

SOME q -SUPERCONGRUENCES MODULO THE SQUARE AND CUBE OF A CYCLOTOMIC POLYNOMIAL

VICTOR J. W. GUO AND MICHAEL J. SCHLOSSER

ABSTRACT. Two q -supercongruences of truncated basic hypergeometric series containing two free parameters are established by employing specific identities for basic hypergeometric series. The results partly extend two q -supercongruences that were earlier conjectured by the same authors and involve q -supercongruences modulo the square and the cube of a cyclotomic polynomial. One of the newly proved q -supercongruences is even conjectured to hold modulo the fourth power of a cyclotomic polynomial.

1. INTRODUCTION

In 1914, Ramanujan [25] listed a number of representations of $1/\pi$, including

$$\sum_{k=0}^{\infty} (6k+1) \frac{(\frac{1}{2})_k^3}{k!^3 4^k} = \frac{4}{\pi}, \quad (1.1)$$

where $(a)_n = a(a+1) \cdots (a+n-1)$ denotes the Pochhammer symbol. Ramanujan's formulas gained unprecedented popularity in the 1980's when they were discovered to provide fast algorithms for calculating decimal digits of π . See, for instance, the monograph [2] by the Borwein brothers.

In 1997, Van Hamme [29] conjectured 13 intriguing p -adic analogues of Ramanujan-type formulas, such as

$$\sum_{k=0}^{(p-1)/2} (6k+1) \frac{(\frac{1}{2})_k^3}{k!^3 4^k} \equiv p(-1)^{(p-1)/2} \pmod{p^4}, \quad (1.2)$$

where $p > 3$ is a prime. Van Hamme himself supplied proofs for three of them. Supercongruences like (1.2) are called Ramanujan-type supercongruences (see [33]). The proof of the supercongruence (1.2) was first given by Long [22]. As of today, all of Van Hamme's 13 supercongruences have been confirmed by various techniques (see [24, 28]).

In recent years, q -congruences and q -supercongruences have been established by different authors (see, for example, [5–13, 15–21, 23, 27, 30–32, 34]). In particular, the present

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authors [9] proved that, for any odd integer $d \geq 5$,

$$\sum_{k=0}^{n-1} [2dk+1] \frac{(q; q^d)_k^d}{(q^d; q^d)_k^d} q^{d(d-3)k/2} \equiv \begin{cases} 0 \pmod{\Phi_n(q)^2}, & \text{if } n \equiv -1 \pmod{d}, \\ 0 \pmod{\Phi_n(q)^3}, & \text{if } n \equiv -1/2 \pmod{d}. \end{cases} \quad (1.3)$$

Here and in what follows, we adopt the standard q -notation: $[n] = 1 + q + \cdots + q^{n-1}$ is the q -integer; $(a; q)_n = (1-a)(1-aq) \cdots (1-aq^{n-1})$ is the q -shifted factorial, with the compact notation $(a_1, a_2, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n$ used for their products; and $\Phi_n(q)$ denotes the n -th cyclotomic polynomial in q , which may be defined as

$$\Phi_n(q) = \prod_{\substack{1 \leq k \leq n \\ \gcd(k, n) = 1}} (q - \zeta^k),$$

where ζ is an n -th primitive root of unity.

We should point out that the q -congruence (1.3) does not hold for $d = 3$. The present authors [9] also established the following companion of (1.3): for any odd integer $d \geq 3$ and integer $n > 1$,

$$\sum_{k=0}^{n-1} [2dk-1] \frac{(q^{-1}; q^d)_k^d}{(q^d; q^d)_k^d} q^{d(d-1)k/2} \equiv \begin{cases} 0 \pmod{\Phi_n(q)^2}, & \text{if } n \equiv 1 \pmod{d}, \\ 0 \pmod{\Phi_n(q)^3}, & \text{if } n \equiv 1/2 \pmod{d}. \end{cases} \quad (1.4)$$

They also proposed the following conjectures [9, Conjectures 1 and 2], which are generalizations of (1.3) and (1.4).

