

# The Eulerian Distribution on Involutions is Indeed Unimodal

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**Abstract.** Let  $I_{n,k}$  (respectively,  $J_{n,k}$ ) be the number of involutions (respectively, fixed-point free involutions) of  $\{1, \dots, n\}$  with  $k$  descents. Motivated by Brenti's conjecture which states that the sequence  $I_{n,0}, I_{n,1}, \dots, I_{n,n-1}$  is log-concave, we prove that the two sequences  $I_{n,k}$  and  $J_{2n,k}$  are unimodal in  $k$ , for all  $n$ . Furthermore, we conjecture that there are nonnegative integers  $a_{n,k}$  such that

$$\sum_{k=0}^{n-1} I_{n,k} t^k = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k} t^k (1+t)^{n-2k-1}.$$

This statement is stronger than the unimodality of  $I_{n,k}$  but is also interesting in its own right.

*Keywords:* Involutions, Descent number, Unimodality, Eulerian polynomial, Zeilberger's algorithm

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## 1 Introduction

A sequence  $a_0, a_1, \dots, a_n$  of real numbers is said to be *unimodal* if for some  $0 \leq j \leq n$  we have  $a_0 \leq a_1 \leq \dots \leq a_j \geq a_{j+1} \geq \dots \geq a_n$ , and is said to be *log-concave* if  $a_i^2 \geq a_{i-1}a_{i+1}$  for all  $1 \leq i \leq n-1$ . Clearly a log-concave sequence of *positive* terms is unimodal. The reader is referred to Stanley's survey [10] for the surprisingly rich variety of methods to show that a sequence is log-concave or unimodal. As noticed by Brenti [2], even though log-concavity and unimodality have one-line definitions, to prove the unimodality or log-concavity of a sequence can sometimes be a very difficult task requiring the use of intricate combinatorial constructions or of refined mathematical tools.

Let  $\mathfrak{S}_n$  be the set of all permutations of  $[n] := \{1, \dots, n\}$ . We say that a permutation  $\pi = a_1 a_2 \dots a_n \in \mathfrak{S}_n$  has a *descent* at  $i$  ( $1 \leq i \leq n-1$ ) if  $a_i > a_{i+1}$ . The number of descents of  $\pi$  is called its descent number and is denoted by  $d(\pi)$ . A statistic on  $\mathfrak{S}_n$  is said to be *Eulerian*, if it is equidistributed with the descent number statistic. Recall that the polynomial

$$A_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{1+d(\pi)} = \sum_{k=1}^n A(n, k) t^k$$

is called an *Eulerian polynomial*. It is well-known that the *Eulerian numbers*  $A(n, k)$  ( $1 \leq k \leq n$ ) form a unimodal sequence, of which several proofs have been published: such

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as the analytical one by showing that the polynomial  $A_n(t)$  has only real zeros [3, p. 294], by induction based on the recurrence relation of  $A(n, k)$  (see [9]), or by combinatorial techniques (see [7, 11]).

Let  $\mathcal{I}_n$  be the set of all involutions in  $\mathfrak{S}_n$  and  $\mathcal{J}_n$  the set of all fixed-point free involutions in  $\mathfrak{S}_n$ . Define

$$I_n(t) = \sum_{\pi \in \mathcal{I}_n} t^{\text{d}(\pi)} = \sum_{k=0}^{n-1} I_{n,k} t^k,$$

$$J_n(t) = \sum_{\pi \in \mathcal{J}_n} t^{\text{d}(\pi)} = \sum_{k=0}^{n-1} J_{n,k} t^k.$$

The first values of these polynomials are given in Table 1.

Table 1: The polynomials  $I_n(t)$  and  $J_n(t)$  for  $n \leq 6$ .

