International Journal of Number Theory

A further common q-extension of two supercongruences

Victor J. W. Guo
School of Mathematics, Hangzhou Normal University
Hangzhou 311121, People's Republic of China
jwguo@hznu.edu.cn

Yi-Xin Xiao
School of Mathematics, Hangzhou Normal University
Hangzhou 311121, People's Republic of China
yixinxiao@stu.hznu.edu.cn

Received (15 February 2025)
Accepted (15 July 2025)

In 2020, the first author and Zudilin gave a q-supercongruence of which the special cases $q=-1$ and $q=1$ correspond to the (B.2) and (H.2) supercongruences of Van Hamme. In this paper, we present a further generalization of this q-supercongruence with an extra parameter.

Keywords: q-Dixon sum; q-congruence; supercongruence; creative microscoping.

Mathematics Subject Classification 2010: 33D15, 11A07, 11B65

1. Introduction

The formula of Bauer [1] from 1859,

$$\sum_{k=0}^{\infty}(-1)^k\frac{4k+1}{64^k}\binom{2k}{k}^3=\frac{2}{\pi} \tag{1.1}$$

became famous after Ramanujan [15] found a number of similar looking infinite series for the constant but with a faster convergence in 1914. The formula (1.1) can be deduced from a ${}_4F_3$ summation (known to Ramanujan). But there are several other proofs of it without using hypergeometric functions, including the original one of Bauer [1]. In 1994, Ekhad and Zeilberger [2] gave a remarkable computer proof of (1.1) by making use of the Wilf–Zeilberger (WZ) method of creative telescoping.

It was Van Hamme [21] who observed in 1997 that many Ramanujan-type series have beautiful p-adic analogues. For instance, the supercongruence

$$\sum_{k=0}^{(p-1)/2}(-1)^k\frac{4k+1}{64^k}\binom{2k}{k}^3\equiv p(-1)^{(p-1)/2} \pmod{p^3} \tag{1.2}$$

(labeled (B.2) on Van Hamme's list) holds for any odd prime p and corresponds to the identity (1.1). The supercongruence (1.2) was first confirmed by Mortenson [14] employing a ${}_6F_5$ transformation. It later received a WZ proof by [24] using the very same WZ pair as in [2]. In 2012, using the WZ method again, Sun [20] gave a generalization of (1.1) modulo p^4 for primes $p > 3$.

Another entry on Van Hamme's list [21], labeled (H.2), is the supercongruence: for any odd prime p,

$$\sum_{k=0}^{(p-1)/2} \frac{1}{64^k} \binom{2k}{k}^3 \equiv \begin{cases} -\Gamma_p(\frac{1}{4})^4 \pmod{p^2} & \text{if } p \equiv 1 \pmod 4, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod 4, \end{cases} \tag{1.3}$$

where $\Gamma_p(x)$ is the p-adic Gamma function. Van Hamme [21] himself not only observed but also confirmed (1.3). Nowadays, many authors have given different generalizations of (1.3) (see [6,8,9,16,17,18,10,11,12]). For example, Long and Ramakrishna [12, Theorem 3] established the following extension of (1.3):

$$\sum_{k=0}^{(p-1)/2} \frac{1}{64^k} \binom{2k}{k}^3 \equiv \begin{cases} -\Gamma_p(\frac{1}{4})^4 \pmod{p^3} & \text{if } p \equiv 1 \pmod 4, \\ -\frac{p^2}{16}\Gamma_p(\frac{1}{4})^4 \pmod{p^3} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

The first author and Zudilin [9, Theorem 1.1] gave a common q-analogue of the supercongruences (1.2) and (1.3): for any positive odd integer n,

$$\sum_{k=0}^{(n-1)/2} \frac{(1+q^{4k+1})\,(q^2;q^4)_k^3}{(1+q)\,(q^4;q^4)_k^3}\, q^k \equiv \frac{[n]_{q^2}(q^3;q^4)_{(n-1)/2}}{(q^5;q^4)_{(n-1)/2}}\, q^{(1-n)/2} \begin{cases} \pmod{\Phi_n(q)^2\Phi_n(-q)^3} & \text{if } n \equiv 1 \pmod 4, \\ \pmod{\Phi_n(q)^3\Phi_n(-q)^3} & \text{if } n \equiv 3 \pmod 4. \end{cases} \tag{1.4}$$

Here and throughout the paper, $\Phi_n(q)$ stands for the n-th *cyclotomic polynomial* in q; the *q-shifted factorial* is defined as $(a;q)_0 = 1$ and $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n \geqslant 1$ or $n = \infty$. For simplicity, we will also adopt the condensed notation

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n(a_2;q)_n \cdots (a_m;q)_n$$

for $n \geqslant 0$ or $n = \infty$. Moreover, $[n] = [n]_q = (1-q^n)/(1-q)$ denotes the *q-integer*. Note that $\Phi_n(q)\Phi_n(-q) = \Phi_n(q^2)$ for all positive odd integers n.

A purpose of this note is to present the following generalization of (1.4).

Theorem 1.1. *Let n be a positive odd integer, and let $0 \leqslant s \leqslant (n-1)/4$. Then*

$$\sum_{k=s}^{(n-1)/2+s} \frac{(1+q^{4k+1})(q^2;q^4)_{k-s}(q^2;q^4)_{k+s}(q^2;q^4)_k}{(1+q)(q^4;q^4)_{k-s}(q^4;q^4)_{k+s}(q^4;q^4)_k} q^k$$
$$\equiv (-1)^s [n]_{q^2} \frac{(q^3;q^4)_{(n-1)/2-s}}{(q^5;q^4)_{(n-1)/2+s}} (q;q^2)_{2s} q^{2s(n-s)-(n-1)/2}$$
$$\begin{cases} \pmod{\Phi_n(q)^2\Phi_n(-q)^3} & \text{if } n \equiv 1 \pmod 4, \\ \pmod{\Phi_n(q)^3\Phi_n(-q)^3} & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Letting $n = p^r$ be a prime power and taking $q \to -1$ in Theorem 1.1, we arrive at the following generalization of (1.2), which was already obtained by the first author [4].