Conjecture 1. *Let $d \geq 5$ be an odd integer. Then*

$$\sum_{k=0}^{n-1} [2dk+1] \frac{(q; q^d)_k^d}{(q^d; q^d)_k^d} q^{d(d-3)k/2} \equiv \begin{cases} 0 \pmod{\Phi_n(q)^3}, & \text{if } n \equiv -1 \pmod{d}, \\ 0 \pmod{\Phi_n(q)^4}, & \text{if } n \equiv -1/2 \pmod{d}. \end{cases}$$

Conjecture 2. *Let $d \geq 5$ be an odd integer and let $n > 1$. Then*

$$\sum_{k=0}^{n-1} [2dk-1] \frac{(q^{-1}; q^d)_k^d}{(q^d; q^d)_k^d} q^{d(d-1)k/2} \equiv \begin{cases} 0 \pmod{\Phi_n(q)^3}, & \text{if } n \equiv 1 \pmod{d}, \\ 0 \pmod{\Phi_n(q)^4}, & \text{if } n \equiv 1/2 \pmod{d}. \end{cases}$$

q -Supercongruences such as those above (modulo a third and even fourth power of a cyclotomic polynomial) are rather special. In fact, concrete results for truncated basic hypergeometric sums being congruent to 0 modulo a high power of a cyclotomic polynomial are very rare. See [8, 10–12, 14, 18] for recent papers featuring such results. The main goal of this paper is to add two complete two-parameter families of q -supercongruences to the list of such q -supercongruences (see Theorems 1 and 2).

We shall prove that the respective first cases of Conjectures 1 and 2 are true by establishing the following more general result.

Theorem 1. *Let d and r be odd integers satisfying $d \geq 3$, $r \leq d - 4$ (in particular, r may be negative) and $\gcd(d, r) = 1$. Let n be an integer such that $n \geq d - r$ and $n \equiv -r \pmod{d}$. Then*

$$\sum_{k=0}^M [2dk + r] \frac{(q^r; q^d)_k^d}{(q^d; q^d)_k^d} q^{d(d-r-2)k/2} \equiv 0 \pmod{[n]\Phi_n(q)^2}, \quad (1.5)$$

where $M = (dn - n - r)/d$ or $n - 1$.

We shall also prove the following q -supercongruences.

Theorem 2. *Let d and r be odd integers satisfying $d \geq 3$, $r \leq d - 4$ (in particular, r may be negative) and $\gcd(d, r) = 1$. Let n be an integer such that $n \geq (d - r)/2$ and $n \equiv -r/2 \pmod{d}$. Then*

$$\sum_{k=0}^M [2dk + r] \frac{(q^r; q^d)_k^d}{(q^d; q^d)_k^d} q^{d(d-r-2)k/2} \equiv 0 \pmod{[n]\Phi_n(q)}, \quad (1.6)$$

where $M = (dn - 2n - r)/d$ or $n - 1$.

The following generalization of the respective second cases of Conjectures 1 and 2 should be true.

Conjecture 3. *The q -supercongruence (1.6) holds modulo $[n]\Phi_n(q)^3$ for $d \geq 5$.*

We shall prove Theorems 1 and 2 in Sections 2 and 3, respectively, by making use of Andrews' multiserries extension (2.2) of the Watson transformation [1, Theorem 4], along with Gasper's very-well-poised Karlsson–Minton type summation [3, Eq. (5.13)]. It should be pointed out that Andrews' transformation plays an important part in combinatorics and number theory (see [7] and the introduction of [12] for more such examples).

2. PROOF OF THEOREM 1

We need a simple q -congruence modulo $\Phi_n(q)^2$, which was already used in [10, 12].