$n$	$I_n(t)$	$J_n(t)$
1	1	0
2	$1 + t$	$t$
3	$1 + 2t + t^2$	0
4	$1 + 4t + 4t^2 + t^3$	$t + t^2 + t^3$
5	$1 + 6t + 12t^2 + 6t^3 + t^4$	0
6	$1 + 9t + 28t^2 + 28t^3 + 9t^4 + t^5$	$t + 3t^2 + 7t^3 + 3t^4 + t^5$

As one may notice from Table 1 that the coefficients of  $I_n(t)$  and  $J_n(t)$  are *symmetric* and *unimodal* for  $1 \leq n \leq 6$ . Actually, the symmetries had been conjectured by Dumont and were first proved by Strehl [12]. Recently, Brenti (see [5]) conjectured that the coefficients of the polynomial  $I_n(t)$  are *log-concave* and Dukes [5] has obtained some partial results on the unimodality of the coefficients of  $I_n(t)$  and  $J_{2n}(t)$ . Note that, in contrast to Eulerian polynomials  $A_n(t)$ , the polynomials  $I_n(t)$  and  $J_{2n}(t)$  may have *non-real zeros*.

In this paper we will prove that for  $n \geq 1$ , the two sequences  $I_{n,0}, I_{n,1}, \dots, I_{n,n-1}$  and  $J_{2n,1}, J_{2n,2}, \dots, J_{2n,2n-1}$  are unimodal. Our starting point is the known generating functions of polynomials  $I_n(t)$  and  $J_n(t)$ :

$$\sum_{n=0}^{\infty} I_n(t) \frac{u^n}{(1-t)^{n+1}} = \sum_{r=0}^{\infty} \frac{t^r}{(1-u)^{r+1} (1-u^2)^{r(r+1)/2}}, \quad (1.1)$$

$$\sum_{n=0}^{\infty} J_n(t) \frac{u^n}{(1-t)^{n+1}} = \sum_{r=0}^{\infty} \frac{t^r}{(1-u^2)^{r(r+1)/2}}, \quad (1.2)$$

which have been obtained by Désarménien and Foata [4] and Gessel and Reutenauer [8] using different methods. We first derive linear recurrence formulas for  $I_{n,k}$  and  $J_{2n,k}$  in the next section and then prove the unimodality by induction in Section 3. We end this paper with further conjectures beyond the unimodality of the two sequences  $I_{n,k}$  and  $J_{2n,k}$ .

## 2 Linear recurrence formulas for $I_{n,k}$ and $J_{2n,k}$

Since the recurrence formula for the numbers  $I_{n,k}$  is a little more complicated than  $J_{2n,k}$ , we shall first prove it for the latter.

**Theorem 2.1.** *For  $n \geq 2$  and  $k \geq 0$ , the numbers  $J_{2n,k}$  satisfy the following recurrence formula:*

$$2nJ_{2n,k} = [k(k+1) + 2n-2]J_{2n-2,k} + 2[(k-1)(2n-k-1) + 1]J_{2n-2,k-1} \\ + [(2n-k)(2n-k+1) + 2n-2]J_{2n-2,k-2}. \quad (2.1)$$

Here and in what follows  $J_{2n,k} = 0$  if  $k < 0$ .

*Proof.* Equating the coefficients of  $u^{2n}$  in (1.2), we obtain

$$\frac{J_{2n}(t)}{(1-t)^{2n+1}} = \sum_{r=0}^{\infty} \binom{r(r+1)/2 + n - 1}{n} t^r. \quad (2.2)$$

Since

$$\binom{r(r+1)/2 + n - 1}{n} = \frac{r(r-1)/2 + r + n - 1}{n} \binom{r(r+1)/2 + n - 2}{n-1},$$

it follows from (2.2) that

$$\frac{J_{2n}(t)}{(1-t)^{2n+1}} = \frac{t^2}{2n} \left( \frac{J_{2n-2}(t)}{(1-t)^{2n-1}} \right)'' + \frac{t}{n} \left( \frac{J_{2n-2}(t)}{(1-t)^{2n-1}} \right)' + \frac{n-1}{n} \frac{J_{2n-2}(t)}{(1-t)^{2n-1}},$$

or

$$J_{2n}(t) = \frac{t^2(1-t)^2}{2n} J_{2n-2}''(t) + \left[ \frac{(2n-1)t^2(1-t)}{n} + \frac{t(1-t)^2}{n} \right] J_{2n-2}'(t) \\ + \left[ (2n-1)t^2 + \frac{(2n-1)(1-t)t}{n} + \frac{(n-1)(1-t)^2}{n} \right] J_{2n-2}(t) \\ = \frac{t^4 - 2t^3 + t^2}{2n} J_{2n-2}''(t) + \left[ \frac{(2-2n)t^3}{n} + \frac{(2n-3)t^2}{n} + \frac{t}{n} \right] J_{2n-2}'(t) \\ + \left[ (2n-2)t^2 + \frac{t}{n} + \frac{n-1}{n} \right] J_{2n-2}(t). \quad (2.3)$$

