A FAMILY OF q-SUPERCONGRUENCES MODULO THE CUBE OF A CYCLOTOMIC POLYNOMIAL

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Abstract. We establish a family of q-supercongruences modulo the cube of a cyclotomic polynomial for truncated basic hypergeometric series. This confirms a weaker form of a previous conjecture of the present authors. Our proof employs a very-well-poised Karlsson–Minton type summation due to Gasper, together with the 'creative microscoping' method introduced by the first author in recent joint work with Zudilin.

1. Introduction

In 1914, Ramanujan [11] mysteriously stated some representations of $1/\pi$, such as

$$\sum_{k=0}^{\infty}(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 4^k}=\frac{4}{\pi}, \tag{1.1}$$

where $(a)_n=a(a+1)\cdots(a+n-1)$ denotes the rising factorial. In 1997, Van Hamme [13] conjectured 13 interesting p-adic analogues of Ramanujan-type formulas. For example,

$$\sum_{k=0}^{(p-1)/2}(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 4^k}\equiv p(-1)^{(p-1)/2} \pmod{p^4}, \tag{1.2}$$

where $p>3$ is a prime. Van Hamme himself gave proofs for three of them. Supercongruences of the form (1.2) are now called Ramanujan-type supercongruences (see [16]). The proof of (1.2) was first provided by Long [9]. See [10] for historical remarks of Van Hamme's 13 supercongruences.

Recently, q-supercongruences have been investigated by different authors (see, for example, [3–8, 14, 15]). In particular, the present authors [3] proved that, for odd integers $d\geqslant 5$,

$$\sum_{k=0}^{n-1}[2dk+1]\frac{(q;q^d)_k^d}{(q^d;q^d)_k^d}q^{d(d-3)k/2}\equiv\begin{cases}0 \pmod{\Phi_n(q)^2}, & \text{if } n\equiv -1 \pmod{d},\\ 0 \pmod{\Phi_n(q)^3}, & \text{if } n\equiv -1/2 \pmod{d}.\end{cases} \tag{1.3}$$

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Here, we adopt the standard q-notation: $[n]=1+q+\cdots+q^{n-1}$ is the *q-integer*; $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ is the *q-shifted factorial*, with the abbreviated notation $(a_1,a_2,\dots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n$; and $\Phi_n(q)$ stands for the n-th *cyclotomic polynomial* in q, which may be defined as

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}}(q-\zeta^k),$$

where ζ is an n-th primitive root of unity.

It is worth mentioning that the q-congruence (1.3) is not true for $d=3$. The present authors [3] also gave the following companion of (1.3): for odd integers $d\geqslant 3$ and integers $n>1$,

$$\sum_{k=0}^{n-1}[2dk-1]\frac{(q^{-1};q^d)_k^d}{(q^d;q^d)_k^d}q^{d(d-1)k/2}\equiv\begin{cases}0 \pmod{\Phi_n(q)^2}, & \text{if } n\equiv 1 \pmod{d},\\ 0 \pmod{\Phi_n(q)^3}, & \text{if } n\equiv 1/2 \pmod{d}.\end{cases}\tag{1.4}$$

In this paper, we shall prove the following q-supercongruence, which is a generalization of the respective second cases of (1.3) and (1.4).

Theorem 1.1. *Let d and r be odd integers satisfying $d\geqslant 3$, $r\leqslant d-4$ (in particular, r may be negative) and $\gcd(d,r)=1$. Let n be an integer such that $n\geqslant (d-r)/2$ and $n\equiv -r/2 \pmod{d}$. Then*

$$\sum_{k=0}^{M}[2dk+r]\frac{(q^r;q^d)_k^d}{(q^d;q^d)_k^d}q^{d(d-r-2)k/2}\equiv 0 \pmod{\Phi_n(q)^3},\tag{1.5}$$

where $M=(dn-2n-r)/d$ or $n-1$.

Note that the present authors [5, Theorem 2] already proved that (1.5) is true modulo $\Phi_n(q)^2$, and further conjectured that it is also true modulo $\Phi_n(q)^4$ for $d\geqslant 5$ (see [5, Conjecture 3]). We believe that the full conjecture is rather difficult to prove.