Corollary 1.2. *For any odd prime p, positive integer r, and non-negative integer $s \leqslant (p^r - 1)/4$, there holds*

$$\sum_{k=s}^{(p^r-1)/2+s} \frac{4k+1}{(-64)^k} \binom{2k-2s}{k-s} \binom{2k+2s}{k+s} \binom{2k}{k} \equiv p^r (-1)^{(p-1)r/2+s} \pmod{p^3}.$$

Similarly, letting $n = p^r$ be a prime power and taking $q \to 1$ in Theorem 1.1, we get the following generalization of (1.3).

Corollary 1.3. *For any odd prime p, positive integer r, and non-negative integer $s \leqslant (p^r - 1)/4$, there holds*

$$\sum_{k=s}^{(p^r-1)/2+s} \frac{1}{64^k} \binom{2k-2s}{k-s} \binom{2k+2s}{k+s} \binom{2k}{k}$$
$$\equiv \frac{p^r(-1)^s(\frac{3}{4})_{(p^r-1)/2-s}(\frac{1}{2})_{2s}}{4^s(\frac{5}{4})_{(p^r-1)/2+s}} \begin{cases} \pmod{p^2} & \text{if } p \equiv 1 \pmod 4, \\ \pmod{p^3} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Mao and Pan [13] (see also [19, Theorem 1.3]) proved that, for any prime $p \equiv 1 \pmod 4$,

$$\sum_{k=0}^{(p+1)/2} \frac{(-\frac{1}{2})_k^3}{k!^3} \equiv 0 \pmod{p^2}. \tag{1.5}$$

On the other hand, the $r = -1$, $d = 2$ and $a = 1$ case of [7, Theorem 4.9] implies that, for any odd prime p,

$$\sum_{k=0}^{(p+1)/2} (-1)^k (4k-1) \frac{(-\frac{1}{2})_k^3}{k!^3} \equiv p(-1)^{(p+1)/2} \pmod{p^3}. \tag{1.6}$$

The first author and Zudilin [9, Theorem 1.2] also gave a common q-analogue of the supercongruences (1.5) and (1.6) as follows: for any positive odd integer n,

$$\sum_{k=0}^{(n+1)/2} \frac{(1+q^{4k-1})\,(q^{-2};q^4)_k^3}{(1+q)\,(q^4;q^4)_k^3}\,q^{7k} \equiv \frac{[n]_{q^2}(q;q^4)_{(n-1)/2}}{(q^7;q^4)_{(n-1)/2}}\,q^{(n-3)/2} \begin{cases} \pmod{\Phi_n(q)^3\Phi_n(-q)^3} & \text{if } n\equiv 1 \pmod 4, \\ \pmod{\Phi_n(q)^2\Phi_n(-q)^3} & \text{if } n\equiv 3 \pmod 4. \end{cases} \tag{1.7}$$

In this note, we shall establish the following generalization of (1.7).

Theorem 1.4. *Let n be a positive odd integer, and let $0 \leqslant s \leqslant (n-3)/4$. Then*

$$\sum_{k=s}^{(n+1)/2+s} \frac{(1+q^{4k-1})(q^{-2};q^4)_{k-s}(q^{-2};q^4)_{k+s}(q^{-2};q^4)_k}{(1+q)(q^4;q^4)_{k-s}(q^4;q^4)_{k+s}(q^4;q^4)_k}\,q^{7k}$$
$$\equiv (-1)^s \frac{q^{2s(n-s+2)+(n-9)/2}[n]_{q^2}[4s+1](q;q^4)_{(n-1)/2-s}}{[2s+1]_{q^2}[2s-1]_{q^2}[4s-1](q^7;q^4)_{(n-1)/2+s}}(q;q^2)_{2s}$$
$$\begin{cases} \pmod{\Phi_n(q)^3\Phi_n(-q)^3} & \text{if } n\equiv 1 \pmod 4, \\ \pmod{\Phi_n(q)^2\Phi_n(-q)^3} & \text{if } n\equiv 3 \pmod 4. \end{cases}$$

Letting n be a prime power and $q\to -1$ in Theorem 1.4, we get the following generalization of (1.5).

Corollary 1.5. *For any odd prime p, positive integer r, and non-negative integer $s \leqslant (p^r-3)/4$, there holds*

$$\sum_{k=s}^{(p^r+1)/2+s} (-1)^k(4k-1)\frac{(-\frac12)_{k-s}(-\frac12)_{k+s}(-\frac12)_k}{(k-s)!(k+s)!k!} \equiv p^r\,\frac{(-1)^{(p-1)r/2+s}}{4s^2-1} \pmod{p^3}.$$

Meanwhile, letting n be a prime power and $q\to 1$ in Theorem 1.4, we are led to the following generalization of (1.6).