Lemma 1. *Let α, r be integers and n a positive integer. Then*

$$(q^{r-\alpha n}, q^{r+\alpha n}; q^d)_k \equiv (q^r; q^d)_k^2 \pmod{\Phi_n(q)^2}. \quad (2.1)$$

We will further utilize a powerful transformation formula due to Andrews [1, Theorem 4], which may be stated as follows:

$$\begin{aligned}
& \sum_{k \geq 0} \frac{(a, q\sqrt{a}, -q\sqrt{a}, b_1, c_1, \dots, b_m, c_m, q^{-N}; q)_k}{(q, \sqrt{a}, -\sqrt{a}, aq/b_1, aq/c_1, \dots, aq/b_m, aq/c_m, aq^{N+1}; q)_k} \left(\frac{a^m q^{m+N}}{b_1 c_1 \dots b_m c_m} \right)^k \\
&= \frac{(aq, aq/b_m c_m; q)_N}{(aq/b_m, aq/c_m; q)_N} \sum_{j_1, \dots, j_{m-1} \geq 0} \frac{(aq/b_1 c_1; q)_{j_1} \dots (aq/b_{m-1} c_{m-1}; q)_{j_{m-1}}}{(q; q)_{j_1} \dots (q; q)_{j_{m-1}}} \\
&\quad \times \frac{(b_2, c_2; q)_{j_1} \dots (b_m, c_m; q)_{j_1 + \dots + j_{m-1}}}{(aq/b_1, aq/c_1; q)_{j_1} \dots (aq/b_{m-1}, aq/c_{m-1}; q)_{j_1 + \dots + j_{m-1}}} \\
&\quad \times \frac{(q^{-N}; q)_{j_1 + \dots + j_{m-1}}}{(b_m c_m q^{-N}/a; q)_{j_1 + \dots + j_{m-1}}} \frac{(aq)^{j_{m-2} + \dots + (m-2)j_1} q^{j_1 + \dots + j_{m-1}}}{(b_2 c_2)^{j_1} \dots (b_{m-1} c_{m-1})^{j_1 + \dots + j_{m-2}}}. \quad (2.2)
\end{aligned}$$

This transformation is a multiseries generalization of Watson's ${}_8\phi_7$ transformation formula (listed in [4, Appendix (III.18)]); cf. [4, Chapter 1] for the notation of a basic hypergeometric ${}_r\phi_s$ series we are using),

$$\begin{aligned}
& {}_8\phi_7 \left[\begin{matrix} a, & qa^{\frac{1}{2}}, & -qa^{\frac{1}{2}}, & b, & c, & d, & e, & q^{-n} \\ & a^{\frac{1}{2}}, & -a^{\frac{1}{2}}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1} \end{matrix} ; q, \frac{a^2 q^{n+2}}{bcde} \right] \\
&= \frac{(aq, aq/de; q)_n}{(aq/d, aq/e; q)_n} {}_4\phi_3 \left[\begin{matrix} aq/bc, & d, & e, & q^{-n} \\ & aq/b, & aq/c, & deq^{-n}/a \end{matrix} ; q, q \right], \quad (2.3)
\end{aligned}$$

to which it reduces for $m = 2$.

Next, we require a very-well-poised Karlsson–Minton type summation due to Gasper [3, Eq. (5.13)] (see also [4, Ex. 2.33 (i)]):

$$\begin{aligned}
& \sum_{k=0}^{\infty} \frac{(a, q\sqrt{a}, -q\sqrt{a}, b, a/b, d, e_1, aq^{n_1+1}/e_1, \dots, e_m, aq^{n_m+1}/e_m; q)_k}{(q, \sqrt{a}, -\sqrt{a}, aq/b, bq, aq/d, aq/e_1, e_1 q^{-n_1}, \dots, aq/e_m, e_m q^{-n_m}; q)_k} \left(\frac{q^{1-\nu}}{d} \right)^k \\
&= \frac{(q, aq, aq/bd, bq/d; q)_{\infty}}{(bq, aq/b, aq/d, q/d; q)_{\infty}} \prod_{j=1}^m \frac{(aq/be_j, bq/e_j; q)_{n_j}}{(aq/e_j, q/e_j; q)_{n_j}}, \quad (2.4)
\end{aligned}$$

where n_1, \dots, n_m are non-negative integers, $\nu = n_1 + \dots + n_m$, and the convergence condition $|q^{1-\nu}/d| < 1$ is required if the series does not terminate. We point out that an elliptic extension of the terminating $d = q^{-\nu}$ case of (2.4) can be found in [26, Eq. (1.7)].