Equating the coefficients of  $t^n$  in (2.3) yields

$$J_{2n,k} = \frac{(k-2)(k-3)}{2n} J_{2n-2,k-2} - \frac{(k-1)(k-2)}{n} J_{2n-2,k-1} + \frac{k(k-1)}{2n} J_{2n-2,k} \\ + \frac{(2-2n)(k-2)}{n} J_{2n-2,k-2} + \frac{(2n-3)(k-1)}{n} J_{2n-2,k-1} + \frac{k}{n} J_{2n-2,k} \\ + (2n-2)J_{2n-2,k-2} + \frac{1}{n} J_{2n-2,k-1} + \frac{n-1}{n} J_{2n-2,k}.$$

After simplification, we obtain (2.1). ■

**Theorem 2.2.** For  $n \geq 3$  and  $k \geq 0$ , the numbers  $I_{n,k}$  satisfy the following recurrence formula:

$$\begin{aligned} nI_{n,k} &= (k+1)I_{n-1,k} + (n-k)I_{n-1,k-1} + [(k+1)^2 + n-2]I_{n-2,k} \\ &\quad + [2k(n-k-1) - n+3]I_{n-2,k-1} + [(n-k)^2 + n-2]I_{n-2,k-2}. \end{aligned} \quad (2.4)$$

Here and in what follows  $I_{n,k} = 0$  if  $k < 0$ .

*Proof.* Extracting the coefficients of  $u^{2n}$  in (1.1), we obtain

$$\frac{I_n(t)}{(1-t)^{n+1}} = \sum_{r=0}^{\infty} t^r \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{r(r+1)/2 + k - 1}{k} \binom{r+n-2k}{n-2k}. \quad (2.5)$$

Let

$$T(n, k) := \binom{x+k-1}{k} \binom{y+n-2k}{n-2k},$$

and

$$s(n) := \sum_{k=0}^{\lfloor n/2 \rfloor} T(n, k).$$

Applying Zeilberger's algorithm, the Maple package `ZeilbergerRecurrence(T,n,k,s,0..n)` gives

$$(2x+y+n+1)s(n) + (y+1)s(n+1) - (n+2)s(n+2) = 0, \quad (2.6)$$

i.e.,

$$s(n) = \frac{y+1}{n}s(n-1) + \frac{2x+y+n-1}{n}s(n-2).$$

When  $x = r(r+1)/2$  and  $y = r$ , we get

$$s(n) = \frac{r+1}{n}s(n-1) + \frac{r(r-1) + 3r + n - 1}{n}s(n-2). \quad (2.7)$$

Now, from (2.5) and (2.7) follows the identity

$$\begin{aligned} \frac{nI_n(t)}{(1-t)^{n+1}} &= t \left( \frac{I_{n-1}(t)}{(1-t)^n} \right)' + \frac{I_{n-1}(t)}{(1-t)^n} + t^2 \left( \frac{I_{n-2}(t)}{(1-t)^{n-1}} \right)'' + 3t \left( \frac{I_{n-2}(t)}{(1-t)^{n-1}} \right)' \\ &\quad + (n-1) \frac{I_{n-2}(t)}{(1-t)^{n-1}}, \end{aligned}$$

or

$$\begin{aligned} nI_n(t) &= (t-t^2)I'_{n-1}(t) + [1 + (n-1)t]I_{n-1}(t) + t^2(1-t)^2I''_{n-2}(t) \\ &\quad + t(1-t)[3 + (2n-5)t]I'_{n-2}(t) + (n-1)[1+t+(n-2)t^2]I_{n-2}(t). \end{aligned} \quad (2.8)$$