In this paper we apply the method of creative microscoping, recently introduced in a paper by the first author with Zudilin [6], to prove Theorem 1.1. In our application of this method here we suitably introduce the parameter a (such that the series satisfies the symmetry $a\leftrightarrow a^{-1}$) into the terms of the series and prove that the congruence holds modulo $\Phi_n(q)$, modulo $1-aq^n$, and modulo $a-q^n$. Thus, by the Chinese remainder theorem for coprime polynomials, the congruence holds modulo the product $\Phi_n(q)(1-aq^n)(a-q^n)$. By letting $a=1$ the congruence is established modulo $\Phi_n(q)^3$.

Our paper is organized as follows: In Section 2 we list some tools we require in our proof of Theorem 1.1. These consist of a Lemma about an elementary q-congruence modulo a cyclotomic polynomial $\Phi_n(q)$, and a very-well-poised Karlsson–Minton type summation by Gasper of which we need a special case. In Section 3 we first prove Theorem 3.1, a parametric generalization of Theorem 1.1 that involves the insertion of different powers of the parameter a, appearing in geometric sequences, in the respective q-shifted factorials.

Afterwards we show how Theorem 1.1 follows from Theorem 3.1. We conclude with Section 4 where we elaborate on the merits and limits of the method of creative microscoping employed here in the quest of proving [5, Conjecture 3] (which remains open).

2. Preliminaries

We need the following result, which is due to the present authors [4, Lemma 2.1]. In order to make the paper self-contained, we include its short proof here.

Lemma 2.1. *Let d, m and n be positive integers with $m \leqslant n-1$. Let r be an integer satisfying $dm \equiv -r \pmod{n}$. Then, for $0 \leqslant k \leqslant m$ and any indeterminate a, we have*

$$\frac{(aq^r;q^d)_{m-k}}{(q^d/a;q^d)_{m-k}} \equiv (-a)^{m-2k}\frac{(aq^r;q^d)_k}{(q^d/a;q^d)_k}q^{m(dm-d+2r)/2+(d-r)k} \pmod{\Phi_n(q)}.$$

If $\gcd(d,n)=1$, then the above q-congruence also holds for $a=1$.

Proof. We first assume that a is an indeterminate. Since $q^{dm+r} \equiv q^n \equiv 1 \pmod{\Phi_n(q)}$, we have

$$\begin{aligned}\frac{(aq^r;q^d)_m}{(q^d/a;q^d)_m} &= \frac{(1-aq^r)(1-aq^{d+r})\cdots(1-aq^{dm-d+r})}{(1-q^d/a)(1-q^{2d}/a)\cdots(1-q^{dm}/a)}\\ &\equiv \frac{(1-aq^r)(1-aq^{d+r})\cdots(1-aq^{dm-d+r})}{(1-q^{d-dm-r}/a)(1-q^{2d-dm-r}/a)\cdots(1-q^{-r}/a)}\\ &= (-a)^m q^{m(dm-d+2r)/2} \pmod{\Phi_n(q)}. \end{aligned}\tag{2.1}$$

Moreover, modulo $\Phi_n(q)$, we get

$$\begin{aligned}&\frac{(aq^r;q^d)_{m-k}}{(q^d/a;q^d)_{m-k}}\\ &= \frac{(aq^r;q^d)_m}{(q^d/a;q^d)_m}\frac{(1-q^{dm-dk+d}/a)(1-q^{dm-dk+2d}/a)\cdots(1-q^{dm}/a)}{(1-aq^{dm-dk+r})(1-aq^{dm-dk+d+r})\cdots(1-aq^{dm-d+r})}\\ &\equiv \frac{(aq^r;q^d)_m}{(q^d/a;q^d)_m}\frac{(1-q^{d-dk-r}/a)(1-q^{2d-dk-r}/a)\cdots(1-q^{-r}/a)}{(1-aq^{-dk})(1-aq^{d-dk})\cdots(1-aq^{-d})}\\ &= \frac{(aq^r;q^d)_m}{(q^d/a;q^d)_m}\frac{(aq^r;q^d)_k}{(q^d/a;q^d)_k}a^{-2k}q^{(d-r)k}.\end{aligned}$$

Substituting (2.1) into the above q-congruence, we obtain the desired q-congruence.