In particular, we note that for $d = bq$ the right-hand side of (2.4) vanishes. Putting in addition $b = q^{-N}$ we get the following terminating summation formula:

$$\sum_{k=0}^N \frac{(a, q\sqrt{a}, -q\sqrt{a}, e_1, aq^{n_1+1}/e_1, \dots, e_m, aq^{n_m+1}/e_m, q^{-N}; q)_k}{(q, \sqrt{a}, -\sqrt{a}, aq/e_1, e_1 q^{-n_1}, \dots, aq/e_m, e_m q^{-n_m}, aq^{N+1}; q)_k} q^{(N-\nu)k} = 0, \quad (2.5)$$

which is valid for $N > \nu = n_1 + \dots + n_m$.

A suitable combination of (2.2) and (2.5) yields the following multi-series summation formula, derived in [12, Lemma 2] (whose proof we nevertheless give here, to make the paper self-contained):

Lemma 2. *Let $m \geq 2$. Let q, a and e_1, \dots, e_{m+1} be arbitrary parameters with $e_{m+1} = e_1$, and let n_1, \dots, n_m and N be non-negative integers such that $N > n_1 + \dots + n_m$. Then*

$$\begin{aligned} 0 = & \sum_{j_1, \dots, j_{m-1} \geq 0} \frac{(e_1 q^{-n_1}/e_2; q)_{j_1} \cdots (e_{m-1} q^{-n_{m-1}}/e_m; q)_{j_{m-1}}}{(q; q)_{j_1} \cdots (q; q)_{j_{m-1}}} \\ & \times \frac{(aq^{n_2+1}/e_2, e_3; q)_{j_1} \cdots (aq^{n_m+1}/e_m, e_{m+1}; q)_{j_1+\dots+j_{m-1}}}{(e_1 q^{-n_1}, aq/e_2; q)_{j_1} \cdots (e_{m-1} q^{-n_{m-1}}, aq/e_m; q)_{j_1+\dots+j_{m-1}}} \\ & \times \frac{(q^{-N}; q)_{j_1+\dots+j_{m-1}}}{(e_1 q^{n_m-N+1}/e_m; q)_{j_1+\dots+j_{m-1}}} \frac{(aq)^{j_{m-2}+\dots+(m-2)j_1} q^{j_1+\dots+j_{m-1}}}{(aq^{n_2+1}e_3/e_2)_{j_1} \cdots (aq^{n_{m-1}+1}e_m/e_{m-1})_{j_1+\dots+j_{m-2}}}. \end{aligned} \quad (2.6)$$

Proof. By specializing the parameters in the multi-sum transformation (2.2) by $b_i \mapsto aq^{n_i+1}/e_i$, $c_i \mapsto e_{i+1}$, for $1 \leq i \leq m$ (where $e_{m+1} = e_1$), and dividing both sides of the identity by the prefactor of the multi-sum, we obtain that the series on the right-hand side of (2.6) equals

$$\begin{aligned} & \frac{(e_m q^{-n_m}, aq/e_1; q)_N}{(aq, e_m q^{-n_m}/e_1; q)_N} \\ & \times \sum_{k=0}^N \frac{(a, q\sqrt{a}, -q\sqrt{a}, e_1, aq^{n_1+1}/e_1, \dots, e_m, aq^{n_m+1}/e_m, q^{-N}; q)_k}{(q, \sqrt{a}, -\sqrt{a}, aq/e_1, e_1 q^{-n_1}, \dots, aq/e_m, e_m q^{-n_m}, aq^{N+1}; q)_k} q^{(N-\nu)k}, \end{aligned}$$

with $\nu = n_1 + \dots + n_m$. Now the last sum vanishes by the special case of Gasper's summation stated in (2.5). \square

Using [11, Lemma 2.1], we can prove the following result which is similar to [11, Lemma 2.2].

Lemma 3. *Let d, n be positive integers with $\gcd(d, n) = 1$. Let r be an integer. Then*

$$\begin{aligned} \sum_{k=0}^m [2dk + r] \frac{(q^r; q^d)_k^d}{(q^d; q^d)_k^d} q^{d(d-r-2)k/2} &\equiv 0 \pmod{[n]}, \\ \sum_{k=0}^{n-1} [2dk + r] \frac{(q^r; q^d)_k^d}{(q^d; q^d)_k^d} q^{d(d-r-2)k/2} &\equiv 0 \pmod{[n]}, \end{aligned}$$

where $0 \leq m \leq n-1$ and $dm \equiv -r \pmod{n}$.