Comparing the coefficients of  $t^k$  in both sides of (2.8), we obtain

$$\begin{aligned}
nI_{n,k} &= kI_{n-1,k} - (k-1)I_{n-1,k-1} + I_{n-1,k} + (n-1)I_{n-1,k-1} \\
&\quad + k(k-1)I_{n-2,k} - 2(k-1)(k-2)I_{n-2,k-1} + (k-2)(k-3)I_{n-2,k-2} \\
&\quad + 3kI_{n-2,k} + (2n-8)(k-1)I_{n-2,k-1} - (2n-5)(k-2)I_{n-2,k-2} \\
&\quad + (n-1)I_{n-2,k} + (n-1)I_{n-2,k-1} + (n-1)(n-2)I_{n-2,k-2},
\end{aligned}$$

which, after simplification, equals the right-hand side of (2.4). ■

*Remark.* The recurrence formula (2.6) can also be proved by hand as follows. It is easy to see that the generating function of  $s(n)$  is

$$\sum_{n=0}^{\infty} s(n)u^n = (1-u^2)^{-x}(1-u)^{-y-1}. \quad (2.9)$$

Differentiating (2.9) with respect to  $u$  implies that

$$\sum_{n=0}^{\infty} ns(n)u^{n-1} = \left( \frac{2ux}{1-u^2} + \frac{y+1}{1-u} \right) (1-u^2)^{-x}(1-u)^{-y-1},$$

consequently,

$$\begin{aligned}
(1-u^2) \sum_{n=0}^{\infty} ns(n)u^{n-1} &= [(2x+y+1)u + y+1](1-u^2)^{-x}(1-u)^{-y-1} \\
&= [(2x+y+1)u + y+1] \sum_{n=0}^{\infty} s(n)u^n. \quad (2.10)
\end{aligned}$$

Comparing the coefficients of  $u^{n+1}$  in both sides of (2.10), we obtain

$$(n+2)s(n+2) - ns(n) = (2x+y+1)s(n) + (y+1)s(n+1),$$

which is equivalent to (2.6).

Note that the right-hand side of (2.1) (respectively, (2.4)) is invariant under the substitution  $k \rightarrow 2n-k$  (respectively,  $k \rightarrow n-1-k$ ), provided that the sequence  $I_{n-1,k}$  (respectively,  $J_{2n-2,k}$ ) is symmetric. Thus, by induction we derive immediately the symmetry properties of  $J_{2n,k}$  and  $I_{n,k}$  (see [4, 8, 12]).

**Corollary 2.3.** *For  $n, k \in \mathbb{N}$ , we have*

$$I_{n,k} = I_{n,n-1-k}, \quad J_{2n,k} = J_{2n,2n-k}.$$

It would be interesting to find a combinatorial proof of the recurrence formulas (2.1) and (2.4), since such a proof could hopefully lead to a combinatorial proof of the unimodality of these two sequences.

### 3 Unimodality of the sequences $I_{n,k}$ and $J_{2n,k}$

The following observation is crucial in our inductive proof of the unimodality of the sequences  $I_{n,k}$  ( $0 \leq k \leq n-1$ ) and  $J_{2n,k}$  ( $1 \leq k \leq 2n-1$ ).

**Lemma 3.1.** *Let  $x_0, x_1, \dots, x_n$  and  $a_0, a_1, \dots, a_n$  be real numbers such that  $x_0 \geq x_1 \geq \dots \geq x_n \geq 0$  and  $a_0 + a_1 + \dots + a_k \geq 0$  for all  $k = 0, 1, \dots, n$ . Then*

$$\sum_{i=0}^n a_i x_i \geq 0.$$

Indeed, the above inequality follows from the identity:

$$\sum_{i=0}^n a_i x_i = \sum_{k=0}^n (x_k - x_{k+1})(a_0 + a_1 + \dots + a_k),$$

where  $x_{n+1} = 0$ .