Corollary 1.6. *For any odd prime p, positive integer r, and non-negative integer $s \leqslant (p^r-3)/4$, there holds*

$$\sum_{k=s}^{(p^r+1)/2+s} \frac{(-\frac12)_{k-s}(-\frac12)_{k+s}(-\frac12)_k}{(k-s)!(k+s)!k!} \equiv \frac{p^r(-1)^s(4s+1)(\frac14)_{(p^r-1)/2-s}(\frac12)_{2s}}{4^s(4s^2-1)(4s-1)(\frac74)_{(p^r-1)/2+s}} \begin{cases} \pmod{p^3} & \text{if } p\equiv 1 \pmod 4, \\ \pmod{p^2} & \text{if } p\equiv 3 \pmod 4. \end{cases}$$

We shall prove Theorems 1.1 and 1.4 in Sections 2 and 3, respectively.

2. Proof of Theorem 1.1

The following easily proved q-congruence (see [5, Lemma 3.1]) will play an important part in our derivation of Theorems 1.1 and 1.4.

Lemma 2.1. *Let n be a positive odd integer. Then, for $0 \leqslant k \leqslant (n-1)/2$, we have*

$$\frac{(aq;q^2)_{(n-1)/2-k}}{(q^2/a;q^2)_{(n-1)/2-k}} \equiv (-a)^{(n-1)/2-2k}\frac{(aq;q^2)_k}{(q^2/a;q^2)_k}\, q^{(n-1)^2/4+k} \pmod{\Phi_n(q)}.$$

We shall also use the q-Dixon sum (see [3, Appendix (II.13)]), which can be written as

$${}_4\phi_3\left[\begin{matrix} a, -qa^{\frac12}, b, c \\ -a^{\frac12}, aq/b, aq/c \end{matrix}; q, \frac{qa^{\frac12}}{bc}\right] = \frac{(aq, qb^{-1}a^{\frac12}, qc^{-1}a^{\frac12}, aq/bc;q)_\infty}{(aq/b, aq/c, qa^{\frac12}, qa^{\frac12}/bc;q)_\infty}, \tag{2.1}$$

where the *basic hypergeometric ${}_{r+1}\phi_r$ series* with $r+1$ upper parameters $a_1,\dots,a_{r+1}$, r lower parameters $b_1,\dots,b_r$, base q and argument z is defined as

$${}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix}; q, z\right] := \sum_{k=0}^{\infty} \frac{(a_1,\dots,a_{r+1};q)_k}{(q,b_1,\dots,b_r;q)_k} z^k.$$

We first build the following q-congruence with an additional parameter a.

Theorem 2.2. *Let n be a positive odd integer and $0 \leqslant s \leqslant (n-1)/4$. Let a be an indeterminate. Then*

$$\begin{aligned}
&\sum_{k=s}^{(n-1)/2+s} \frac{(1+q^{4k+1})(aq^2;q^4)_{k-s}(q^2;q^4)_{k+s}(q^2/a;q^4)_k}{(1+q)(q^4;q^4)_{k-s}(q^4/a;q^4)_{k+s}(aq^4;q^4)_k} q^k \\
&\equiv \frac{(q^2;q^4)_{(n+1)/2}(q^3;q^4)_{(n-1)/2-s}(q^2/a, aq^3;q^4)_s}{(q^{2-4s};q^4)_{(n-1)/2+s}(q^{5+4s};q^4)_{(n-1)/2}(1+q)(1-q^{1+4s})} q^{(4s-1)((n-1)/2-s)/2+s} \\
&\quad \begin{cases} \pmod{\Phi_n(-q)(1-aq^{2n})(a-q^{2n})} & \text{if } n\equiv 1 \pmod 4, \\ \pmod{\Phi_n(q^2)(1-aq^{2n})(a-q^{2n})} & \text{if } n\equiv 3 \pmod 4. \end{cases}
\end{aligned} \tag{2.2}$$

Proof. For $a = q^{-2n}$, the left-hand side of (2.2) can be written as

$$\begin{aligned}
&\sum_{k=s}^{(n-1)/2+s} \frac{(1+q^{4k+1})(q^{2-2n};q^4)_{k-s}(q^2;q^4)_{k+s}(q^{2+2n};q^4)_k}{(1+q)(q^4;q^4)_{k-s}(q^{4+2n};q^4)_{k+s}(q^{4-2n};q^4)_k} q^k \\
&= \sum_{k=0}^{(n-1)/2} \frac{(1+q^{1+4k+4s})(q^{2-2n};q^4)_k(q^2;q^4)_{k+2s}(q^{2+2n};q^4)_{k+s}}{(1+q)(q^4;q^4)_k(q^{4+2n};q^4)_{k+2s}(q^{4-2n};q^4)_{k+s}} q^{k+s} \\
&= q^s \frac{(q^2;q^4)_{2s}(q^{2+2n};q^4)_s}{(q^{4+2n};q^4)_{2s}(q^{4-2n};q^4)_s} \\
&\quad \times \sum_{k=0}^{(n-1)/2} \frac{(1+q^{1+4k+4s})(q^{2-2n};q^4)_k(q^{2+8s};q^4)_k(q^{2+4s+2n};q^4)_k}{(1+q)(q^4;q^4)_k(q^{4+8s+2n};q^4)_k(q^{4+4s-2n};q^4)_k} q^k.
\end{aligned} \tag{2.3}$$

Making the parameter substitutions $q \mapsto q^4$, $a \mapsto q^{2+8s}$, $b \mapsto q^{2+4s}/a$ and $c \mapsto aq^2$ in (2.1), we get

$$\sum_{k=0}^{\infty} \frac{(q^{2+8s}, -q^{5+4s}, q^{2+4s}/a, aq^2; q^4)_k}{(q^4, -q^{1+4s}, aq^{4+4s}, q^{4+8s}/a; q^4)_k} q^k = \frac{(q^{6+8s}, aq^3, q^{3+4s}/a, q^{2+4s}; q^4)_\infty}{(aq^{4+4s}, q^{4+8s}/a, q^{5+4s}, q; q^4)_\infty}. \quad (2.4)$$

