiv (-1)^{(n-1)/2}[n](q^{2d};q^{2d})_{(n-1)/2-(n-r)/(2d)}(q^{d(1-n)+r};q^{2d})_{(n-1)/2}\\
&\quad\times (q^{r};q^{2d})_{(n-r)/(2d)}q^{-(n-1)^2d/4-(n-r)/2}\\
&\equiv (-1)^{(n-1)/2}[n](q^{2d};q^{2d})_{n-1}q^{-(n-1)^2d/4-(n-r)/2}\\
&\equiv (-1)^{(n-1)/2}[n]nq^{-(n-1)^2d/4-(n-r)/2} \pmod{\Phi_n(q)^2}.\end{aligned}$$

where we have used the q-congruence (2.8) in the last step. This together with (2.7) imply that

$$[r]\frac{(q^{r+2d},q^{2d-r};q^{2d})_{(n-1)/2}}{(aq^{2d},q^{2d}/a;q^{2d})_{(n-1)/2}} \equiv \frac{[n]n(1-a)a^{(n-1)/2}}{(1-a^n)q^{(n-r)/2}} \pmod{\Phi_n(q)^2}. \tag{2.13}$$

Letting $b=1$ in (2.12) and applying (2.13) and the identity

$$(1-q^n)(1+a^2-a-aq^n)=(1-a)^2+(1-aq^n)(a-q^n), \tag{2.14}$$

we conclude that, modulo $\Phi_n(q)^2(1-aq^n)(a-q^n)$,

$$\begin{aligned}&\sum_{k=0}^{(n-r)/d}[3dk+r]\frac{(q^r;q^d)_k(aq^r,q^r/a,q^d;q^{2d})_k}{(q^{2d};q^{2d})_k(q^d/a,aq^d,q^r;q^d)_k}q^{k(d-dk-2r)/2}\\
&\quad\equiv q^{(r-n)/2}[n]+q^{(r-n)/2}[n]\frac{(1-aq^n)(a-q^n)}{(1-a)^2}\left(1-\frac{n(1-a)a^{(n-1)/2}}{1-a^n}\right).\end{aligned} \tag{2.15}$$

By the L'Hôpital rule, we obtain

$$\lim_{a\to1}\frac{(1-aq^n)(a-q^n)}{(1-a)^2}\frac{(1-a^n-n(1-a)a^{(n-1)/2})}{(1-a^n)}=\frac{(n^2-1)(1-q)^2}{24}[n]^2, \tag{2.16}$$

which was first utilized by the first author [6]. Thereby, taking $a\to1$ in (2.15) and applying the above limit, we prove the truth of (1.5) modulo $\Phi_n(q)^4$. By (1.4), the q-congruence (1.5) also holds true modulo $[n]$. Noting that the least common multiple of $\Phi_n(q)^4$ and $[n]$ is just $[n]\Phi_n(q)^3$, we accomplish the proof. □

3. Proof of Theorem 1.5

Recall that He and Wang [14, Lemmas 5.1 and 5.2] have proved the following results.

Lemma 3.1. *Let $n\equiv 1 \pmod 4$ be a positive integer. Let a be an indeterminate. Then*

$$\sum_{k=0}^{n-1}[6k+1]\frac{(aq;q^4)_k(q/a;q^4)_k(q^2;q^4)_k}{(aq^2;q^2)_k(q^2/a;q^2)_k(q^4;q^4)_k}q^{-k^2}\equiv 0 \pmod{[n]}, \tag{3.1}$$

$$\sum_{k=0}^{n-1}[6k+1]\frac{(q^{1-n};q^4)_k(q^{1+n};q^4)_k(q^2;q^4)_k}{(q^{2+n};q^2)_k(q^{2-n};q^2)_k(q^4;q^4)_k}q^{-k^2}=q^{(1-n)/2}[n]. \tag{3.2}$$

We now give a parametric version of Theorem 1.5.