**Theorem 3.2.** *The sequence  $J_{2n,1}, J_{2n,2}, \dots, J_{2n,2n-1}$  is unimodal.*

*Proof.* By the symmetry of  $J_{2n,k}$ , it is enough to show that  $J_{2n,k} \geq J_{2n,k-1}$  for all  $2 \leq k \leq n$ . We proceed by induction on  $n$ . The  $n = 2$  case is clear from Table 1. Suppose the sequence  $J_{2n-2,k}$  is unimodal in  $k$ . By Theorem 2.1, one has

$$2n(J_{2n,k} - J_{2n,k-1}) = A_0 J_{2n-2,k} + A_1 J_{2n-2,k-1} + A_2 J_{2n-2,k-2} + A_3 J_{2n-2,k-3}, \quad (3.1)$$

where

$$A_0 = k^2 + k + 2n - 2, \quad A_1 = 4nk - 3k^2 - 6n + k + 6,$$

$$A_2 = 3k^2 + 4n^2 - 8nk - 5k + 12n - 4, \quad A_3 = 3k - k^2 + 4nk - 4n^2 - 8n.$$

We have the following two cases:

- If  $2 \leq k \leq n-1$ , then

$$J_{2n-2,k} \geq J_{2n-2,k-1} \geq J_{2n-2,k-2} \geq J_{2n-2,k-3}$$

by the induction hypothesis, and clearly

$$A_0 \geq 0, \quad A_0 + A_1 = 2(k-1)(2n-k) + 4 \geq 0,$$

$$A_0 + A_1 + A_2 = (2n-k)^2 - 3k + 8n \geq 0, \quad A_0 + A_1 + A_2 + A_3 = 0.$$

Therefore, by Lemma 3.1, we have

$$J_{2n,k} - J_{2n,k-1} \geq 0.$$

- If  $k = n$ , then

$$J_{2n-2,n-1} \geq J_{2n-2,n} = J_{2n-2,n-2} \geq J_{2n-2,n-3}$$

by symmetry and the induction hypothesis. In this case, we have  $A_1 = (n-2)(n-3) \geq 0$  and thus the corresponding condition of Lemma 3.1 is satisfied. Therefore, we have

$$J_{2n,n} - J_{2n,n-1} \geq 0.$$

This completes the proof. ■

**Theorem 3.3.** *The sequence  $I_{n,0}, I_{n,1}, \dots, I_{n,n-1}$  is unimodal.*

*Proof.* By the symmetry of  $I_{n,k}$ , it suffices to show that  $I_{n,k} \geq I_{n,k-1}$  for all  $1 \leq k \leq (n-1)/2$ . From Table 1, it is clear that the sequences  $I_{n,k}$  are unimodal in  $k$  for  $1 \leq n \leq 6$ .

Now suppose  $n \geq 7$  and the sequences  $I_{n-1,k}$  and  $I_{n-2,k}$  are unimodal in  $k$ . Replacing  $k$  by  $k-1$  in (2.4), we obtain

$$\begin{aligned} nI_{n,k-1} &= kI_{n-1,k-1} + (n-k+1)I_{n-1,k-2} + (k^2+n-2)I_{n-2,k-1} \\ &\quad + [2(k-1)(n-k) - n + 3]I_{n-2,k-2} + [(n-k+1)^2 + n - 2]I_{n-2,k-3}. \end{aligned} \quad (3.2)$$

Combining (2.4) and (3.2) yields

$$\begin{aligned} n(I_{n,k} - I_{n,k-1}) &= B_0I_{n-1,k} + B_1I_{n-1,k-1} + B_2I_{n-1,k-2} \\ &\quad + C_0I_{n-2,k} + C_1I_{n-2,k-1} + C_2I_{n-2,k-2} + C_3I_{n-2,k-3}, \end{aligned} \quad (3.3)$$

where

$$\begin{aligned} B_0 &= k+1, \quad B_1 = n-2k, \quad B_2 = -(n-k+1), \\ C_0 &= (k+1)^2 + n-2, \quad C_1 = 2nk - 3k^2 - 2k - 2n + 5, \\ C_2 &= n^2 - 4nk + 3k^2 + 4n - 2k - 5, \quad C_3 = -(n-k+1)^2 - n + 2. \end{aligned}$$