Since n is odd, taking $a = q^{-2n}$ in (2.4), we know that the left-hand side terminates and is equal to

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{(q^{2+8s}, -q^{5+4s}, q^{2+4s+2n}, q^{2-2n}; q^4)_k}{(q^4, -q^{1+4s}, q^{4+4s-2n}, q^{4+8s+2n}; q^4)_k} q^k \\ &\quad = \sum_{k=0}^{(n-1)/2} \frac{(1+q^{1+4k+4s})(q^{2-2n}, q^{2+8s}, q^{2+4s+2n}; q^4)_k}{(1+q^{1+4s})(q^4, q^{4+8s+2n}, q^{4+4s-2n}; q^4)_k} q^k, \end{aligned}$$

while the right-hand side becomes

$$\frac{(q^{6+8s}, q^{3-2n}, q^{3+4s+2n}, q^{2+4s}; q^4)_\infty}{(q^{4+4s-2n}, q^{4+8s+2n}, q^{5+4s}, q; q^4)_\infty} = \frac{(q^{6+8s}, q^{3-2n}; q^4)_{(n-1)/2}}{(q^{4+4s-2n}, q^{5+4s}; q^4)_{(n-1)/2}}.$$

It follows that

$$\begin{aligned} &\sum_{k=0}^{(n-1)/2} \frac{(1+q^{1+4k+4s})(q^{2-2n}, q^{2+8s}, q^{2+4s+2n}; q^4)_k}{(1+q^{1+4s})(q^4, q^{4+8s+2n}, q^{4+4s-2n}; q^4)_k} q^k \\ &\quad = \frac{(q^{6+8s}, q^{3-2n}; q^4)_{(n-1)/2}}{(q^{4+4s-2n}, q^{5+4s}; q^4)_{(n-1)/2}}. \end{aligned} \quad (2.5)$$

Substituting (2.5) into (2.3) and making some simplifications, we obtain

$$\begin{aligned} &\sum_{k=s}^{(n-1)/2+s} \frac{(1+q^{4k+1})(q^{2-2n}; q^4)_{k-s}(q^2; q^4)_{k+s}(q^{2+2n}; q^4)_k}{(1+q)(q^4; q^4)_{k-s}(q^{4+2n}; q^4)_{k+s}(q^{4-2n}; q^4)_k} q^k \\ &= \frac{(1+q^{1+4s})(q^2; q^4)_{2s}(q^{2+2n}; q^4)_s(q^{6+8s}, q^{3-2n}; q^4)_{(n-1)/2}}{(1+q)(q^{4+2n}; q^4)_{2s}(q^{4-2n}; q^4)_s(q^{4+4s-2n}, q^{5+4s}; q^4)_{(n-1)/2}} \\ &= \frac{(q^2; q^4)_{(n+1)/2}(q^3; q^4)_{(n-1)/2-s}(q^{2+2n}, q^{3-2n}; q^4)_s}{(q^{2-4s}; q^4)_{(n-1)/2+s}(q^{5+4s}; q^4)_{(n-1)/2}(1+q)(1-q^{1+4s})} q^{(4s-1)((n-1)/2-s)/2+s}. \end{aligned}$$

This proves that the q-congruence (2.2) is true modulo $1 - aq^{2n}$.

Similarly, for $a = q^{2n}$, the left-hand side of (2.4) can be written as

$$\begin{aligned}
&\sum_{k=s}^{(n-1)/2+s} \frac{(1+q^{4k+1})(q^{2+2n};q^4)_{k-s}(q^2;q^4)_{k+s}(q^{2-2n};q^4)_k}{(1+q)(q^4;q^4)_{k-s}(q^{4-2n};q^4)_{k+s}(q^{4+2n};q^4)_k} q^k \\
&= \sum_{k=0}^{(n-1)/2} \frac{(1+q^{1+4k+4s})(q^{2+2n};q^4)_k(q^2;q^4)_{k+2s}(q^{2-2n};q^4)_{k+s}}{(1+q)(q^4;q^4)_k(q^{4-2n};q^4)_{k+2s}(q^{4+2n};q^4)_{k+s}} q^{k+s} \\
&= q^s \frac{(q^2;q^4)_{2s}(q^{2-2n};q^4)_s}{(q^{4-2n};q^4)_{2s}(q^{4+2n};q^4)_s} \\
&\quad \times \sum_{k=0}^{(n-1)/2} \frac{(1+q^{1+4k+4s})(q^{2+2n};q^4)_k(q^{2+8s};q^4)_k(q^{2+4s-2n};q^4)_k}{(1+q)(q^4;q^4)_k(q^{4+8s-2n};q^4)_k(q^{4+4s+2n};q^4)_k} q^k. \qquad (2.6)
\end{aligned}$$

Letting $a = q^{2n}$ in (2.4), we conclude that the left-hand side is equal to

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(q^{2+8s}, -q^{5+4s}, q^{2+4s-2n}, q^{2+2n};q^4)_k}{(q^4, -q^{1+4s}, q^{4+4s+2n}, q^{4+8s-2n};q^4)_k} q^k \\
&= \sum_{k=0}^{(n-1)/2-s} \frac{(1+q^{1+4k+4s})(q^{2+2n}, q^{2+8s}, q^{2+4s-2n};q^4)_k}{(1+q^{1+4s})(q^4, q^{4+8s-2n}, q^{4+4s+2n};q^4)_k} q^k \\
&= \sum_{k=0}^{(n-1)/2} \frac{(1+q^{1+4k+4s})(q^{2+2n}, q^{2+8s}, q^{2+4s-2n};q^4)_k}{(1+q^{1+4s})(q^4, q^{4+8s-2n}, q^{4+4s+2n};q^4)_k} q^k,
\end{aligned}$$

while the right-hand side becomes

$$\frac{(q^{6+8s}, q^{3+2n}, q^{3+4s-2n}, q^{2+4s};q^4)_\infty}{(q^{4+4s+2n}, q^{4+8s-2n}, q^{5+4s}, q;q^4)_\infty} = \frac{(q^{6+8s}, q^{3+4s-2n};q^4)_{(n-1)/2-s}}{(q^{4+8s-2n}, q^{5+4s};q^4)_{(n-1)/2-s}}.$$

