Four parametric q-supercongruences from a quadratic summation of Gasper and Rahman

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Abstract. Recently, He and Wang [Proc. Amer. Math. Soc. (2024), 4775–4784] obtained four q-supercongruences by employing the creative microscoping method and a quadratic summation of Gasper and Rahman. In this paper, using the same method, we give a generalization of these four q-supercongruences with an additional parameter. As conclusions, we deduce four supercongruences modulo p^3 for odd primes r, such as

$$\sum_{k=2s}^{(p^r-1)/2+s}(6k+1)\frac{(\frac{1}{4})_{k+s}(\frac{1}{4})_k(\frac{1}{2})_{k-s}}{k!(k-2s)!(k-s)!}4^{k-s}\equiv(-1)^s p^r(\tfrac{1}{4})_s(\tfrac{1}{4})_{2s} \pmod{p^3},$$

where $(x)_k=x(x+1)\cdots(x+k-1)$, $p^r\equiv 1 \pmod 4$, and $0\leqslant s\leqslant (p^r-1)/2$.

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1. Introduction

In 1914, Ramanujan [10] created a number of celebrated infinite series for $1/\pi$ (see also [1, p. 352]). Although the following is not in Ramanujan's list, it provides such an example:

$$\sum_{k=0}^{\infty}(-1)^k(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 8^k}=\frac{2\sqrt{2}}{\pi},$$

which may be deemed a special case of Gosper's ${}_4F_3$ summation [4]. Here $(x)_a=\Gamma(x+a)/\Gamma(x)$ is the *Pochhammer symbol* also for a not being a non-negative integer ($\Gamma(x)$ is the classical Gamma function). In 1997, motivated by Ramanujan's work, Van Hamme [12] noticed 13 amazing supercongruences, which he labeled as (A.2)–(M.2) in sequence, such as: for any odd prime p,

$$\sum_{k=0}^{(p-1)/2}(-1)^k(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 8^k}\equiv p\left(\frac{-2}{p}\right) \pmod{p^3}, \tag{1.1}$$

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where $(\frac{\cdot}{p})$ denotes the Legendre symbol modulo p. By means of Long's method [9], Swisher [11] confirmed the supercongruence (1.1). Besides, a WZ (Wilf–Zeilberger) proof of (1.1) modulo p^2 was given by Chen, Xie, and He [2] shortly afterwards.

In 2019, the first author and Zudilin [6] introduced a method called "creative microscoping" to give a q-analogue of (1.1): for any positive odd integer n,

$$\sum_{k=0}^{(n-1)/2}(-1)^k[6k+1]\frac{(q;q^2)_k^3}{(q^4;q^4)_k^3}\equiv(-q)^{(1-n)(n+5)/8}[n] \pmod{[n]\Phi_n(q)^2}. \tag{1.2}$$

Here and in what follows, $[n]=(1-q^n)/(1-q)$ is the *q-integer*,

$$(a;q)_n=\begin{cases}(1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if } n=1,2,\dots,\\ 1, & \text{if } n=0,\\ \dfrac{1}{(1-aq^{-1})(1-q^{-2})\cdots(1-aq^{n})}, & \text{if } n=-1,-2,\dots\end{cases}$$

is the *q-shifted factorial*, and $(a;q)_\infty=\lim_{n\to\infty}(a;q)_n$. For the sake of simplicity, we will also use the abbreviated notation: $(a_1,a_2,\dots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n$ for $n\in\mathbb{Z}$ or $n=\infty$. Moreover, $\Phi_n(q)$ stands for the n-th *cyclotomic polynomial* in q, which can be written as

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}}(q-\zeta^k),$$

where ζ is any n-th primitive root of unity.

Recently, employing the creative microscoping method along with a quadratic summation of Gasper and Rahman, He and Wang [7, Theorem 2.4] obtained the following q-supercongruence similar to (1.2): for any positive integer n with $n\equiv 1 \pmod 4$,

$$\sum_{k=0}^{(n-1)/2}[6k+1]\frac{(q;q^4)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{-k^2}\equiv q^{(1-n)/2}[n] \pmod{[n]\Phi_n(q)^2}. \tag{1.3}$$

In this paper, we shall give a generalization of (1.3) in the modulus $\Phi_n(q)^3$ case as follows.

Theorem 1.1. *Let $n>1$ be an integer with $n\equiv 1 \pmod 4$ and let $0\leqslant s\leqslant (n-1)/2$. Then*

$$\begin{aligned}\sum_{k=2s}^{(n-1)/2+s}&[6k+1]\frac{(q;q^4)_{k+s}(q;q^4)_k(q^2;q^4)_{k-s}}{(q^2;q^2)_k(q^2;q^2)_{k-2s}(q^4;q^4)_{k-s}}q^{-(k+2s)^2}\\ &\equiv(-1)^s q^{3s-10s^2+(1-n)/2}(q;q^4)_s(q;q^4)_{2s}[n] \pmod{\Phi_n(q)^3}. \end{aligned}\tag{1.4}$$

For n prime, dividing both sides of (1.4) by the polynomial $(q^4;q^4)_{3s}$ and taking the limits as $q\to 1$, we get the following supercongruence: for any prime power $p^r\equiv 1 \pmod 4$ and non-negative integer $s\leqslant (p^r-1)/2$,

$$\sum_{k=2s}^{(p^r-1)/2+s}(6k+1)\frac{(\frac14)_{k+s}(\frac14)_k(\frac12)_{k-s}}{k!(k-2s)!(k-s)!}4^{k-s}\equiv(-1)^s p^r(\tfrac14)_s(\tfrac14)_{2s}\pmod{p^3}.$$

He and Wang [7, Theorem 2.4] also proved that, for any positive integer n with $n\equiv 3 \pmod 4$,

$$\sum_{k=0}^{n-1}[6k+1]\frac{(q;q^4)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{-k^2}\equiv q^{(1-3n)/2}[3n]\pmod{[n]\Phi_n(q)^2}. \tag{1.5}$$

Similarly, we have the following generalization of (1.5) in the modulus $\Phi_n(q)^3$ case.