We now assume that $\gcd(d,n)=1$ and $a=1$. Then the desired result follows from the same argument. □

We will further utilize a very-well-poised Karlsson–Minton type summation due to Gasper [1, Eq. (5.13)] (see also [2, Ex. 2.33 (i)]):

$$\sum_{k=0}^{\infty} \frac{(a, q\sqrt{a}, -q\sqrt{a}, b, a/b, d, e_1, aq^{n_1+1}/e_1, \dots, e_m, aq^{n_m+1}/e_m; q)_k}{(q, \sqrt{a}, -\sqrt{a}, aq/b, bq, aq/d, aq/e_1, e_1q^{-n_1}, \dots, aq/e_m, e_mq^{-n_m}; q)_k} \left(\frac{q^{1-\nu}}{d}\right)^k$$
$$= \frac{(q, aq, aq/bd, bq/d; q)_\infty}{(bq, aq/b, aq/d, q/d; q)_\infty} \prod_{j=1}^{m} \frac{(aq/be_j, bq/e_j; q)_{n_j}}{(aq/e_j, q/e_j; q)_{n_j}}, \qquad (2.2)$$

where $n_1, \dots, n_m$ are non-negative integers, $\nu = n_1 + \cdots + n_m$, and the convergence condition $|q^{1-\nu}/d| < 1$ is needed when the series does not terminate. We point out that an elliptic extension of the terminating $d = q^{-\nu}$ case of (2.2) was given by Rosengren and the second author [12, Eq. (1.7)].

In particular, we notice that the right-hand side of (2.2) vanishes for $d = bq$. Further taking $b = q^{-N}$ we get the following summation formula:

$$\sum_{k=0}^{N} \frac{(a, q\sqrt{a}, -q\sqrt{a}, e_1, aq^{n_1+1}/e_1, \dots, e_m, aq^{n_m+1}/e_m, q^{-N}; q)_k}{(q, \sqrt{a}, -\sqrt{a}, aq/e_1, e_1q^{-n_1}, \dots, aq/e_m, e_mq^{-n_m}, aq^{N+1}; q)_k} q^{(N-\nu)k} = 0, \qquad (2.3)$$

provided that $N > \nu = n_1 + \cdots + n_m$.

3. A parametric generalization and proof of Theorem 1.1

We now give a parametric generalization of Theorem 1.1.

Theorem 3.1. *Let d and r be odd integers satisfying $d \geqslant 3$, $r \leqslant d-4$ (in particular, r may be negative) and $\gcd(d,r) = 1$. Let n be an integer such that $n \geqslant (d-r)/2$ and $n \equiv -r/2 \pmod{d}$. Then modulo $\Phi_n(q)(1-aq^n)(a-q^n)$,*

$$\sum_{k=0}^{M} [2dk+r] \frac{(a^{d-2}q^r, a^{d-4}q^r, \dots, aq^r; q^d)_k}{(a^{d-2}q^d, a^{d-4}q^d, \dots, aq^d; q^d)_k}$$
$$\times \frac{(a^{2-d}q^r, a^{4-d}q^r, \dots, a^{-1}q^r; q^d)_k (q^r; q^d)_k}{(a^{2-d}q^d, a^{4-d}q^d, \dots, a^{-1}q^d; q^d)_k (q^d; q^d)_k} q^{d(d-r-2)k/2} \equiv 0, \qquad (3.1)$$

where $(dn-2n-r)/d \leqslant M \leqslant n-1$.

Proof. It is easy to see that $\gcd(d,n) = 1$ and thereby none of the numbers $d, 2d, \dots, (n-1)d$ are multiples of n. This means that the denominators of the left-hand side of (3.1) do not contain the factor $1 - aq^n$ nor $1 - a^{-1}q^n$. Thus, for $a = q^{-n}$ or $a = q^n$, the left-hand side of (3.1) can be written as

$$\sum_{k=0}^{(dn-2n-r)/d} [2dk+r] \frac{(q^{r-(d-2)n}, q^{r-(d-4)n}, \dots, q^{r-n}; q^d)_k}{(q^{d-(d-2)n}, q^{d-(d-4)n}, \dots, q^{d-n}; q^d)_k}$$
$$\times \frac{(q^{(d-2)n+r}, q^{(d-4)n+r}, \dots, q^{n+r}; q^d)_k (q^r; q^d)_k}{(q^{(d-2)n+d}, q^{(d-4)n+d}, \dots, q^{n+d}; q^d)_k (q^d; q^d)_k} q^{d(d-r-2)k/2}, \qquad (3.2)$$

where we have used $(q^{r-(d-2)n};q^d)_k=0$ for $k>(dn-2n-r)/d$. Specializing the parameters in (2.3) by $N=(dn-2n-r)/d$, $a=q^r$, $q\mapsto q^d$, $m=(d-1)/2$, $e_i=q^{r-(d-2i-2)n}$ $(1\leqslant i\leqslant m-1)$, $e_m=q^{(d+r)/2}$, $n_1=\cdots=n_{m-1}=(2n+r-d)/d$ and $n_m=(2n+r-d)/(2d)$ and noticing $N-(n_1+\cdots+n_m)=(d-r-2)/2>0$, we see that (3.2) is equal to 0. This proves that (3.1) holds modulo $(1-aq^n)(a-q^n)$.