We have collected enough ingredients which enables us to prove Theorem 1.

Proof of Theorem 1. The q -congruence (1.5) modulo $[n]$ follows from Lemma 3 immediately. In what follows, we shall prove the modulus $\Phi_n(q)^3$ case of (1.5).

For $M = (dn - n - r)/d$, the left-hand side of (1.5) can be written as the following multiple of a terminating $_{d+5}\phi_{d+4}$ series:

$$[r] \sum_{k=0}^{(dn-n-r)/d} \frac{(q^r, q^{d+r/2}, -q^{d+r/2}, q^r, \dots, q^r, q^{(d+r)/2}, q^{d+(d-1)n}, q^{r-(d-1)n}; q^d)_k}{(q^d, q^{r/2}, -q^{r/2}, q^d, \dots, q^d, q^{(d+r)/2}, q^{r-(d-1)n}, q^{d+(d-1)n}; q^d)_k} q^{d(d-r-2)k/2}.$$

Here, the q^r, \dots, q^r in the numerator means $d-1$ instances of q^r , and similarly, the q^d, \dots, q^d in the denominator means $d-1$ instances of q^d . By Andrews' transformation (2.2), we may rewrite the above expression as

$$\begin{aligned} [r] & \frac{(q^{d+r}, q^{(r-d)/2-(d-1)n}; q^d)_{(dn-n-r)/d}}{(q^{(d+r)/2}, q^{r-(d-1)n}; q^d)_{(dn-n-r)/d}} \sum_{j_1, \dots, j_{m-1} \geq 0} \frac{(q^{d-r}; q^d)_{j_1} \cdots (q^{d-r}; q^d)_{j_{m-1}}}{(q^d; q^d)_{j_1} \cdots (q^d; q^d)_{j_{m-1}}} \\ & \times \frac{(q^r, q^r; q^d)_{j_1} \cdots (q^r, q^r; q^d)_{j_1+\dots+j_{m-2}} (q^{(d+r)/2}, q^{d+(d-1)n}; q^d)_{j_1+\dots+j_{m-1}}}{(q^d, q^d; q^d)_{j_1} \cdots (q^d, q^d; q^d)_{j_1+\dots+j_{m-1}}} \\ & \times \frac{(q^{r-(d-1)n}; q^d)_{j_1+\dots+j_{m-1}}}{(q^{(3d+r)/2}; q^d)_{j_1+\dots+j_{m-1}}} q^{(d-r)(j_{m-2}+\dots+(m-2)j_1)+d(j_1+\dots+j_{m-1})}, \end{aligned} \quad (2.7)$$

where $m = (d+1)/2$.

It is easy to see that the q -shifted factorial $(q^{d+r}; q^d)_{(dn-n-r)/d}$ contains the factor $1 - q^{(d-1)n}$ which is a multiple of $1 - q^n$. Moreover, since none of $(r-d)/2$, $(d+r)/2$ and $(d+r)/2 + dn - n - r - d$ are multiples of n , the q -shifted factorials

$$(q^{(r-d)/2-(d-1)n}; q^d)_{(dn-n-r)/d} \quad \text{and} \quad (q^{(d+r)/2}; q^d)_{(dn-n-r)/d}$$

have the same number (0 or 1) of factors of the form $1 - q^{\alpha n}$ ($\alpha \in \mathbb{Z}$). Besides, the q -shifted factorial $(q^{r-(d-1)n}; q^d)_{(dn-n-r)/d}$ is relatively prime to $\Phi_n(q)$. Thus we conclude that the fraction before the multi-sum in (2.7) is congruent to 0 modulo $\Phi_n(q)$.