It follows that

$$\begin{aligned}
&\sum_{k=0}^{(n-1)/2} \frac{(1+q^{1+4k+4s})(q^{2+2n}, q^{2+8s}, q^{2+4s-2n};q^4)_k}{(1+q^{1+4s})(q^4, q^{4+8s-2n}, q^{4+4s+2n};q^4)_k} q^k \\
&= \frac{(q^{6+8s}, q^{3+4s-2n};q^4)_{(n-1)/2-s}}{(q^{4+8s-2n}, q^{5+4s};q^4)_{(n-1)/2-s}}. \qquad (2.7)
\end{aligned}$$

Substituting (2.7) into (2.6), we arrive at

$$\begin{aligned}
&\sum_{k=s}^{(n-1)/2+s} \frac{(1+q^{4k+1})(q^{2+2n};q^4)_{k-s}(q^2;q^4)_{k+s}(q^{2-2n};q^4)_k}{(1+q)(q^4;q^4)_{k-s}(q^{4-2n};q^4)_{k+s}(q^{4+2n};q^4)_k} q^k \\
&= \frac{(1+q^{1+4s})(q^2;q^4)_{2s}(q^{2-2n};q^4)_s(q^{6+8s}, q^{3+4s-2n};q^4)_{(n-1)/2-s}}{(1+q)(q^{4-2n};q^4)_{2s}(q^{4+2n};q^4)_s(q^{4+8s-2n}, q^{5+4s};q^4)_{(n-1)/2-s}} \\
&= \frac{(q^2;q^4)_{(n+1)/2}(q^3;q^4)_{(n-1)/2-s}(q^{2-2n}, q^{3+2n};q^4)_s}{(q^{2-4s};q^4)_{(n-1)/2+s}(q^{5+4s};q^4)_{(n-1)/2}(1+q)(1-q^{1+4s})} q^{(4s-1)((n-1)/2-s)/2+s}.
\end{aligned}$$

This proves that the q-congruence (2.2) is true modulo $a - q^{2n}$.

In view of Lemma 2.1 we can check that, for $m=(n-1)/2$ and $s\leqslant k\leqslant m-s$,

$$\frac{(1+q^{4m-4k+1})(aq^2;q^4)_{m-k-s}(q^2;q^4)_{m-k+s}(q^2/a;q^4)_{m-k}}{(1+q)(q^4;q^4)_{m-k-s}(q^4/a;q^4)_{m-k+s}(aq^4;q^4)_{m-k}}q^{m-k}$$
$$\equiv -\frac{(1+q^{4k+1})(aq^2;q^4)_{k-s}(q^2;q^4)_{k+s}(q^2/a;q^4)_{k}}{(1+q)(q^4;q^4)_{k-s}(q^4/a;q^4)_{k+s}(aq^4;q^4)_{k}}q^{k}$$
$$\begin{cases}\pmod{\Phi_n(-q)} & \text{if } n\equiv 1 \pmod 4,\\ \pmod{\Phi_n(q^2)} & \text{if } n\equiv 3 \pmod 4.\end{cases}$$

Furthermore, for s in the range $(n-1)/2-s<k\leqslant(n-1)/2$, the summand indexed k on the left-hand side of (2.2) is congruent to 0 modulo $\Phi_n(q^2)$ because the factor $(q^2;q^4)_{k+s}$ appears in the numerator. This means that

$$\sum_{k=s}^{(n-1)/2+s}\frac{(1+q^{4k+1})(aq^2;q^4)_{k-s}(q^2;q^4)_{k+s}(q^2/a;q^4)_{k}}{(1+q)(q^4;q^4)_{k-s}(q^4/a;q^4)_{k+s}(aq^4;q^4)_{k}}q^{k}$$
$$\equiv 0\begin{cases}\pmod{\Phi_n(-q)} & \text{if } n\equiv 1 \pmod 4,\\ \pmod{\Phi_n(q^2)} & \text{if } n\equiv 3 \pmod 4.\end{cases}$$

Namely, the q-congruence (2.2) is true modulo $\Phi_n(-q)$ if $n\equiv 1 \pmod 4$, and is true modulo $\Phi_n(q^2)$ if $n\equiv 3 \pmod 4$. The proof then follows from the fact that $(\Phi_n(-q))$ (or $\Phi_n(q)^2$), $1-aq^{2n}$, and $a-q^{2n}$ are pairwise coprime polynomials in q. □