For $M=(dn-2n-r)/d$, by Lemma 2.1, we can easily check that

$$
\begin{aligned}
&[2d(M-k)+r]\frac{(a^{d-2}q^r,a^{d-4}q^r,\dots,aq^r;q^d)_{M-k}}{(a^{d-2}q^d,a^{d-4}q^d,\dots,aq^d;q^d)_{M-k}}\\
&\times\frac{(a^{2-d}q^r,a^{4-d}q^r,\dots,a^{-1}q^r;q^d)_{M-k}(q^r;q^d)_{M-k}}{(a^{2-d}q^d,a^{4-d}q^d,\dots,a^{-1}q^d;q^d)_{M-k}(q^d;q^d)_{M-k}}q^{d(d-r-2)(M-k)/2}\\
&\quad\equiv-[2dk+r]\frac{(a^{d-2}q^r,a^{d-4}q^r,\dots,aq^r;q^d)_k}{(a^{d-2}q^d,a^{d-4}q^d,\dots,aq^d;q^d)_k}\\
&\qquad\times\frac{(a^{2-d}q^r,a^{4-d}q^r,\dots,a^{-1}q^r;q^d)_k(q^r;q^d)_k}{(a^{2-d}q^d,a^{4-d}q^d,\dots,a^{-1}q^d;q^d)_k(q^d;q^d)_k}q^{d(d-r-2)k/2}\pmod{\Phi_n(q)}.
\end{aligned}
$$

It now becomes evident that the k-th and $(M-k)$-th summands on the left-hand side of (3.1) cancel each other modulo $\Phi_n(q)$. Therefore, the left-hand side of (3.1) is congruent to 0 modulo $\Phi_n(q)$ for $M=(dn-2n-r)/d$. Furthermore, for any k in the range $(dn-2n-r)/d<k\leqslant n-1$, we have $(q^r;q^d)_k/(q^d;q^d)_k\equiv 0\pmod{\Phi_n(q)}$. Hence, the q-congruence (3.1) also holds modulo $\Phi_n(q)$ for $(dn-2n-r)/d<M\leqslant n-1$. □

Proof of Theorem 1.1. Since $\gcd(n,d)=1$ and $0\leqslant k\leqslant n-1$, the factors related to a in the denominators of the left-hand side of (3.1) are relatively prime to $\Phi_n(q)$ when $a=1$. On the other hand, the polynomial $(1-aq^n)(a-q^n)$ has the factor $\Phi_n(q)^2$ when $a=1$. Thus, letting $a=1$ in (3.1), we see that (1.5) holds modulo $\Phi_n(q)^3$. □

4. Concluding remarks

We have inserted different powers of the parameter a, appearing in geometric sequences, in the respective q-shifted factorials on the left-hand side of (1.5), in order to establish the desired generalized congruence modulo $(1-aq^n)(a-q^n)$. The proof of Theorem 1.1 is similar to the proofs in the paper [3] but is quite different from those in [6], where the parameter a is inserted in a more standard way (without higher powers of a).

While the method of creative microscoping enabled us to strengthen [5, Theorem 2] to the congruence modulo $\Phi_n(q)^3$ in Theorem 1.1, we believe that it is rather unlikely to prove that (1.5) is true modulo $\Phi_n(q)^4$ for $d\geqslant 5$ [5, Conjecture 3] by the method of creative microscoping, since the parametric generalization in (3.1) does not hold modulo $\Phi_n(q)^2(1-aq^n)(a-q^n)$ in general. For this reason, the proof of (1.5) modulo $\Phi_n(q)^2$ given in [5] still has its virtue. Recall that the present authors [5] wrote the left-hand side of (1.5) as a product of two rational functions X and Y, and showed that X is congruent to 0 modulo $\Phi_n(q)^2$. Hence, to prove [5, Conjecture 3], it remains to prove that Y is also congruent to 0 modulo $\Phi_n(q)^2$. We hope that an interested reader can shed light on this problem and settle the conjecture.

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