Theorem 1.2. *Let n be a positive integer with $n\equiv 3 \pmod 4$ and $0\leqslant s\leqslant (n-1)/2$. Then*

$$\begin{aligned}\sum_{k=2s}^{n-1}[6k+1]&\frac{(q;q^4)_{k+s}(q;q^4)_k(q^2;q^4)_{k-s}}{(q^2;q^2)_k(q^2;q^2)_{k-2s}(q^4;q^4)_{k-s}}q^{-(k+2s)^2}\\&\equiv(-1)^s q^{3s-10s^2+(1-3n)/2}(q;q^4)_s(q;q^4)_{2s}[3n]\pmod{\Phi_n(q)^3}.\end{aligned} \tag{1.6}$$

For n prime, dividing both sides of (1.6) by $(q^4;q^4)_{3s}$ and letting $q\to 1$, we arrive at the following conclusion: for any prime power $p^r\equiv 3 \pmod 4$ and $0\leqslant s\leqslant (p^r-1)/2$,

$$\sum_{k=2s}^{p^r-1}(6k+1)\frac{(\frac14)_{k+s}(\frac14)_k(\frac12)_{k-s}}{k!(k-2s)!(k-s)!}4^{k-s}\equiv(-1)^s 3p^r(\tfrac14)_s(\tfrac14)_{2s}\pmod{p^3}.$$

The other two results of [7, Theorem 2.4] can be stated as follows: for any positive integer $n\equiv 3 \pmod 4$,

$$\sum_{k=0}^{(n+1)/2}[6k-1]\frac{(q^{-1};q^4)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{2k-k^2}\equiv q^{-(n+1)/2}[n]\pmod{[n]\Phi_n(q)^2}, \tag{1.7}$$

and for any integer $n>1$ with $n\equiv 1 \pmod 4$,

$$\sum_{k=0}^{n-1}[6k-1]\frac{(q^{-1};q^4)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{2k-k^2}\equiv q^{-(3n+1)/2}[3n]\pmod{[n]\Phi_n(q)^2}. \tag{1.8}$$

Like before, we shall build the following two theorems, which are generalizations of (1.7) and (1.8) modulo $\Phi_n(q)^3$, respectively.

Theorem 1.3. *Let n be a positive integer with $n \equiv 3 \pmod 4$ and $0 \leqslant s \leqslant (n-3)/2$. Then*

$$\sum_{k=2s}^{(n+1)/2+s} [6k-1] \frac{(q^{-1};q^4)_{k+s}(q^{-1};q^4)_k(q^2;q^4)_{k-s}}{(q^2;q^2)_k(q^2;q^2)_{k-2s}(q^4;q^4)_{k-s}} q^{2k-(k+2s)^2}$$
$$\equiv (-1)^s q^{5s-10s^2-(1+n)/2}(q^{-1};q^4)_{2s}(q^{-1};q^4)_s[n] \pmod{\Phi_n(q)^3}. \tag{1.9}$$

From (1.9) we can deduce that, for any prime power p^r satisfying $p^r \equiv 3 \pmod 4$ and $0 \leqslant s \leqslant (p^r-3)/2$,

$$\sum_{k=2s}^{(p^r+1)/2+s} (6k-1) \frac{(-\frac{1}{4})_{k+s}(-\frac{1}{4})_k(\frac{1}{2})_{k-s}}{k!(k-2s)!(k-s)!} 4^{k-s} \equiv (-1)^s p^r (-\tfrac{1}{4})_{2s}(-\tfrac{1}{4})_s \pmod{p^3}.$$

Theorem 1.4. *Let n be a positive integer with $n > 1$ and $n \equiv 1 \pmod 4$, and $0 \leqslant s \leqslant (n-1)/2$. Then*

$$\sum_{k=2s}^{n-1} [6k-1] \frac{(q^{-1};q^4)_{k+s}(q^{-1};q^4)_k(q^2;q^4)_{k-s}}{(q^2;q^2)_k(q^2;q^2)_{k-2s}(q^4;q^4)_{k-s}} q^{2k-(k+2s)^2}$$
$$\equiv (-1)^s q^{5s-10s^2-(1+3n)/2}(q^{-1};q^4)_{2s}(q^{-1};q^4)_s[3n] \pmod{\Phi_n(q)^3}. \tag{1.10}$$

As a conclusion of (1.10), we have the following supercongruence: for any prime power $p^r \equiv 1 \pmod 4$ and $0 \leqslant s \leqslant (p^r-1)/2$,

$$\sum_{k=2s}^{p^r-1} (6k-1) \frac{(-\frac{1}{4})_{k+s}(-\frac{1}{4})_k(\frac{1}{2})_{k-s}}{k!(k-2s)!(k-s)!} 4^{k-s} \equiv (-1)^s 3p^r (-\tfrac{1}{4})_{2s}(-\tfrac{1}{4})_s \pmod{p^3}.$$

Recall that Gasper and Rahman's quadratic summation (see [3, eq. (3.8.12)]) may be stated as follows:

$$\sum_{k=0}^{\infty} \frac{1-aq^{3k}}{1-a} \frac{(a,b,q/b;q)_k(d,f,a^2q/df;q^2)_k q^k}{(aq/d,aq/f,df/a;q)_k(q^2,aq^2/b,abq;q^2)_k}$$
$$+ \frac{(aq,f/a,b,q/b;q)_\infty(d,a^2q/df,fq^2/d,df^2q/a^2;q^2)_\infty}{(a/f,fq/a,aq/d,df/a;q)_\infty(aq^2/b,abq,fq/ab,bf/a;q^2)_\infty}$$
$$\times \sum_{k=0}^{\infty} \frac{(f,bf/a,fq/ab;q^2)_k q^{2k}}{(q^2,fq^2/d,df^2q/a^2;q^2)_k}$$
$$= \frac{(aq,f/a;q)_\infty(aq^2/bd,abq/d,bdf/a,dfq/ab;q^2)_\infty}{(aq/d,df/a;q)_\infty(aq^2/b,abq,bf/a,fq/ab;q^2)_\infty}. \tag{1.11}$$

It plays an important part in the study of q-supercongruences. Wei [13] first applied (1.11) to derive a q-analogue of a supercongruence of Liu [8]. Motivated by Wei's work, the first author [5] proved more supercongruences related to (1.11). We shall prove Theorems 1.1–1.4 in Sections 2–5, respectively, by making use of the creative microscoping method and Gasper and Rahman's quadratic summation (1.11) again.