Proof of Theorem 1.1. Suppose that $n>1$, for the $n=1$ case is trivial. Both sides of (2.2) are coprime with $\Phi_n(q^2)$ when $a=1$, since k is in the range $s\leqslant k\leqslant (n-1)/2+s$. On the other hand, when $a=1$, the polynomial $(1-aq^{2n})(a-q^{2n})=(1-q^{2n})^2$ has the factor $\Phi_n(q^2)^2$. Thus, letting $a=1$ in (2.2) and performing some simplifications, we are led to

$$\sum_{k=s}^{(n-1)/2+s}\frac{(1+q^{4k+1})(q^2;q^4)_{k-s}(q^2;q^4)_{k+s}(q^2;q^4)_{k}}{(1+q)(q^4;q^4)_{k-s}(q^4;q^4)_{k+s}(q^4;q^4)_{k}}q^{k}$$
$$\equiv \frac{(q^2;q^4)_{(n+1)/2}(q^3;q^4)_{(n-1)/2-s}(q^2,q^3;q^4)_s}{(q^{2-4s};q^4)_{(n-1)/2+s}(q^{5+4s};q^4)_{(n-1)/2}(1+q)(1-q^{1+4s})}q^{(4s-1)((n-1)/2-s)/2+s}$$
$$=(-1)^s[n]_{q^2}\frac{(q^3;q^4)_{(n-1)/2-s}}{(q^5;q^4)_{(n-1)/2+s}}(q,q^2)_{2s}q^{2s(n-s)-(n-1)/2}$$
$$\begin{cases}\pmod{\Phi_n(q)^2\Phi_n(-q)^3} & \text{if } n\equiv 1 \pmod 4,\\ \pmod{\Phi_n(q)^3\Phi_n(-q)^3} & \text{if } n\equiv 3 \pmod 4,\end{cases}$$

as desired. □

3. Proof of Theorem 1.4

Likewise, we need the following parametric generalization of Theorem 1.4.

Theorem 3.1. *Let n be a positive odd integer and $0 \leqslant s \leqslant (n-3)/4$. Let a be an indeterminate. Then*

$$\begin{aligned}&\sum_{k=s}^{(n+1)/2+s} \frac{(1+q^{4k-1})(aq^{-2};q^4)_{k-s}(q^{-2};q^4)_{k+s}(q^{-2}/a;q^4)_k}{(1+q)(q^4;q^4)_{k-s}(q^4/a;q^4)_{k+s}(aq^4;q^4)_k} q^{7k} \\ &\equiv \frac{(q^{6+4s};q^4)_{(n-1)/2-s}(q^{1-4s};q^4)_{(n-1)/2}(aq^5,q^{-2}/a;q^4)_s}{(q^6;q^4)_{(n-3)/2}(q^7;q^4)_{(n-1)/2+s}(1+q)(q^{-1+4s}-1)} q^{(2n+5)s+(n-5)/2} \\ &\begin{cases} \pmod{\Phi_n(q^2)(1-aq^{2n})(a-q^{2n})} & \text{if } n \equiv 1 \pmod 4, \\ \pmod{\Phi_n(-q)(1-aq^{2n})(a-q^{2n})} & \text{if } n \equiv 3 \pmod 4. \end{cases}\end{aligned} \tag{3.1}$$

Proof. For $a = q^{-2n}$, the left-hand side of (3.1) can be written as

$$\begin{aligned}&\sum_{k=s}^{(n+1)/2+s} \frac{(1+q^{-1+4k})(aq^{-2};q^4)_{k-s}(q^{-2};q^4)_{k+s}(q^{-2}/a;q^4)_k}{(1+q)(q^4;q^4)_{k-s}(q^4/a;q^4)_{k+s}(aq^4;q^4)_k} q^{7k} \\ &= \sum_{k=s}^{(n+1)/2+s} \frac{(1+q^{-1+4k})(q^{-2-2n};q^4)_{k-s}(q^{-2};q^4)_{k+s}(q^{-2+2n};q^4)_k}{(1+q)(q^4;q^4)_{k-s}(q^{4+2n};q^4)_{k+s}(q^{4-2n};q^4)_k} q^{7k} \\ &= \sum_{k=0}^{(n+1)/2} \frac{(1+q^{-1+4k+4s})(q^{-2-2n};q^4)_k(q^{-2};q^4)_{k+2s}(q^{-2+2n};q^4)_{k+s}}{(1+q)(q^4;q^4)_k(q^{4+2n};q^4)_{k+2s}(q^{4-2n};q^4)_{k+s}} q^{7k+7s}.\end{aligned} \tag{3.2}$$

Performing the parameter substitutions $q \mapsto q^4$, $a \mapsto q^{-2+8s}$, $b \mapsto q^{-2+4s}/a$ and $c \mapsto aq^{-2}$ in (2.1), we obtain

$$\begin{aligned}&\sum_{k=0}^{\infty} \frac{(q^{-2+8s},-q^{3+4s},q^{-2+4s}/a,aq^{-2};q^4)_k}{(q^4,-q^{-1+4s},aq^{4+4s},q^{4+8s}/a;q^4)_k} q^{7k} \\ &= \frac{(q^{2+8s},aq^5,q^{5+4s}/a,q^{6+4s};q^4)_\infty}{(aq^{4+4s},q^{4+8s}/a,q^{3+4s},q^7;q^4)_\infty}.\end{aligned} \tag{3.3}$$

Putting $a = q^{-2n}$ in (3.3), we obtain

$$\begin{aligned}&\sum_{k=0}^{(n+1)/2} \frac{(1+q^{-1+4k+4s})(q^{-2-2n},q^{-2+8s},q^{-2+4s+2n};q^4)_k}{(1+q^{-1+4s})(q^4,q^{4+8s+2n},q^{4+4s-2n};q^4)_k} q^{7k} \\ &= \frac{(q^{5-2n},q^{6+4s};q^4)_{(n-1)/2+s}}{(q^7,q^{4+4s-2n};q^4)_{(n-1)/2+s}}.\end{aligned} \tag{3.4}$$