Lemma 3.2. *Let $n \equiv 1 \pmod 4$ be an integer with $n > 1$ and let $r \geqslant 1$. Let a be an indeterminate. Then, modulo*

$$[n^r] \prod_{j=0}^{n^{r-1}-1} (1 - aq^{(4j+1)n})(a - q^{(4j+1)n}),$$

we have

$$\sum_{k=0}^{n^r-1} [6k+1] \frac{(aq;q^4)_k (q/a;q^4)_k (q^2;q^4)_k}{(aq^2;q^2)_k (q^2/a;q^2)_k (q^4;q^4)_k} q^{-k^2}$$

$$\equiv q^{(1-n)/2}[n] \sum_{k=0}^{n^{r-1}-1} [6k+1]_{q^n} \frac{(aq^n;q^{4n})_k (q^n/a;q^{4n})_k (q^{2n};q^{4n})_k}{(aq^{2n};q^{2n})_k (q^{2n}/a;q^{2n})_k (q^{4n};q^{4n})_k} q^{-nk^2}. \tag{3.3}$$

Proof. The $n \mapsto n^r$ case of (3.1) means that the left-hand side of (3.3) is congruent to 0 modulo $[n^r]$. Similarly, the $r \mapsto r-1$ and $q \mapsto q^n$ case of (3.1) implies that the summation on the right-hand side of (3.3) discarding the prefactor is congruent to 0 modulo $[n^{r-1}]_{q^n}$. Furthermore, it is easy to see that $[n]$ is coprime with the denominators of the sum on the right-hand side of (3.3). Thus, the right-hand side of (3.3) is congruent to 0 modulo $[n][n^{r-1}]_{q^n} = [n^r]$. This proves that the q-congruence (3.3) holds modulo $[n^r]$.

In order to prove (3.3) modulo

$$\prod_{j=0}^{n^{r-1}-1} (1 - aq^{(4j+1)n})(a - q^{(4j+1)n}), \tag{3.4}$$

we need only to show that the two sides of (3.3) are identical for $a = q^{\pm(4j+1)n}$ $(0 \leqslant j \leqslant n^{r-1}-1)$. In order words, we need to prove that

$$\sum_{k=0}^{n^r-1} [6k+1] \frac{(q^{1-(4j+1)n};q^4)_k (q^{1+(4j+1)n};q^4)_k (q^2;q^4)_k}{(q^{2-(4j+1)n};q^2)_k (q^{2+(4j+1)n};q^2)_k (q^4;q^4)_k} q^{-k^2}$$

$$= q^{(1-n)/2}[n] \sum_{k=0}^{n^{r-1}-1} [6k+1]_{q^n} \frac{(q^{-4jn};q^{4n})_k (q^{(4j+2)n};q^{4n})_k (q^{2n};q^{4n})_k}{(q^{(1-4j)n};q^{2n})_k (q^{(4j+3)n};q^{2n})_k (q^{4n};q^{4n})_k} q^{-nk^2}. \tag{3.5}$$

First observe that $n^r - 1 \geqslant ((4j+1)n-1)/4$ for $0 \leqslant j \leqslant n^{r-1}-1$, and $(q^{1-(4j+1)n};q^4)_k = 0$ for $k > ((4j+1)n-1)/4$. By (3.2), the left-hand side of (3.5) equals $q^{(1-(4j+1)n)/2}[(4j+1)n]$. Similarly, the right-hand side of (3.5) is given by

$$q^{(1-n)/2}[n] \cdot q^{-2jn}[4j+1]_{q^n} = q^{(1-(4j+1)n)/2}[(4j+1)n].$$

This proves the identity (3.5). Namely, the q-congruence (3.3) holds modulo (3.4). Since $[n^r]$ and (3.4) are coprime polynomials in q, we complete the proof of (3.3). □

Proof of Theorem 1.5. It is obvious that the limit of (3.4) as $a \to 1$ has the factor $\prod_{j=1}^{r}\Phi_{n^j}(q)^{2n^{r-j}}$. The factor of the common denominator of the two sides of (3.3) involving a is $(aq^2;q^2)_{n^r-1}(q^2/a;q^2)_{n^r-1}$. When $a=1$ it merely possesses the factor $\prod_{j=1}^{r}\Phi_{n^j}(q)^{2n^{r-j}-2}$ related to $\Phi_n(q), \Phi_{n^2}(q), \dots, \Phi_{n^r}(q)$. Hence, letting $a=1$ in (3.3), we conclude that (1.10) is true modulo $\prod_{j=1}^{r}\Phi_{n^j}(q)^3$, one factor $\prod_{j=1}^{r}\Phi_{n^j}(q)$ coming from $[n^r]$.