2. Proof of Theorem 1.1

Letting $b=q^{-2n}$ and then taking $n\to\infty$ in (1.11), we obtain the following summation:

$$\sum_{k=0}^{\infty}\frac{1-aq^{3k}}{1-a}\frac{(a;q)_k(d,f,a^2q/df;q^2)_k q^{(k-k^2)/2}}{(q^2;q^2)_k(aq/d,aq/f,df/a;q)_k}a^{-k}=\frac{(aq^2,dq/a,fq/a,aq^2/df;q^2)_\infty}{(q/a,aq^2/d,aq^2/f,dfq/a;q^2)_\infty}, \quad (2.1)$$

which was noticed by He and Wang [7, eq. (5.2)].

We first give a q-congruence modulo $\Phi_n(q)$ with the help of (2.1).

Lemma 2.1. *Let n be an odd integer with $n>1$. Let a be an indeterminate, and $0\leqslant s\leqslant (n-1)/2$. Then*

$$\sum_{k=s}^{(n-1)/2+s}[6k+1]\frac{(aq;q^4)_{k+s}(q/a;q^4)_k(q^2;q^4)_{k-s}}{(aq^2;q^2)_k(q^2/a;q^2)_{k-2s}(q^4;q^4)_{k-s}}q^{-(k+2s)^2}\equiv 0 \pmod{\Phi_n(q)}. \quad (2.2)$$

Proof. It is easy to see that the left-hand side of (2.2) can be written as

$$\begin{aligned}&\sum_{k=s}^{(n-1)/2+s}[6k+1]\frac{(aq;q^4)_{k+s}(q/a;q^4)_k(q^2;q^4)_{k-s}}{(aq^2;q^2)_k(q^2/a;q^2)_{k-2s}(q^4;q^4)_{k-s}}q^{-(k+2s)^2}\\&=q^{-9s^2}\frac{(aq;q^4)_{2s}(q/a;q^4)_s(1-q^{6s+1})}{(aq^2;q^2)_s(q^2/a;q^2)_{-s}(1-q)}\\&\quad\times\sum_{k=0}^{(n-1)/2}\frac{(1-q^{6k+6s+1})(aq^{8s+1},q^{4s+1}/a,q^2;q^4)_k}{(1-q^{6s+1})(aq^{2s+2},q^{-2s+2}/a;q^2)_k(q^4;q^4)_k}q^{-k^2-6ks}. \quad (2.3)\end{aligned}$$

If $0\le s<(n-1)/6$, making the parameter substitutions $q\mapsto q^2$, $a=q^{6s-n+1}$, $d=aq^{8s+1}$, and $f=q^{4s+1}/a$ in (2.1), we get

$$\begin{aligned}&\sum_{k=0}^{(n-1)/2}\frac{(1-q^{6k+6s-n+1})(q^{6s-n+1};q^2)_k(aq^{8s+1},q^{4s+1}/a,q^{-2n+2};q^4)_k}{(1-q^{6s-n+1})(q^4;q^4)_k(q^{-2s-n+2}/a,aq^{2s-n+2},q^{6s+n+1};q^2)_k}q^{-k^2-6ks+kn}\\&=\frac{(q^{6s-n+5},aq^{2s+n+2},q^{-2s+n+2}/a,q^{-6s-n+3};q^4)_\infty}{(q^{-6s+n+1},q^{-2s-n+4}/a,aq^{2s-n+4},q^{6s+n+3};q^4)_\infty}\\&=\frac{(q^{6s-n+5},q^{-6s-n+3};q^4)_{(n-1)/2}}{(q^{-2s-n+4}/a,aq^{2s-n+4};q^4)_{(n-1)/2}},\end{aligned}$$

where we have used the fact $(q^{-2n+2};q^4)_k=0$ for $k>(n-1)/2$. It is not difficult to see that $(q^{6s-n+5},q^{-6s-n+3};q^4)_{(n-1)/2}$ is congruent to 0 modulo $\Phi_n(q)$. Since $q^n\equiv 1 \pmod{\Phi_n(q)}$, we conclude that the right-hand side of (2.3) vanishes modulo $\Phi_n(q)$.

If $s=(n-1)/6$, we note that

$$\frac{1-q^{6s+1}}{1-q}=\frac{1-q^n}{1-q}=[n].$$

By definition, $[n]$ is divisible by $\Phi_n(q)$. Thus, the lemma holds in this case as well.

If $(n-1)/6 < s \le (n-1)/2$,

$$\frac{(q^{6s-n+5}, q^{-6s-n+3}; q^4)_{(n-1)/2}}{(q^{-2s-n+4}/a, aq^{2s-n+4}; q^4)_{(n-1)/2}} \equiv \frac{(q^{6s-3n+5}, q^{-6s+n+3}; q^4)_{(n-1)/2}}{(q^{-2s-n+4}/a, aq^{2s-n+4}; q^4)_{(n-1)/2}} \pmod{\Phi_n(q)},$$

and at this moment $(q^{6s-3n+5}, q^{-6s+n+3}; q^4)_{(n-1)/2} = 0$. This proves the lemma. □

We now give the following parametric version of Theorem 1.1.