Substituting (3.4) into (3.2), we get

$$\begin{aligned}&\sum_{k=s}^{(n+1)/2+s}\frac{(1+q^{-1+4k})(q^{-2-2n};q^4)_{k-s}(q^2;q^4)_{k+s}(q^{-2+2n};q^4)_k}{(1+q)(q^4;q^4)_{k-s}(q^{4+2n};q^4)_{k+s}(q^{4-2n};q^4)_k}q^{7k}\\&=q^{7s}\frac{(1+q^{-1+4s})(q^{-2};q^4)_{2s}(q^{-2+2n};q^4)_s(q^{5-2n},q^{6+4s};q^4)_{(n-1)/2+s}}{(1+q)(q^{4+2n};q^4)_{2s}(q^{4-2n};q^4)_s(q^7,q^{4+4s-2n};q^4)_{(n-1)/2+s}}\\&=\frac{(q^{6+4s};q^4)_{(n-1)/2-s}(q^{1-4s};q^4)_{(n-1)/2}(q^{5-2n},q^{-2+2n};q^4)_s}{(q^6;q^4)_{(n-3)/2}(q^7;q^4)_{(n-1)/2+s}(1+q)(q^{-1+4s}-1)}q^{(2n+5)s+(n-5)/2}.\end{aligned}$$

This proves that the q-congruence (3.1) is true modulo $1-aq^{2n}$.

Similarly, for $a=q^{2n}$, the left-hand side of (3.1) can be written as

$$\begin{aligned}&\sum_{k=s}^{(n+1)/2+s}\frac{(1+q^{-1+4k})(aq^{-2};q^4)_{k-s}(q^{-2};q^4)_{k+s}(q^{-2}/a;q^4)_k}{(1+q)(q^4;q^4)_{k-s}(q^4/a;q^4)_{k+s}(aq^4;q^4)_k}q^{7k}\\&=\sum_{k=s}^{(n+1)/2+s}\frac{(1+q^{-1+4k})(q^{-2+2n};q^4)_{k-s}(q^{-2};q^4)_{k+s}(q^{-2-2n};q^4)_k}{(1+q)(q^4;q^4)_{k-s}(q^{4-2n};q^4)_{k+s}(q^{4+2n};q^4)_k}q^{7k}\\&=\sum_{k=0}^{(n+1)/2}\frac{(1+q^{-1+4k+4s})(q^{-2+2n};q^4)_k(q^{-2};q^4)_{k+2s}(q^{-2-2n};q^4)_{k+s}}{(1+q)(q^4;q^4)_k(q^{4-2n};q^4)_{k+2s}(q^{4+2n};q^4)_{k+s}}q^{7k+7s}.\end{aligned}\tag{3.5}$$

We now put $a=q^{2n}$ in (3.3) to get the following identity:

$$\begin{aligned}&\sum_{k=0}^{(n+1)/2}\frac{(1+q^{-1+4k+4s})(q^{-2+2n},q^{-2+8s},q^{-2+4s-2n};q^4)_k}{(1+q^{-1+4s})(q^4,q^{4+8s-2n},q^{4+4s+2n};q^4)_k}q^{7k}\\&=\frac{(q^{5+4s-2n},q^{6+4s};q^4)_{(n-1)/2}}{(q^7,q^{4+8s-2n};q^4)_{(n-1)/2}}.\end{aligned}\tag{3.6}$$

Substituting (3.6) into (3.5), we get

$$\begin{aligned}&\sum_{k=s}^{(n+1)/2+s}\frac{(1+q^{-1+4k})(aq^{-2};q^4)_{k-s}(q^{-2};q^4)_{k+s}(q^{-2}/a;q^4)_k}{(1+q)(q^4;q^4)_{k-s}(q^4/a;q^4)_{k+s}(aq^4;q^4)_k}q^{7k}\\&=q^{7s}\frac{(1+q^{-1+4s})(q^{-2};q^4)_{2s}(q^{-2-2n};q^4)_s(q^{5+4s-2n},q^{6+4s};q^4)_{(n-1)/2}}{(1+q)(q^{4-2n};q^4)_{2s}(q^{4+2n};q^4)_s(q^7,q^{4+8s-2n};q^4)_{(n-1)/2}}\\&=\frac{(q^{6+4s};q^4)_{(n-1)/2-s}(q^{1-4s};q^4)_{(n-1)/2}(q^{5+2n},q^{-2-2n};q^4)_s}{(q^6;q^4)_{(n-3)/2}(q^7;q^4)_{(n-1)/2+s}(1+q)(q^{-1+4s}-1)}q^{(2n+5)s+(n-5)/2}.\end{aligned}$$

This shows that the q-congruence (3.1) is true modulo $a-q^{2n}$.

In light of Lemma 2.1, we can verify that, for $m=(n+1)/2$ and $s\leqslant k\leqslant m-s$,

$$\begin{aligned}&\frac{(1+q^{4m-4k-1})(aq^{-2};q^4)_{m-k-s}(q^{-2};q^4)_{m-k+s}(q^{-2}/a;q^4)_{m-k}}{(1+q)(q^4;q^4)_{m-k-s}(q^4/a;q^4)_{m-k+s}(aq^4;q^4)_{m-k}}q^{7(m-k)}\\&\equiv-\frac{(1+q^{4k-1})(aq^{-2};q^4)_{k-s}(q^{-2};q^4)_{k+s}(q^{-2}/a;q^4)_k}{(1+q)(q^4;q^4)_{k-s}(q^4/a;q^4)_{k+s}(aq^4;q^4)_k}q^{7k}\\&\quad\begin{cases}\pmod{\Phi_n(q^2)} & \text{if } n\equiv 1 \pmod 4,\\ \pmod{\Phi_n(-q)} & \text{if } n\equiv 3 \pmod 4.\end{cases}\end{aligned}$$