Moreover, the proof of (3.3) modulo $[n^r]$ is also valid for $a=1$. That is, the q-congruence (1.10) is true modulo $[n^r]$. Since the least common multiple of $\prod_{j=1}^{r}\Phi_{n^j}(q)^3$ and $[n^r]$ is $[n^r]\prod_{j=1}^{r}\Phi_{n^j}(q)^2$, we finish the proof. □

4. Proof of Theorem 1.6

Similarly as before, we first give a parametric version of Theorem 1.6.

Lemma 4.1. *Let $n \equiv 3 \pmod 4$ be an integer with $n>1$ and let $r \geqslant 2$. Then, modulo*

$$[n^r]\prod_{j=0}^{n^{r-2}-1}(1-aq^{(4j+1)n^2})(a-q^{(4j+1)n^2}),$$

we have

$$\sum_{k=0}^{n^r-1}[6k+1]\frac{(aq;q^4)_k(q/a;q^4)_k(q^2;q^4)_k}{(aq^2;q^2)_k(q^2/a;q^2)_k(q^4;q^4)_k}q^{-k^2}$$
$$\equiv q^{(1-n^2)/2}[n^2]\sum_{k=0}^{n^{r-2}-1}[6k+1]_{q^{n^2}}\frac{(aq^{n^2};q^{4n^2})_k(q^{n^2}/a;q^{4n^2})_k(q^{2n^2};q^{4n^2})_k}{(aq^{2n^2};q^{2n^2})_k(q^{2n^2}/a;q^{2n^2})_k(q^{4n^2};q^{4n^2})_k}q^{-n^2k^2}. \tag{4.1}$$

Proof. He and Wang [14, Theorem 2.4] also proved that (3.1) also holds for $n \equiv 3 \pmod 4$. This enables us to show that (4.1) is true modulo $[n^r]$ whenever r is odd or even.

In order to prove (4.1) modulo

$$\prod_{j=0}^{n^{r-2}-1}(1-aq^{(4j+1)n^2})(a-q^{(4j+1)n^2}), \tag{4.2}$$

it suffices to show that, for $0 \leqslant j \leqslant n^{r-2}-1$,

$$\sum_{k=0}^{n^r-1}[6k+1]\frac{(q^{1-(4j+1)n^2};q^4)_k(q^{1+(4j+1)n^2};q^4)_k(q^2;q^4)_k}{(q^{2-(4j+1)n^2};q^2)_k(q^{2+(4j+1)n^2};q^2)_k(q^4;q^4)_k}q^{-k^2}$$
$$= q^{(1-n^2)/2}[n^2]\sum_{k=0}^{n^{r-2}-1}[6k+1]_{q^{n^2}}\frac{(q^{-4jn^2};q^{4n^2})_k(q^{(4j+2)n^2};q^{4n^2})_k(q^{2n^2};q^{4n^2})_k}{(q^{(1-4j)n^2};q^{2n^2})_k(q^{(4j+3)n^2};q^{2n^2})_k(q^{4n^2};q^{4n^2})_k}q^{-n^2k^2}. \tag{4.3}$$

In view of (3.2), we can easily show that the both sides of (4.3) are equal to

$$q^{(1-n^2)/2}[n^2]\cdot q^{-2jn^2}[4j+1]_{q^{n^2}} = q^{(1-(4j+1)n^2)/2}[(4j+1)n^2].$$