Theorem 2.2. *Let $n > 1$ be an integer with $n \equiv 1 \pmod 4$ and $0 \leqslant s \leqslant (n-1)/2$. Let a be an indeterminate. Then*

$$\sum_{k=s}^{(n-1)/2+s} [6k+1] \frac{(aq; q^4)_{k+s}(q/a; q^4)_k (q^2; q^4)_{k-s}}{(aq^2; q^2)_k (q^2/a; q^2)_{k-2s} (q^4; q^4)_{k-s}} q^{-(k+2s)^2}$$
$$\equiv (-a)^{-s}(aq; q^4)_{2s}(q/a; q^4)_s q^{3s-10s^2+(1-n)/2}[n] \pmod{\Phi_n(q)(1-aq^n)(a-q^n)}. \quad (2.4)$$

Proof. For $a = q^n$, the left-hand side of (2.4) can be written as

$$\sum_{k=s}^{(n-1)/2+s} [6k+1] \frac{(q^{n+1}; q^4)_{k+s}(q^{-n+1}; q^4)_k (q^2; q^4)_{k-s}}{(q^{n+2}; q^2)_k (q^{-n+2}; q^2)_{k-2s} (q^4; q^4)_{k-s}} q^{-(k+2s)^2}$$
$$= \sum_{k=0}^{(n-1)/2} [6k+6s+1] \frac{(q^{n+1}; q^4)_{k+2s}(q^{-n+1}; q^4)_{k+s} (q^2; q^4)_k}{(q^{n+2}; q^2)_{k+s} (q^{-n+2}; q^2)_{k-s} (q^4; q^4)_k} q^{-(k+3s)^2}$$
$$= q^{-9s^2} \frac{(q^{n+1}; q^4)_{2s}(q^{-n+1}; q^4)_s (1-q^{6s+1})}{(q^{n+2}; q^2)_s (q^{-n+2}; q^2)_{-s} (1-q)}$$
$$\times \sum_{k=0}^{(n-1)/2} \frac{(1-q^{6k+6s+1})(q^{8s+n+1}, q^{4s-n+1}, q^2; q^4)_k}{(1-q^{6s+1})(q^{2s+n+2}, q^{-2s-n+2}; q^2)_k (q^4; q^4)_k} q^{-k^2-6ks}. \quad (2.5)$$

If $(n-1)/4 < s \leqslant (n-1)/2$, then the right-hand side of (2.5) vanishes, because it contains the factor $(q^{-n+1}; q^4)_s$ in the numerator. In this case, both sides of (2.4) are both equal to 0. We now assume that $0 \leqslant s \leqslant (n-1)/4$. Performing the parameter substitutions $q \mapsto q^2$, $a = q^{6s+1}$, $d = aq^{8s+1}$ and $f = q^{4s+1}/a$ in (2.1), we have

$$\sum_{k=0}^{\infty} \frac{(1-q^{6k+6s+1})(aq^{8s+1}, q^{4s+1}/a, q^2; q^4)_k q^{-k^2-6ks}}{(1-q^{6s+1})(q^4; q^4)_k (q^{-2s+2}/a, aq^{2s+2}; q^2)_k} = \frac{(q^{6s+5}, aq^{2s+2}, q^{-2s+2}/a, q^{-6s+3}; q^4)_\infty}{(q^{-6s+1}, q^{-2s+4}/a, aq^{2s+4}, q^{6s+3}; q^4)_\infty}. \quad (2.6)$$

Since $n \equiv 1 \pmod 4$, putting $a = q^n$ in (2.6) we obtain

$$\sum_{k=0}^{(n-1)/2} \frac{(1-q^{6k+6s+1})(q^{8s+n+1}, q^{4s-n+1}, q^2; q^4)_k q^{-k^2-6ks}}{(1-q^{6s+1})(q^{2s+n+2}, q^{-2s-n+2}; q^2)_k (q^4; q^4)_k}$$
$$= \frac{(q^{-6s+3}; q^4)_{3s}(q^{-2s-n+2}; q^4)_{s+(n+1)/2}}{(q^{-6s+1}; q^4)_{3s+1}(q^{-2s-n+4}; q^4)_{s+(n-1)/2}}, \quad (2.7)$$

where we have used the fact $(q^{4s-n+1};q^4)_k=0$ for $k>(n-1)/4-s$. In view of (2.7), the right-hand side of (2.5) is equal to

$$
\begin{aligned}
&q^{-9s^2}\frac{(q^{n+1};q^4)_{2s}(q^{-n+1};q^4)_s(1-q^{6s+1})(q^{-6s+3};q^4)_{3s}(q^{-2s-n+2};q^4)_{s+(n+1)/2}}{(q^{n+2};q^2)_s(q^{-n+2};q^2)_{-s}(1-q)(q^{-6s+1};q^4)_{3s+1}(q^{-2s-n+4};q^4)_{s+(n-1)/2}}\\
&\quad=(-q)^{-ns}(q^{n+1};q^4)_{2s}(q^{-n+1};q^4)_s q^{3s-10s^2+(1-n)/2}[n].
\end{aligned}
$$

Namely, both sides of (2.4) are equal for $a=q^n$. This proves that the q-congruence (2.4) is true modulo $a-q^n$ whether $s>(n-1)/4$ or not.

For $a=q^{-n}$, the left-hand side of (2.4) can be written as

$$
\begin{aligned}
&\sum_{k=s}^{(n-1)/2+s}[6k+1]\frac{(q^{-n+1};q^4)_{k+s}(q^{n+1};q^4)_k(q^2;q^4)_{k-s}}{(q^{-n+2};q^2)_k(q^{n+2};q^2)_{k-2s}(q^4;q^4)_{k-s}}q^{-(k+2s)^2}\\
&\quad=q^{-9s^2}\frac{(q^{-n+1};q^4)_{2s}(q^{n+1};q^4)_s(1-q^{6s+1})}{(q^{-n+2};q^2)_s(q^{n+2};q^2)_{-s}(1-q)}\\
&\qquad\times\sum_{k=0}^{(n-1)/2}\frac{(1-q^{6k+6s+1})(q^{8s-n+1},q^{4s+n+1},q^2;q^4)_k}{(1-q^{6s+1})(q^{2s-n+2},q^{-2s+n+2};q^2)_k(q^4;q^4)_k}q^{-k^2-6ks}.
\end{aligned}
\tag{2.8}
$$