Note that the non-zero terms in the multi-summation in (2.7) are those indexed by (j_1, \dots, j_{m-1}) that satisfy the inequality $j_1 + \dots + j_{m-1} \leq (dn - n - r)/d$ because the factor $(q^{r-(d-1)n}; q^d)_{j_1+\dots+j_{m-1}}$ appears in the numerator. None of the factors appearing in the denominator of the multi-sum of (2.7) contain a factor of the form $1 - q^{\alpha n}$ (and are therefore relatively prime to $\Phi_n(q)$), except for $(q^{(3d+r)/2}; q^d)_{j_1+\dots+j_{m-1}}$ when

$$(dn - d - n - r)/(2d) \leq j_1 + \dots + j_{m-1} \leq (dn - n - r)/d.$$

Since

$$\frac{(q^{(d+r)/2}; q^d)_{j_1+\dots+j_{m-1}}}{(q^{(3d+r)/2}; q^d)_{j_1+\dots+j_{m-1}}} = \frac{1 - q^{(d+r)/2}}{1 - q^{(d+r)/2+(j_1+\dots+j_{m-1})d}},$$

the denominator of the above fraction contains a factor of the form $1 - q^{\alpha n}$ if and only if $j_1 + \dots + j_{m-1} = (dn - d - n - r)/(2d)$ (in this case, the denominator contains the factor $1 - q^{(d-1)n/2}$). Writing $n = ad - r$ (with $a \geq 1$), we have $j_1 + \dots + j_{m-1} = a(d-1)/2 - (r+1)/2$. Noticing that $m-1 = (d-1)/2$ and $r \leq d-4$, there must exist an i such that $j_i \geq a$. Then $(q^{d-r}; q^d)_{j_i}$ has the factor $1 - q^{d-r+d(a-1)} = 1 - q^n$ which is divisible by $\Phi_n(q)$. Hence the denominator of the reduced form of the multi-sum in (2.7)

is relatively prime to $\Phi_n(q)$. It remains to show that the multi-sum in (2.7), without the previous fraction, is congruent to 0 modulo $\Phi_n(q)^2$.

By repeated applications of Lemma 1, the multi-sum in (2.7) (without the previous fraction), modulo $\Phi_n(q)^2$, is congruent to

$$\begin{aligned} & \sum_{j_1, \dots, j_{m-1} \geq 0} q^{(d-r)(j_{m-2} + \dots + (m-2)j_1) + d(j_1 + \dots + j_{m-1})} \frac{(q^{d-r}; q^d)_{j_1} \cdots (q^{d-r}; q^d)_{j_{m-1}}}{(q^d; q^d)_{j_1} \cdots (q^d; q^d)_{j_{m-1}}} \\ & \times \frac{(q^{r+(m+1)n}, q^{r-(m+1)n}; q^d)_{j_1} \cdots (q^{r+(2m-2)n}, q^{r-(2m-2)n}; q^d)_{j_1 + \dots + j_{m-2}}}{(q^{d-mn}, q^{d+mn}; q^d)_{j_1} \cdots (q^{d-(2m-3)n}, q^{d+(2m-3)n}; q^d)_{j_1 + \dots + j_{m-2}}} \\ & \times \frac{(q^{d+(d-1)n}, q^{(d+r)/2}; q^d)_{j_1 + \dots + j_{m-1}} (q^{r-(d-1)n}; q^d)_{j_1 + \dots + j_{m-1}}}{(q^{d-(2m-2)n}, q^{d+(2m-2)n}; q^d)_{j_1 + \dots + j_{m-1}} (q^{(3d+r)/2}; q^d)_{j_1 + \dots + j_{m-1}}}, \end{aligned}$$

where $m = (d+1)/2$. However, this sum vanishes in light of the $m = (d+1)/2$, $q \mapsto q^d$, $a = q^r$, $e_1 = q^{(d+r)/2}$, $e_m = q^{r-(2m-2)n}$, $e_i = q^{r-(m+i-2)n}$, $n_1 = (dn - d + n + r)/(2d)$, $n_m = 0$, $n_i = (n + r - d)/d$, $2 \leq i \leq m-1$, $N = (dn - n - r)/d$ case of Lemma 2. (It is easy to verify that $N - n_1 - \dots - n_m = d(d-r-2)/2 > 0$.) This proves that (1.5) holds modulo $\Phi_n(q)^3$ for $M = (dn - n - r)/d$.