Moreover, for $(n+1)/2-s<k\leqslant(n+1)/2$, the summand indexed k on the left-hand side of (3.1) is congruent to 0 modulo $\Phi_n(q^2)$. This proves that

$$\begin{aligned}&\sum_{k=s}^{(n+1)/2+s}\frac{(1+q^{4k-1})(aq^{-2};q^4)_{k-s}(q^{-2};q^4)_{k+s}(q^{-2}/a;q^4)_k}{(1+q)(q^4;q^4)_{k-s}(q^4/a;q^4)_{k+s}(aq^4;q^4)_k}q^{7k}\\&\equiv 0\begin{cases}\pmod{\Phi_n(q^2)} & \text{if } n\equiv 1 \pmod 4,\\ \pmod{\Phi_n(-q)} & \text{if } n\equiv 3 \pmod 4.\end{cases}\end{aligned}$$

Namely, the q-congruence (3.1) is true modulo $\Phi_n(q^2)$ if $n\equiv 1 \pmod 4$, and is true modulo $\Phi_n(-q)$ if $n\equiv 3 \pmod 4$. Since $\Phi_n(q)^2$, $1-aq^{2n}$, and $a-q^{2n}$ are pairwise coprime polynomials, we complete the proof. □

Proof of Theorem 1.4. Letting $a=1$ in (3.1) and making some simplifications, we finish the proof. □

ORCID

Victor J. W. Guo, https://orcid.org/0000-0002-4153-715X
Yi-Xin Xiao, https://orcid.org/0009-0008-2194-9836

References

[1] G. Bauer, Von den coefficienten der Reihen von Kugelfunctionen einer variabeln, *J. Reine Angew. Math.* **56** (1859), 101–121.

[2] S. B. Ekhad and D. Zeilberger, A WZ proof of Ramanujan's formula for π, in: *Geometry, Analysis, and Mechanics*, pp. 107–108, J. M. Rassias (ed.), (World Scientific, Singapore, 1994)

[3] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, 2nd edition, Encyclopedia of Mathematics and Its Applications, Vol. 96 (Cambridge University Press, Cambridge, 2004).

[4] V. J. W. Guo, A further common q-generalization of the (B.2) and (C.2) supercongruences of Van Hamme, *Int. J. Number Theory* **21** (2025) 605–614.

[5] V. J. W. Guo and M. J. Schlosser, Some q-supercongruences from transformation formulas for basic hypergeometric series, *Constr. Approx.* **53** (2021) 155–200.

[6] V. J. W. Guo and J. Zeng, Some q-supercongruences for truncated basic hypergeometric series, *Acta Arith.* **171** (2015) 309–326.

[7] V. J. W. Guo and W. Zudilin, A q-microscope for supercongruences, *Adv. Math.* **346** (2019) 329–358.

[8] V. J. W. Guo and W. Zudilin, On a q-deformation of modular forms, *J. Math. Anal. Appl.* **475** (2019) 1636–646.

[9] V. J. W. Guo and W. Zudilin, A common q-analogue of two supercongruences, *Results Math.* **75** (2020) Article ID:46.

[10] J.-C. Liu, Some supercongruences on truncated ${}_3F_2$ hypergeometric series, *J. Difference Equ. Appl.* **24** (2018) 438–451.

[11] J.-C. Liu, On Van Hamme's (A.2) and (H.2) supercongruences, *J. Math. Anal. Appl.* **471** (2019) 613–622.

[12] L. Long and R. Ramakrishna, Some supercongruences occurring in truncated hypergeometric series, *Adv. Math.* **290** (2016) 773–808.

[13] G.-S. Mao and H. Pan, On the divisibility of some truncated hypergeometric series, *Acta Arith.* **195** (2020) 199–206.

[14] E. Mortenson, A p-adic supercongruence conjecture of van Hamme, *Proc. Amer. Math. Soc.* **136** (2008) 4321–4328.

[15] S. Ramanujan, Modular equations and approximations to π, *Quart. J. Math. Oxford Ser. (2)* **45** (1914) 350–372.

[16] H. Song and C. Wang, Further generalizations of the (A.2) and (H.2) supercongruences of Van Hamme, *Results Math.* **79** (2024) Article ID:147.

[17] Z.-H. Sun, Congruences concerning Legendre polynomials II, *J. Number Theory* **133** (2013) 1950–1976.

[18] Z.-H. Sun, Generalized Legendre polynomials and related supercongruences, *J. Number Theory* **143** (2014) 293–319.

[19] Z.-W. Sun, On sums of Apéry polynomials and related congruences, *J. Number Theory* **132** (2012) 2673–2690.

[20] Z.-W. Sun, A refinement of a congruence result by van Hamme and Mortenson, *Illinois J. Math.* **56** (2012) 967–979.

[21] L. Van Hamme, Some conjectures concerning partial sums of generalized hypergeometric series, in: *p-Adic Functional Analysis (Nijmegen, 1996)*, pp. 223–236, Lecture Notes in Pure and Appl. Math. 192, (Dekker, New York, 1997)

[22] C. Wang, A new q-extension of the (H.2) congruence of Van Hamme for primes $p \equiv 1 \pmod 4$, *Results Math.* **76** (2021) Article ID:205.

[23] C. Wei, A further q-analogue of Van Hamme's (H.2) supercongruence for any prime $p \equiv 1 \pmod 4$, *Results Math.* **76** (2021) Article ID:92.

[24] W. Zudilin, Ramanujan-type supercongruences, *J. Number Theory* **129** (2009) 1848–1857.