If $(n-1)/8<s\leqslant(n-1)/2$, then the right-hand side of (2.8) vanishes, because of the factor $(q^{-n+1};q^4)_{2s}$. Thus, the two sides of (2.4) are equal to 0. We now assume that $0\leqslant s\leqslant(n-1)/8$. Letting $a=q^{-n}$ in (2.6) leads to

$$
\begin{aligned}
&\sum_{k=0}^{(n-1)/2}\frac{(1-q^{6k+6s+1})(q^{8s-n+1},q^{4s+n+1},q^2;q^4)_k}{(1-q^{6s+1})(q^{2s-n+2},q^{-2s+n+2};q^2)_k(q^4;q^4)_k}q^{-k^2-6ks}\\
&\quad=\frac{(q^{-6s+3};q^4)_{3s}(q^{2s-n+2};q^4)_{(n+1)/2-s}}{(q^{-6s+1};q^4)_{3s+1}(q^{2s-n+4};q^4)_{(n-1)/2-s}},
\end{aligned}
\tag{2.9}
$$

where we have used $(q^{8s-n+1};q^4)_k=0$ for $k>(n-1)/8-s$. In light of (2.9), the right-hand side of (2.8) is equal to

$$
\begin{aligned}
&q^{-9s^2}\frac{(q^{-n+1};q^4)_{2s}(q^{n+1};q^4)_s(1-q^{6s+1})(q^{-6s+3};q^4)_{3s}(q^{2s-n+2};q^4)_{(n+1)/2-s}}{(q^{-n+2};q^2)_s(q^{n+2};q^2)_{-s}(1-q)(q^{-6s+1};q^4)_{3s+1}(q^{2s-n+4};q^4)_{(n-1)/2-s}}\\
&\quad=(-q)^{ns}(q^{-n+1};q^4)_{2s}(q^{n+1};q^4)_s q^{3s-10s^2+(1-n)/2}[n].
\end{aligned}
$$

This means that (2.4) is true modulo $1-aq^n$ whether $s>(n-1)/8$ or not.

Since $[n]\equiv 0 \pmod{\Phi_n(q)}$, Lemma 2.1 implies that (2.4) is also true modulo $\Phi_n(q)$. The proof then follows from the fact that $1-aq^n$, $a-q^n$, and $\Phi_n(q)$ are pairwise coprime polynomials in q. □

Proof of Theorem 1.1. When $a=1$, the denominators of the left-hand side of (2.4) are coprime with $\Phi_n(q)$. In the same time, the polynomial $(1-aq^n)(a-q^n)=(1-q^n)^2$ incorporates the factor $\Phi_n(q)^2$. Thus, taking the limits as $a\to 1$ in (2.4), we are led to the desired q-supercongruence (1.4). □

3. Proof of Theorem 1.2

The proof is similar to that of Theorem 1.1. We first establish the following parametric generalization of Theorem 1.2. Noticing that $1/(q^4;q^4)_k = 0$ for any negative integer k, we can summing over k from s to $n-1$ on the left-hand side of (1.6).

Theorem 3.1. *Let n be a positive integer with $n \equiv 3 \pmod 4$ and let $0 \leqslant s \leqslant (n-1)/2$. Let a be an indeterminate. Then, modulo $\Phi_n(q)(1-aq^{3n})(a-q^{3n})$,*

$$\sum_{k=s}^{n-1}[6k+1]\frac{(aq;q^4)_{k+s}(q/a;q^4)_k(q^2;q^4)_{k-s}}{(aq^2;q^2)_k(q^2/a;q^2)_{k-2s}(q^4;q^4)_{k-s}}q^{-(k+2s)^2}$$
$$\equiv (-a)^{-s}(aq;q^4)_{2s}(q/a;q^4)_s q^{3s-10s^2+(1-3n)/2}[3n]. \tag{3.1}$$

Proof. For $a=q^{3n}$, the left-hand side of (3.1) can be written as

$$\sum_{k=s}^{n-1}[6k+1]\frac{(q^{3n+1};q^4)_{k+s}(q^{-3n+1};q^4)_k(q^2;q^4)_{k-s}}{(q^{3n+2};q^2)_k(q^{-3n+2};q^2)_{k-2s}(q^4;q^4)_{k-s}}q^{-(k+2s)^2}$$
$$= q^{-9s^2}\frac{(q^{3n+1};q^4)_{2s}(q^{-3n+1};q^4)_s(1-q^{6s+1})}{(q^{3n+2};q^2)_s(q^{-3n+2};q^2)_{-s}(1-q)}$$
$$\times \sum_{k=0}^{n-1-s}\frac{(1-q^{6k+6s+1})(q^{8s+3n+1},q^{4s-3n+1},q^2;q^4)_k}{(1-q^{6s+1})(q^{2s+3n+2},q^{-2s-3n+2};q^2)_k(q^4;q^4)_k}q^{-k^2-6ks}. \tag{3.2}$$

Since $3n \equiv 1 \pmod 4$ and $s \leqslant (3n-1)/4$, using the $n \to 3n$ version of (2.7), we see that the right-hand side of (3.2) is equal to

$$q^{-9s^2}\frac{(q^{3n+1};q^4)_{2s}(q^{-3n+1};q^4)_s(1-q^{6s+1})(q^{-6s+3};q^4)_{3s}(q^{-2s-3n+2};q^4)_{s+(3n+1)/2}}{(q^{3n+2};q^2)_s(q^{-3n+2};q^2)_{-s}(1-q)(q^{-6s+1};q^4)_{3s+1}(q^{-2s-3n+4};q^4)_{s+(3n-1)/2}}$$
$$= (-q)^{-3ns}(q^{3n+1};q^4)_{2s}(q^{-3n+1};q^4)_s q^{3s-10s^2+(1-3n)/2}[3n].$$

This indicates that (3.1) holds modulo $a-q^{3n}$.