Since $(q^r; q^d)_k / (q^d; q^d)_k$ is congruent to 0 modulo $\Phi_n(q)$ for $(dn - n - r)/d < k \leq n-1$, we conclude that (1.5) also holds modulo $\Phi_n(q)^3$ for $M = n-1$. \square

3. PROOF OF THEOREM 2

We first give a simple lemma on a property of certain arithmetic progressions.

Lemma 4. *Let d and r be odd integers satisfying $d \geq 3$, $r \leq d-4$ and $\gcd(d, r) = 1$. Let n be an integer such that $n \geq (d-r)/2$ and $n \equiv -r/2 \pmod{d}$. Then there are no multiples of n in the arithmetic progression*

$$\frac{d+r}{2}, \frac{d+r}{2} + d, \dots, \frac{d+r}{2} + dn - 2n - r - d. \quad (3.1)$$

Proof. By the condition $\gcd(d, r) = 1$, we have $\gcd((d+r)/2, (d-r)/2) = 1$. Suppose that

$$(d+r)/2 + ad = bn \quad (3.2)$$

for some integers a and b with $a \geq 0$. Then $(d+r)/2 + ad > (r-d)/2 \geq -n$ and so $b \geq 0$. Since $n \equiv (d-r)/2 \pmod{d}$, we deduce from (3.2) that $b \equiv -1 \pmod{d}$ and thereby $b \geq d-1$. But we have

$$\frac{d+r}{2} + dn - 2n - r - d = dn - 2n + \frac{d-r}{2} - d \leq (d-1)n - d,$$

thus implying that no number in the arithmetic progression (3.1) is a multiple of n . \square

Proof of Theorem 2. As before, the q -congruence (1.6) modulo $[n]$ can be deduced from Lemma 3. It remains to prove the modulus $\Phi_n(q)^2$ case of (1.6).

For $M = (dn - 2n - r)/d$, the left-hand side of (1.6) can be written as the following multiple of a terminating ${}_d\phi_4$ series (this time we changed the position of $q^{(d+r)/2}$):

$$[r] \sum_{k=0}^{(dn-2n-r)/d} \frac{(q^r, q^{d+r/2}, -q^{d+r/2}, q^{(d+r)/2}, q^r, \dots, q^r, q^{d+(d-2)n}, q^{r-(d-2)n}; q^d)_k}{(q^d, q^{r/2}, -q^{r/2}, q^{(d+r)/2}, q^d, \dots, q^d, q^{r-(d-2)n}, q^{d+(d-2)n}; q^d)_k} q^{d(d-r-2)k/2}.$$

Here, the q^r, \dots, q^r in the numerator stands for $d-1$ instances of q^r , and similarly, the q^d, \dots, q^d in the denominator stands for $d-1$ instances of q^d . By Andrews' transformation (2.2), we may rewrite the above expression as

$$\begin{aligned} [r] \frac{(q^{d+r}, q^{-(d-2)n}; q^d)_{(dn-2n-r)/d}}{(q^d, q^{r-(d-2)n}; q^d)_{(dn-2n-r)/d}} & \sum_{j_1, \dots, j_{m-1} \geq 0} \frac{(q^{(d-r)/2}; q^d)_{j_1} (q^{d-r}; q^d)_{j_2} \cdots (q^{d-r}; q^d)_{j_{m-1}}}{(q^d; q^d)_{j_1} (q^d; q^d)_{j_2} \cdots (q^d; q^d)_{j_{m-1}}} \\ & \times \frac{(q^r, q^r; q^d)_{j_1} \cdots (q^r, q^r; q^d)_{j_1+\dots+j_{m-2}} (q^r, q^{d+(d-2)n}; q^d)_{j_1+\dots+j_{m-1}}}{(q^{(d+r)/2}, q^d; q^d)_{j_1} (q^d, q^d; q^d)_{j_1+j_2} \cdots (q^d, q^d; q^d)_{j_1+\dots+j_{m-1}}} \\ & \times \frac{(q^{r-(d-2)n}; q^d)_{j_1+\dots+j_{m-1}}}{(q^{d+r}; q^d)_{j_1+\dots+j_{m-1}}} q^{(d-r)(j_{m-2}+\dots+(m-2)j_1)+d(j_1+\dots+j_{m-1})}, \end{aligned} \quad (3.3)$$

where $m = (d+1)/2$.