Notice that  $I_{n-1,k} \geq I_{n-1,k-1} \geq I_{n-1,k-2}$  for  $1 \leq k \leq (n-1)/2$ . By Lemma 3.1, we have

$$B_0I_{n-1,k} + B_1I_{n-1,k-1} + B_2I_{n-1,k-2} \geq 0. \quad (3.4)$$

It remains to show that

$$C_0I_{n-2,k} + C_1I_{n-2,k-1} + C_2I_{n-2,k-2} + C_3I_{n-2,k-3} \geq 0, \quad \forall 1 \leq k \leq (n-1)/2. \quad (3.5)$$

We need to consider the following two cases:

- If  $1 \leq k \leq (n-2)/2$ , then

$$I_{n-2,k} \geq I_{n-2,k-1} \geq I_{n-2,k-2} \geq I_{n-2,k-3}$$

by the induction hypothesis, and

$$\begin{aligned} C_0 &= (k+1)^2 + n-2 \geq 0, \quad C_0 + C_1 = (2k-1)(n-k-1) + k+3 \geq 0, \\ C_0 + C_1 + C_2 &= (n-k+1)^2 + n-2 \geq 0, \quad C_0 + C_1 + C_2 + C_3 = 0. \end{aligned}$$

- If  $k = (n-1)/2$ , then by symmetry and the induction hypothesis,

$$I_{n-2,k-1} \geq I_{n-2,k} = I_{n-2,k-2} \geq I_{n-2,k-3}.$$

In this case, we have  $C_1 = (n-3)(n-7)/4 \geq 0$  for  $n \geq 7$ .

Therefore, by Lemma 3.1 the inequality (3.5) holds. It follows from (3.3)–(3.5) that

$$I_{n,k} - I_{n,k-1} \geq 0, \quad \forall 1 \leq k \leq (n-1)/2.$$

This completes the proof. ■

## 4 Further remarks and open problems

Since  $I_{n,k} = I_{n,n-1-k}$ , we can rewrite  $I_n(t)$  as follows:

$$I_n(t) = \sum_{k=0}^{n-1} I_{n,k} t^k = \begin{cases} \sum_{k=0}^{n/2-1} I_{n,k} t^k (1 + t^{n-2k-1}), & \text{if } n \text{ is even,} \\ I_{n,(n-1)/2} t^{(n-1)/2} + \sum_{k=0}^{(n-3)/2} I_{n,k} t^k (1 + t^{n-2k-1}), & \text{if } n \text{ is odd.} \end{cases}$$

Applying the well-known formula

$$x^n + y^n = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n}{n-j} \binom{n-j}{j} (xy)^j (x+y)^{n-2j},$$

we obtain

$$I_n(t) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k} t^k (1+t)^{n-2k-1}, \quad (4.1)$$

where

$$a_{n,k} = \begin{cases} \sum_{j=0}^k (-1)^{k-j} \frac{n-2j-1}{n-k-j-1} \binom{n-k-j-1}{k-j} I_{n,j}, & \text{if } 2k+1 < n, \\ I_{n,k} + \sum_{j=0}^{k-1} (-1)^{k-j} \frac{n-2j-1}{n-k-j-1} \binom{n-k-j-1}{k-j} I_{n,j}, & \text{if } 2k+1 = n. \end{cases}$$

The first values of  $a_{n,k}$  are given in Table 2, which seems to suggest the following conjecture.

**Conjecture 4.1.** *For  $n \geq 1$  and  $k \geq 0$ , the coefficients  $a_{n,k}$  are nonnegative integers.*



Table 2: Values of  $a_{n,k}$  for  $n \leq 16$  and  $0 \leq k \leq \lfloor (n-1)/2 \rfloor$ .

$k \setminus n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1			0	1	2	4	6	9	12	16	20	25	30	36	42	49
2					2	6	18	39	79	141	239	379	579	849	1211	1680
3							0	18	78	272	722	1716	3626	7160	13206	23263
4									20	124	668	2560	8360	23536	59824	139457
5											32	700	4800	24160	95680	325572
6													440	5480	44632	257964
7															2176	44376

Since the coefficients of  $t^k(1+t)^{n-2k-1}$  are symmetric and unimodal with center of symmetry at  $(n-1)/2$ , Conjecture 4.1 is stronger than the fact that the coefficients of  $I_n(t)$  are symmetric and unimodal. A more interesting question is to give a combinatorial interpretation of  $a_{n,k}$ . Note that the Eulerian polynomials can be written as

$$A_n(t) = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} c_{n,k} t^k (1+t)^{n-2k+1},$$

where  $c_{n,k}$  is the number of increasing binary trees on  $[n]$  with  $k$  leaves and no vertices having left children only (see [1, 6, 7]).