For $a=q^{-3n}$, the left-hand side of (3.1) can be written as

$$q^{-9s^2}\frac{(q^{-3n+1};q^4)_{2s}(q^{3n+1};q^4)_s(1-q^{6s+1})}{(q^{-3n+2};q^2)_s(q^{3n+2};q^2)_{-s}(1-q)}$$
$$\times \sum_{k=0}^{n-1-s}\frac{(1-q^{6k+6s+1})(q^{8s-3n+1},q^{4s+3n+1},q^2;q^4)_k}{(1-q^{6s+1})(q^{2s-3n+2},q^{-2s+3n+2};q^2)_k(q^4;q^4)_k}q^{-k^2-6ks}. \tag{3.3}$$

If $(3n-1)/8 < s \leqslant (n-1)/2$, then both sides of (3.1) vanish, because of the factor $(q^{-3n+1};q^4)_{2s}$. We now assume that $0 \leqslant s \leqslant (3n-1)/8$. Using the $n \to 3n$ version of (2.9), we conclude that (3.3) is equal to

$$q^{-9s^2}\frac{(q^{-3n+1};q^4)_{2s}(q^{3n+1};q^4)_s(1-q^{6s+1})(q^{-6s+3};q^4)_{3s}(q^{2s-3n+2};q^4)_{(3n+1)/2-s}}{(q^{-3n+2};q^2)_s(q^{3n+2};q^2)_{-s}(1-q)(q^{-6s+1};q^4)_{3s+1}(q^{2s-3n+4};q^4)_{(3n-1)/2-s}}$$
$$= (-q)^{3ns}(q^{-3n+1};q^4)_{2s}(q^{3n+1};q^4)_s q^{3s-10s^2+(1-3n)/2}[3n].$$

This implies that (3.1) holds modulo $1-aq^{3n}$.

It is obvious that $(q^2;q^4)_{k-s}/(q^4;q^4)_{k-s}$ is congruent to 0 modulo $\Phi_n(q)$ for $(n-1)/2+s<k\le n-1$, and that $[3n]\equiv 0 \pmod{\Phi_n(q)}$. By Lemma 2.1, the q-congruence (3.1) also holds modulo $\Phi_n(q)$. The proof then follows from the property that the polynomials $1-aq^{3n}$, $a-q^{3n}$, and $\Phi_n(q)$ are pairwise coprime. □

Proof of Theorem 1.2. Letting $a=1$ in (3.1), we obtain the desired q-congruence (1.6). □

4. Proof of Theorem 1.3

As before, we first present the following lemma from Gasper and Rahman's summation (1.11).

Lemma 4.1. *Let n be an odd integer with $n>1$. Let a be an indeterminate, and $0\leqslant s\leqslant (n-1)/2$. Then*

$$\sum_{k=s}^{(n+1)/2+s}[6k-1]\frac{(aq^{-1};q^4)_{k+s}(q^{-1}/a;q^4)_k(q^2;q^4)_{k-s}}{(aq^2;q^2)_k(q^2/a;q^2)_{k-2s}(q^4;q^4)_{k-s}}q^{2k-(k+2s)^2}\equiv 0 \pmod{\Phi_n(q)}. \quad (4.1)$$

Proof. It is easy to see that the left-hand side of (4.1) can be written as

$$\begin{aligned}&q^{2s-9s^2}\frac{(aq^{-1};q^4)_{2s}(q^{-1}/a;q^4)_s(1-q^{6s-1})}{(aq^2;q^2)_s(q^2/a;q^2)_{-s}(1-q)}\\&\quad\times\sum_{k=0}^{(n+1)/2}\frac{(1-q^{6k+6s-1})(aq^{8s-1},q^{4s-1}/a,q^2;q^4)_k}{(1-q^{6s-1})(aq^{2s+2},q^{-2s+2}/a;q^2)_k(q^4;q^4)_k}q^{2k-k^2-6ks}.\end{aligned} \quad (4.2)$$

If $0\le s<(n+1)/6$, letting $q\mapsto q^2$, $a=q^{6s-n-1}$, $d=aq^{8s-1}$, and $f=q^{4s-1}/a$ in (2.1), we get

$$\begin{aligned}&\sum_{k=0}^{(n+1)/2}\frac{(1-q^{6k+6s-n-1})(q^{6s-n+1};q^2)_k(aq^{8s-1},q^{4s-1}/a,q^{-2n+2};q^4)_k}{(1-q^{6s-n-1})(q^4;q^4)_k(q^{-2s-n+2}/a,aq^{2s-n+2},q^{6s+n+1};q^2)_k}q^{-k^2-6ks+kn}\\&\quad=\frac{(q^{6s-n+3},aq^{2s+n+2},q^{-2s+n+2}/a,q^{-6s-n+5};q^4)_\infty}{(q^{-6s+n+3},q^{-2s-n+4}/a,aq^{2s-n+4},q^{6s+n+1};q^4)_\infty}\\&\quad=\frac{(q^{6s-n+3},q^{-6s-n+5};q^4)_{(n-1)/2}}{(q^{-2s-n+4}/a,aq^{2s-n+4};q^4)_{(n-1)/2}},\end{aligned}$$

where we have utilized the property $(q^{-2n+2};q^4)_k=0$ for $k>(n-1)/2$. It is not hard to see that $(q^{6s-n+3},q^{-6s-n+5};q^4)_{(n-1)/2}$ is congruent to 0 modulo $\Phi_n(q)$. Since $q^n\equiv 1 \pmod{\Phi_n(q)}$, we conclude that (4.2) vanishes modulo $\Phi_n(q)$.

If $s=(n+1)/6$, we have

$$\frac{(1-q^{6s-1})}{(1-q)}=\frac{(1-q^n)}{(1-q)}=[n],$$

according to the definition, [n] can be divided by $\Phi_n(q)$. Thus, the lemma also holds in this case.

If $(n+1)/6<s\le (n-1)/2$,

$$\frac{(q^{6s-n+3},q^{-6s-n+5};q^4)_{(n-1)/2}}{(q^{-2s-n+4}/a,aq^{2s-n+4};q^4)_{(n-1)/2}}\equiv\frac{(q^{6s-3n+3},q^{-6s+n+5};q^4)_{(n-1)/2}}{(q^{-2s-n+4}/a,aq^{2s-n+4};q^4)_{(n-1)/2}}\pmod{\Phi_n(q)}$$

and at this moment $(q^{6s-3n+3},q^{-6s+n+5};q^4)_{(n-1)/2}=0$. This completes the proof. □

Now we can give a parametric generalization of Theorem 1.3.