It is easily seen that the q -shifted factorial $(q^{d+r}; q^d)_{(dn-2n-r)/d}$ has the factor $1 - q^{(d-2)n}$ which is a multiple of $1 - q^n$. Clearly, the q -shifted factorial $(q^{-(d-2)n}; q^d)_{(dn-2n-r)/d}$ has the factor $1 - q^{-(d-1)n}$ (again being a multiple of $1 - q^n$) since $(dn - 2n - r)/d \geq 1$ holds according to the conditions $d \geq 3$, $r \leq d-4$, and $n \geq (d-r)/2$. This indicates that the q -factorial $(q^{d+r}, q^{-(d-2)n}; q^d)_{(dn-2n-r)/d}$ in the numerator of the fraction before the multi-sum in (3.3) is divisible by $\Phi_n(q)^2$. Further, it is not difficult to see that the q -factorial $(q^d, q^{r-(d-2)n}; q^d)_{(dn-2n-r)/d}$ in the denominator is relatively prime to $\Phi_n(q)$.

Like the proof of Theorem 1, the non-zero terms in the multi-sum in (3.3) are those indexed by (j_1, \dots, j_{m-1}) satisfying the inequality $j_1 + \dots + j_{m-1} \leq (dn - 2n - r)/d$ because of the appearance of the factor $(q^{r-(d-2)n}; q^d)_{j_1+\dots+j_{m-1}}$ in the numerator. By Lemma 4, the q -shifted factorial $(q^{(d+r)/2}, q^d)_{j_1}$ in the denominator does not contain a factor of the form $1 - q^{an}$ for $j_1 \leq (dn - 2n - r)/d$ (and are therefore relatively prime to $\Phi_n(q)$). In addition, none of the other factors appearing in the denominator of the multi-sum of (3.3) contain a factor of the form $1 - q^{an}$, except for $(q^{d+r}; q^d)_{j_1+\dots+j_{m-1}}$ when $j_1 + \dots + j_{m-1} = (dn - 2n - r)/d$ (in this case the denominator contains the factor $1 - q^{(d-2)n}$).

Letting $n = ad + (d-r)/2$ (with $a \geq 0$), we get $j_1 + \dots + j_{m-1} = a(d-2) + (d-r)/2 - 1$. If $j_1 \geq a+1$, then $(q^{(d-r)/2}; q^d)_{j_1}$ contains the factor $1 - q^{(d-r)/2+ad} = 1 - q^n$. If $j_1 \leq a$, then $j_2 + \dots + j_{m-1} \geq a(d-3) + (d-r)/2 - 1$. Since $m-2 = (d-3)/2$, $d \geq 3$, and $r \leq d-4$, there must be an i with $2 \leq i \leq m-1$ and $j_i \geq 2a+1$. Then $(q^{d-r}; q^d)_{j_i}$ contains the factor $1 - q^{d-r+2ad} = 1 - q^{2n}$ which is a multiple of $\Phi_n(q)$. Therefore, the

denominator of the reduced form of the multi-sum in (3.3) is relatively prime to $\Phi_n(q)$. This proves that (3.3) is congruent to 0 modulo $\Phi_n(q)^2$.

For $M = n - 1$, since $(q^r; q^d)_k / (q^d; q^d)_k$ is congruent to 0 modulo $\Phi_n(q)$ for $(dn - 2n - r)/d < k \leq n - 1$, we conclude that (1.6) is also true modulo $\Phi_n(q)^2$ in this case. \square

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SCHOOL OF MATHEMATICS AND STATISTICS, HUAIYIN NORMAL UNIVERSITY, HUAI’AN 223300, JIANGSU, PEOPLE’S REPUBLIC OF CHINA

E-mail address: jwguo@hytc.edu.cn

FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT WIEN, OSKAR-MORGENSTERN-PLATZ 1, A-1090 VIENNA, AUSTRIA

E-mail address: michael.schlosser@univie.ac.at