We now proceed to derive a recurrence relation for  $a_{n,k}$ . Set  $x = x(t) = t/(1+t)^2$  and

$$P_n(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k} x^k.$$

Then we can rewrite (4.1) as

$$I_n(t) = (1+t)^{n-1} P_n(x). \quad (4.2)$$

Differentiating (4.2) with respect to  $t$  we get

$$I'_n(t) = (n-1)(1+t)^{n-2} P_n(x) + (1+t)^{n-1} P'_n(x) x'(t), \quad (4.3)$$

$$\begin{aligned} I''_n(t) &= (n-1)(n-2)(1+t)^{n-3} P_n(x) + 2(n-1)(1+t)^{n-2} P'_n(x) x'(t) \\ &\quad + (1+t)^{n-1} P''_n(x) (x'(t))^2 + (1+t)^{n-1} P'_n(x) x''(t), \end{aligned} \quad (4.4)$$

$$x'(t) = \frac{1-t}{(1+t)^3}, \quad x''(t) = \frac{2t-4}{(1+t)^4}. \quad (4.5)$$

Substituting (4.2)–(4.5) into (2.8), we obtain

$$\begin{aligned}
& n(1+t)^{n-1}P_n(x) \\
&= [1 + (2n-2)t + t^2](1+t)^{n-3}P_{n-1}(x) + t(1-t)^2(1+t)^{n-5}P'_{n-1}(x) \\
&\quad + [-(t^2 + 14t + 1)(1-t)^2 + (1 + 6t - 18t^2 + 6t^3 + t^4)n + 4t^2n^2](1+t)^{n-5}P_{n-2}(x) \\
&\quad + [3t(t^2 - 4t + 1)(1-t)^2 + 4t^2(1-t)^2n](1+t)^{n-7}P'_{n-2}(x) \\
&\quad + t^2(1-t)^4(1+t)^{n-9}P''_{n-2}(x). \tag{4.6}
\end{aligned}$$

Dividing the two sides of (4.6) by  $(1+t)^{n-1}$  and noticing that  $t/(1+t)^2 = x$ , after a little manipulation we get

$$\begin{aligned}
nP_n(x) &= [1 + (2n-4)x]P_{n-1}(x) + (x - 4x^2)P'_{n-1}(x) \\
&\quad + [(n-1) + (2n-8)x + 4(n-3)(n-4)x^2]P_{n-2}(x) \\
&\quad + [3x + (4n-30)x^2 + (72-16n)x^3]P'_{n-2}(x) + (x^2 - 8x^3 + 16x^4)P''_{n-2}(x).
\end{aligned}$$

Extracting the coefficients of  $x^k$  yields

$$\begin{aligned}
na_{n,k} &= a_{n-1,k} + (2n-4)a_{n-1,k-1} + ka_{n-1,k} - 4(k-1)a_{n-1,k-1} \\
&\quad + (n-1)a_{n-2,k} + (2n-8)a_{n-2,k-1} + 4(n-3)(n-4)a_{n-2,k-2} \\
&\quad + 3ka_{n-2,k} + (4n-30)(k-1)a_{n-2,k-1} + (72-16n)(k-2)a_{n-2,k-2} \\
&\quad + k(k-1)a_{n-2,k} - 8(k-1)(k-2)a_{n-2,k-1} + 16(k-2)(k-3)a_{n-2,k-2}.
\end{aligned}$$

After simplification, we obtain the following recurrence formula for  $a_{n,k}$ .

**Theorem 4.2.** *For  $n \geq 3$  and  $k \geq 0$ , there holds*

$$\begin{aligned}
na_{n,k} &= (k+1)a_{n-1,k} + (2n-4k)a_{n-1,k-1} + [k(k+2) + n-1]a_{n-2,k} \\
&\quad + [(k-1)(4n-8k-14) + 2n-8]a_{n-2,k-1} + 4(n-2k)(n-2k+1)a_{n-2,k-2}, \tag{4.7}
\end{aligned}$$

where  $a_{n,k} = 0$  if  $k < 0$  or  $k > (n-1)/2$ .