Theorem 4.2. *Let n be a positive integer with $n\equiv 3 \pmod 4$ and $0\leqslant s\leqslant (n-1)/2$. Then, modulo $\Phi_n(q)(1-aq^n)(a-q^n)$,*

$$\begin{aligned}&\sum_{k=s}^{(n+1)/2+s}[6k-1]\frac{(aq^{-1};q^4)_{k+s}(q^{-1}/a;q^4)_k(q^2;q^4)_{k-s}}{(aq^2;q^2)_k(q^2/a;q^2)_{k-2s}(q^4;q^4)_{k-s}}q^{2k-(k+2s)^2}\\&\quad\equiv(-a)^{-s}(aq^{-1};q^4)_{2s}(q^{-1}/a;q^4)_s q^{5s-10s^2-(1+n)/2}[n].\end{aligned}\tag{4.3}$$

Proof. For $a=q^n$, the left-hand side of (4.3) can be written as

$$\begin{aligned}&\sum_{k=s}^{(n+1)/2+s}[6k-1]\frac{(q^{n-1};q^4)_{k+s}(q^{-n-1};q^4)_k(q^2;q^4)_{k-s}}{(q^{n+2};q^2)_k(q^{-n+2};q^2)_{k-2s}(q^4;q^4)_{k-s}}q^{2k-(k+2s)^2}\\&=q^{2s-9s^2}\frac{(q^{n-1};q^4)_{2s}(q^{-n-1};q^4)_s(1-q^{6s-1})}{(q^{n+2};q^2)_s(q^{-n+2};q^2)_{-s}(1-q)}\\&\quad\times\sum_{k=0}^{(n+1)/2}\frac{(1-q^{6k+6s-1})(q^{8s+n-1},q^{4s-n-1},q^2;q^4)_k}{(1-q^{6s-1})(q^{2s+n+2},q^{-2s-n+2};q^2)_k(q^4;q^4)_k}q^{2k-k^2-6ks}.\end{aligned}\tag{4.4}$$

If $(n+1)/4<s\leqslant(n-1)/2$, then both sides of (4.3) are equal to 0, because of the factor $(q^{-n-1};q^4)_s$. We now assume that $0\leqslant s\leqslant(n+1)/4$. Letting $q\mapsto q^2$, $a=q^{6s-1}$, $d=aq^{8s-1}$ and $f=q^{4s-1}/a$ in (2.1), we have

$$\sum_{k=0}^{\infty}\frac{(1-q^{6k+6s-1})(aq^{8s-1},q^{4s-1}/a,q^2;q^4)_k q^{2k-k^2-6ks}}{(1-q^{6s-1})(q^4;q^4)_k(q^{-2s+2}/a,aq^{2s+2};q^2)_k}=\frac{(q^{6s+3},aq^{2s+2},q^{-2s+2}/a,q^{-6s+5};q^4)_\infty}{(q^{-6s+3},q^{-2s+4}/a,aq^{2s+4},q^{6s+1};q^4)_\infty}.\tag{4.5}$$

Since $n\equiv 3\pmod 4$, taking $a=q^n$ in (4.5) we get

$$\begin{aligned}&\sum_{k=0}^{(n+1)/2}\frac{(1-q^{6k+6s-1})(q^{8s+n-1},q^{4s-n-1},q^2;q^4)_k}{(1-q^{6s-1})(q^{2s+n+2},q^{-2s-n+2};q^2)_k(q^4;q^4)_k}q^{2k-k^2-6ks}\\&\quad=\frac{(q^{-6s+5};q^4)_{3s-1}(q^{-2s-n+2};q^4)_{s+(n+1)/2}}{(q^{-6s+3};q^4)_{3s}(q^{-2s-n+4};q^4)_{s+(n-1)/2}}.\end{aligned}\tag{4.6}$$

Plugging (4.6) into (4.4), we see that the right-hand side of (4.4) is equal to

$$q^{2s-9s^2}\frac{(q^{n-1};q^4)_{2s}(q^{-n-1};q^4)_s(1-q^{6s-1})(q^{-6s+5};q^4)_{3s-1}(q^{-2s-n+2};q^4)_{s+(n+1)/2}}{(q^{n+2};q^2)_s(q^{-n+2};q^2)_{-s}(1-q)(q^{-6s+3};q^4)_{3s}(q^{-2s-n+4};q^4)_{s+(n-1)/2}}$$
$$=(-q)^{-ns}(q^{n-1};q^4)_{2s}(q^{-n-1};q^4)_s q^{5s-10s^2-(1+n)/2}[n],$$

which is the $a=q^n$ case of the right-hand side of (4.3). This demonstrates that the q-congruence (4.3) is true modulo $a-q^n$.

For $a=q^{-n}$, the left-hand side of (4.3) can be written as

$$q^{2s-9s^2}\frac{(q^{-n-1};q^4)_{2s}(q^{n-1};q^4)_s(1-q^{6s-1})}{(q^{-n+2};q^2)_s(q^{n+2};q^2)_{-s}(1-q)}$$
$$\times\sum_{k=0}^{(n+1)/2}\frac{(1-q^{6k+6s-1})(q^{8s-n-1},q^{4s+n-1},q^2;q^4)_k}{(1-q^{6s-1})(q^{2s-n+2},q^{-2s+n+2};q^2)_k(q^4;q^4)_k}q^{2k-k^2-6ks}. \tag{4.7}$$