This proves the truth of (4.1) modulo (4.2). □

Proof of Theorem 1.6. It is not hard to see that the $a=1$ case of (4.2) has the factor $\prod_{j=2}^{r}\Phi_{n^j}(q)^{2n^{r-j}}$. Moreover, the factor of the common denominator of the two sides of (4.1) involving a is $(aq^2;q^2)_{n^r-1}(q^2/a;q^2)_{n^r-1}$. When $a=1$ it only has the factor $\prod_{j=2}^{r}\Phi_{n^j}(q)^{2n^{r-j}-2}$ related to $\Phi_{n^2}(q),\Phi_{n^3}(q),\dots,\Phi_{n^r}(q)$. Thus, taking $a=1$ in (4.1) we conclude that (1.11) holds modulo $\Phi_n(q)\prod_{j=2}^{r}\Phi_{n^j}(q)^3$, where one product $\prod_{j=1}^{r}\Phi_{n^j}(q)$ is from $[n^r]$.

Further, the proof of (4.1) modulo $[n^r]$ is valid for $a=1$ as well. The proof then follows from the fact that the least common multiple of $\Phi_n(q)\prod_{j=2}^{r}\Phi_{n^j}(q)^3$ and $[n^r]$ is just $[n^r]\prod_{j=2}^{r}\Phi_{n^j}(q)^2$.
□

5. Concluding remarks and open problems

In their previous paper [10, Corollary 4.5], the authors proved that, for any prime $p\equiv 3 \pmod 4$ and even integer $r\geqslant 2$,

$$\sum_{k=0}^{(p^r-1)/2}(6k+1)\frac{(\frac12)_k(\frac14)_k^2}{k!^3}4^k\equiv p^2\sum_{k=0}^{(p^{r-2}-1)/2}(6k+1)\frac{(\frac12)_k(\frac14)_k^2}{k!^3}4^k \tag{5.1}$$

modulo p^{2r}, and they [10, Conjecture 5.4] suspected that this supercongruence holds modulo $p^{5r/2}$ for $p^r>3$. However, the method to prove (1.9) cannot be utilized to settle this conjecture. Nevertheless, we have the following two conjectures.

Conjecture 5.1. *For $p>3$, the supercongruence* (5.1) *holds modulo p^{3r-1} or p^{3r-2} depending on whether r is even or odd, respectively.*

He and Wang [14, Theorem 2.4] proved that

$$\sum_{k=0}^{n-1}[6k+1]\frac{(q;q^4)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{-k^2}\equiv q^{(1-n)/2}[n]\pmod{[n]\Phi_n(q)^2}, \tag{5.2}$$

which is also a conclusion of (1.4). Numerics suggests that the following generalization of the above q-supercongruence seems to be true.

Conjecture 5.2. *Let m and n be positive integers with $n\equiv 1 \pmod 4$ and $n>1$. Then, modulo $\Phi_n(q)^3$,*

$$\sum_{k=0}^{mn-1}[6k+1]\frac{(q;q^4)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{-k^2}\equiv q^{(1-n)/2}[n]\sum_{k=0}^{m-1}(6k+1)\frac{(\frac12)_k(\frac14)_k^2}{k!^3}4^k. \tag{5.3}$$

Finally, we believe that (1.4) can be further generalized as follows.

Conjecture 5.3. *Let m and n be positive odd integers with $n\equiv 1 \pmod 4$ and $n>1$. Then, modulo $\Phi_n(q)^4$,*

$$\sum_{k=0}^{(mn-1)/2}[6k+1]\frac{(q;q^4)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{-k^2}$$

$$\equiv q^{(1-n)/2}[n]\left\{1+\frac{(n^2-1)(1-q)^2}{24}[n]^2\right\}$$
$$\times\sum_{k=0}^{(m-1)/2}[6k+1]_{q^{n^2}}\frac{(q^{n^2};q^{4n^2})_k^2(q^{2n^2};q^{4n^2})_k}{(q^{2n^2};q^{2n^2})_k^2(q^{4n^2};q^{4n^2})_k}q^{-n^2k^2}.$$

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6. Declarations

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