Note that, if  $n \geq 2k+3$ , then

$$(k-1)(4n-8k-14) + 2n-8 > 0 \quad \text{for any } k \geq 1,$$

and so are the other coefficients in (4.7). Therefore, Conjecture 4.1 would be proved if one can show that  $a_{2n+1,n} \geq 0$  and  $a_{2n+2,n} \geq 0$ .

Finally, from (4.1) it is easy to see that

$$a_{2n+1,n} = (-1)^n I_{2n+1}(-1) = \sum_{k=0}^{2n} (-1)^{n-k} I_{2n+1,k},$$

$$a_{2n+2,n} = (-1)^n I'_{2n+2}(-1) = \sum_{k=1}^{2n+1} (-1)^{n+1-k} k I_{2n+2,k}.$$

Thus, Conjecture 4.1 is equivalent to the *nonnegativity* of the above two alternating sums.

Since  $J_{2n,k} = J_{2n,2n-k}$ , in the same manner as  $I_n(t)$  we obtain

$$J_{2n}(t) = \sum_{k=1}^n b_{2n,k} t^k (1+t)^{2n-2k},$$

where

$$b_{2n,k} = \begin{cases} \sum_{j=1}^k (-1)^{k-j} \frac{2n-2j}{2n-k-j} \binom{2n-k-j}{k-j} J_{2n,j}, & \text{if } k < n, \\ J_{2n,k} + \sum_{j=1}^{k-1} (-1)^{k-j} \frac{2n-2j}{2n-k-j} \binom{2n-k-j}{k-j} J_{2n,j}, & \text{if } k = n. \end{cases}$$

Now, it follows from (2.2) that

$$J_{2n,k} = \sum_{i=0}^k (-1)^{k-i} \binom{2n+1}{k-i} \binom{i(i+1)/2 + n - 1}{i(i+1)/2 - 1}$$

is a polynomial in  $n$  of degree  $d := k(k+1)/2 - 1$  with leading coefficient  $1/d!$ , and so is  $b_{2n,k}$ . Thus, we have  $\lim_{n \rightarrow +\infty} b_{2n,k} = +\infty$  for any fixed  $k > 1$ .

The first values of  $b_{2n,k}$  are given in Table 3, which seems to suggest

**Conjecture 4.3.** *For  $n \geq 9$  and  $k \geq 1$ , the coefficients  $b_{2n,k}$  are nonnegative integers.*

Similarly to the proof of Theorem 4.2, we can prove the following result.

**Theorem 4.4.** *For  $n \geq 2$  and  $k \geq 1$ , there holds*

$$2nb_{2n,k} = [k(k+1) + 2n - 2]b_{2n-2,k} + [2 + 2(k-1)(4n - 4k - 3)]b_{2n-2,k-1} \\ + 8(n - k + 1)(2n - 2k + 1)b_{2n-2,k-2}.$$

where  $b_{2n,k} = 0$  if  $k < 1$  or  $k > n$ .

Theorem 4.4 allows us to reduce the verification of Conjecture 4.3 to the boundary case  $b_{2n,n} \geq 0$  for  $n \geq 9$ .

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Table 3: Values of  $b_{2n,k}$  for  $2n \leq 24$  and  $1 \leq k \leq n$ .

$k \setminus 2n$	2	4	6	8	10	12	14	16	18	20	22	24
1	1	1	1	1	1	1	1	1	1	1	1	1
2		-1	-1	0	2	5	9	14	20	27	35	44
3			3	12	36	91	201	399	728	1242	2007	3102
4				-7	-10	91	652	2593	7902	20401	46852	98494
5					25	219	1710	10532	50165	194139	639968	1861215
6						-65	249	11319	122571	841038	4377636	18747924
7							283	6586	135545	1737505	15219292	101116704
8								-583	33188	1372734	24412940	277963127
9									4417	379029	16488999	367507439
10										1791	3350211	203698690
11											133107	36903128
12												761785

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