If $(n+1)/8<s\leqslant(n-1)/2$, then both sides of (4.3) vanish in this case. We now assume that $0\leqslant s\leqslant(n+1)/8$. Putting $a=q^{-n}$ in (4.5) yields

$$\sum_{k=0}^{(n+1)/2}\frac{(1-q^{6k+6s-1})(q^{8s-n-1},q^{4s+n-1},q^2;q^4)_k}{(1-q^{6s-1})(q^{2s-n+2},q^{-2s+n+2};q^2)_k(q^4;q^4)_k}q^{2k-k^2-6ks}$$
$$=\frac{(q^{-6s+5};q^4)_{3s-1}(q^{2s-n+2};q^4)_{(n+1)/2-s}}{(q^{-6s+3};q^4)_{3s}(q^{2s-n+4};q^4)_{(n-1)/2-s}}. \tag{4.8}$$

It follows that (4.7) is equal to

$$q^{2s-9s^2}\frac{(q^{-n-1};q^4)_{2s}(q^{n-1};q^4)_s(1-q^{6s-1})(q^{-6s+5};q^4)_{3s-1}(q^{2s-n+2};q^4)_{(n+1)/2-s}}{(q^{-n+2};q^2)_s(q^{n+2};q^2)_{-s}(1-q)(q^{-6s+3};q^4)_{3s}(q^{2s-n+4};q^4)_{(n-1)/2-s}}$$
$$=(-q)^{ns}(q^{-n-1};q^4)_{2s}(q^{n-1};q^4)_s q^{5s-10s^2-(1+n)/2}[n].$$

This confirms that (4.3) is true modulo $1-aq^n$. □

Proof of Theorem 1.3. Letting $a=1$ in (4.3) and assuming that $0\leqslant s\leqslant(n-3)/2$, we are led to (1.9). □

5. Proof of Theorem 1.4

The proof is similar to that of Theorem 1.3. We need to build the following generalization of Theorem 1.3 with an extra parameter a.

Theorem 5.1. *Let n be a positive integer with $n>1$ and $n\equiv 1 \pmod 4$, and $0\leqslant s\leqslant(n-1)/2$. Then, modulo $\Phi_n(q)(1-aq^{3n})(a-q^{3n})$,*

$$\sum_{k=s}^{n-1}[6k-1]\frac{(aq^{-1};q^4)_{k+s}(q^{-1}/a;q^4)_k(q^2;q^4)_{k-s}}{(aq^2;q^2)_k(q^2/a;q^2)_{k-2s}(q^4;q^4)_{k-s}}q^{2k-(k+2s)^2}$$
$$\equiv(-a)^{-s}(aq^{-1};q^4)_{2s}(q^{-1}/a;q^4)_s q^{5s-10s^2-(1+3n)/2}[3n]. \tag{5.1}$$

Proof. For $a = q^{3n}$, the left-hand side of (5.1) can be written as

$$q^{2s-9s^2}\frac{(q^{3n-1};q^4)_{2s}(q^{-3n-1};q^4)_s(1-q^{6s-1})}{(q^{3n+2};q^2)_s(q^{-3n+2};q^2)_{-s}(1-q)}$$
$$\times\sum_{k=0}^{n-1-s}\frac{(1-q^{6k+6s-1})(q^{8s+3n-1},q^{4s-3n-1},q^2;q^4)_k}{(1-q^{6s-1})(q^{2s+3n+2},q^{-2s-3n+2};q^2)_k(q^4;q^4)_k}q^{2k-k^2-6ks}. \tag{5.2}$$

Employing the $n\to 3n$ version of (4.6), we know that (5.2) is equal to

$$q^{2s-9s^2}\frac{(q^{3n-1};q^4)_{2s}(q^{-3n-1};q^4)_s(1-q^{6s-1})(q^{-6s+5};q^4)_{3s-1}(q^{-2s-3n+2};q^4)_{s+(3n+1)/2}}{(q^{3n+2};q^2)_s(q^{-3n+2};q^2)_{-s}(1-q)(q^{-6s+3};q^4)_{3s}(q^{-2s-3n+4};q^4)_{s+(3n-1)/2}}$$
$$=(-q)^{-3ns}(q^{3n-1};q^4)_{2s}(q^{-3n-1};q^4)_s q^{5s-10s^2-(1+3n)/2}[3n].$$

This proves that (5.1) holds modulo $a-q^{3n}$.

For $a = q^{-3n}$, the left-hand side of (5.1) can be written as

$$q^{2s-9s^2}\frac{(q^{-3n-1};q^4)_{2s}(q^{3n-1};q^4)_s(1-q^{6s-1})}{(q^{-3n+2};q^2)_s(q^{3n+2};q^2)_{-s}(1-q)}$$
$$\times\sum_{k=0}^{n-1-s}\frac{(1-q^{6k+6s-1})(q^{8s-3n-1},q^{4s+3n-1},q^2;q^4)_k}{(1-q^{6s-1})(q^{2s-3n+2},q^{-2s+3n+2};q^2)_k(q^4;q^4)_k}q^{2k-k^2-6ks}. \tag{5.3}$$

If $(3n+1)/8 < s \leqslant (n-1)/2$, then both sides of (5.1) is equal to 0 for $a = q^{-3n}$. We now assume that $0 \leqslant s \leqslant (3n+1)/8$. By the $n \to 3n$ version of (4.8), we deduce that (5.3) is equal to

$$q^{2s-9s^2}\frac{(q^{-3n-1};q^4)_{2s}(q^{3n-1};q^4)_s(1-q^{6s-1})(q^{-6s+5};q^4)_{3s-1}(q^{2s-3n+2};q^4)_{(3n+1)/2-s}}{(q^{-3n+2};q^2)_s(q^{3n+2};q^2)_{-s}(1-q)(q^{-6s+3};q^4)_{3s}(q^{2s-3n+4};q^4)_{(3n-1)/2-s}}$$
$$=(-q)^{3ns}(q^{-3n-1};q^4)_{2s}(q^{3n-1};q^4)_s q^{5s-10s^2-(1+3n)/2}[3n].$$

This proves that the q-congruence (5.1) holds modulo $1-aq^{3n}$.

In view of Lemma 4.1 also holds modulo $\Phi_n(q)$. □

Proof of Theorem 1.4. Letting $a \to 1$ in (5.1), we accomplish the proof. □

Data Availability Statements. Data sharing not applicable to this article as no datasets were generated or analysed during the current study

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