A NEW FAMILY OF q-HYPERGEOMETRIC CONGRUENCES FROM ANDREWS' MULTI-SERIES TRANSFORMATION

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Abstract. We deduce a new family of q-hypergeometric congruences modulo the fourth power of a cyclotomic polynomial from George Andrews' multi-series extension of the Watson transformation. A Karlsson–Minton type summation for very-well-poised basic hypergeometric series due to George Gasper also plays an important role in our proof. We put forward two relevant conjectures on supercongruences and q-supercongruences for further study.

1. Introduction

Let $(a)_n = a(a+1)\cdots(a+n-1)$ denote the rising factorial, or the Pochhammer symbol. In 2016, Long and Ramakrishna [19, Thm. 2] established the following supercongruence: for any odd prime p,

$$\sum_{k=0}^{p-1}(6k+1)\frac{(\frac{1}{3})_k^6}{k!^6} \equiv \begin{cases} -p\Gamma_p(\frac{1}{3})^9 \pmod{p^6}, & \text{if } p\equiv 1 \pmod 6,\\[2mm] -\frac{10p^4}{27}\Gamma_p(\frac{1}{3})^9 \pmod{p^6}, & \text{if } p\equiv 5 \pmod 6, \end{cases} \tag{1.1}$$

where $\Gamma_p(x)$ denotes the p-adic Gamma function. For $p\equiv 1 \pmod 6$ this result confirms Van Hamme's (D.2) supercongruence [24], which is a congruence modulo p^4. The supercongruence (1.1) is now called a Ramanujan-type supercongruence. For more such supercongruences, we refer the reader to [24, 27].

Nowadays, a number of supercongruences have been generalized to q-supercongruences by different authors (see, for example, [7, 9, 11, 12, 16, 25]). In particular, the authors [11, Theorem 2.3] gave a partial q-analogue of (1.1):

$$\sum_{k=0}^{n-1}[6k+1]\frac{(q;q^3)_k^6}{(q^3;q^3)_k^6}q^{3k} \equiv \begin{cases} 0 \pmod{[n]}, & \text{if } n\equiv 1 \pmod 3,\\[2mm] 0 \pmod{[n]\Phi_n(q)}, & \text{if } n\equiv 2 \pmod 3. \end{cases} \tag{1.2}$$

Shortly afterwards, they [13, 14] further proved the following result: for any integers d, r, n satisfying $d\geqslant 2$, $r\leqslant d-2$, $n\geqslant d-r$, d and r are coprime, and $n\equiv -r \pmod d$, there

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holds

$$\sum_{k=0}^{n-1}[2dk+r]\frac{(q^r;q^d)_k^{2d}}{(q^d;q^d)_k^{2d}}q^{d(d-1-r)k}\equiv 0 \pmod{[n]\Phi_n(q)^3}. \tag{1.3}$$

Recently, Wei [25] obtained the following complete q-analogue of the second supercongruence in (1.1): for any positive integer $n\equiv 2 \pmod 3$,

$$\sum_{k=0}^{n-1}[6k+1]\frac{(q;q^3)_k^6}{(q^3;q^3)_k^6}q^{3k}\equiv 5[2n]\frac{(q^2;q^3)_{(2n-1)/3}^3}{(q^3;q^3)_{(2n-1)/3}^3} \pmod{[n]\Phi_n(q)^5}, \tag{1.4}$$

He also generalized (1.2) for $n\equiv 1 \pmod 3$ to the modulus $[n]\Phi_n(q)^4$ case. At this point, it is appropriate to recall the standard q-notation (see [3]). For any indeterminates a and q, let $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ be the *q-shifted factorial*. For brevity, we adopt the notation $(a_1,a_2,\dots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n$. Moreover, $[n]=[n]_q=1+q+\cdots+q^{n-1}$ denotes the *q-integer* and $\Phi_n(q)$ the n-th *cyclotomic polynomial* in q, which is irreducible over the integers and may be factorized over the complex numbers as

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(n,k)=1}}(q-\zeta^k),$$

where ζ is any n-th primitive root of unity.

In this paper, we shall establish the following generalization of (1.3).

Theorem 1.1. *Let d,r,s,n be integers satisfying $d\geqslant 2$, $r\leqslant d-2$ (in particular, r is possibly negative), $0\leqslant s\leqslant d-r-2$, and $n\geqslant d-r$, such that $\gcd(d,r)=1$ and $n\equiv -r \pmod d$. Then*

$$\sum_{k=0}^{M}[2dk+r]_{q^2}[2dk+r]^{2s}\frac{(q^{2r};q^{2d})_k^{2d}}{(q^{2d};q^{2d})_k^{2d}}q^{2d(d-r-s-1)k}\equiv 0 \pmod{[n]_{q^2}\Phi_n(q^2)^3}, \tag{1.5}$$

where $M=(dn-n-r)/d$ or $M=n-1$.

Letting $n=p^m$ be a prime power and then taking the limits as $q\to 1$, we get the following supercongruence from (1.5): for any integers d,r,s with $d\geqslant 2$, $r\leqslant d-2$, $0\leqslant s\leqslant d-r-2$ and $\gcd(d,r)=1$, and prime p with $p^m\geqslant d-r$ and $p^m\equiv -r \pmod d$,

$$\sum_{k=0}^{M}(2dk+r)^{2s+1}\frac{(\frac{r}{d})_k^{2d}}{(1)_k^{2d}}\equiv 0 \pmod{p^{m+3}}, \tag{1.6}$$

where $M=(dp^m-p^m-r)/d$ or $M=p^m-1$.

The proof of Theorem 1.1 is similar to that of (1.3). Namely, we need to make a careful use of Andrews' multi-series generalization (2.2) of Watson's ${}_8\Phi_7$ transformation [1, Theorem 4], along with Gasper's very-well-poised Karlsson–Minton type summation [2, Eq. (5.13)]. We remark that Andrews' transformation plays an important part in number theory and combinatorics. For instance, Zudilin [26] utilized Andrews' transformation to solve a problem of Schmidt. Krattenthaler and Rivoal [18] applied it to give an alternative proof of a result of Zudilin relating a very-well-poised hypergeometric series to a Vasilenko–Vasilev-type multiple integral, an important tool in the study of the arithmetic behavior of values of the Riemann zeta function at integers. The couple Hessami Pilehrood [17] also employed this transformation to provide a short proof of a theorem of Zagier. For the application of Andrews' multi-series transformation in q-congruences, see [4–9, 12, 15, 23].

We shall prove the q-congruence (1.5) modulo $\Phi_n(q^2)^4$ in Section 2, and prove it is also true modulo $[n]_{q^2}$ in Section 3. The combination of these two congruences results in a congruence modulo the least common multiple of $\Phi_n(q^2)^4$ and $[n]_{q^2}$, i.e., modulo $[n]_{q^2}\Phi_n(q^2)^3$, which full establishes Theorem 1.1. Finally, we propose two related conjectures in Section 4.

2. Proof of (1.5) modulo $\Phi_n(q^2)^4$

We need a simple q-congruence modulo $\Phi_n(q^2)^2$, which is intrinsically the same as [13, Lemma 1]. For the reader's convenience, we give a new proof here.

Lemma 2.1. *Let α, r be integers and n a positive integer. Then*

$$(q^{2r-2\alpha n}, q^{2r+2\alpha n}; q^{2d})_k \equiv (q^{2r}; q^{2d})_k^2 \pmod{\Phi_n(q^2)^2}. \tag{2.1}$$

Proof. It is easy to see that

$$(aq^{2r}, bq^{2r}; q^{2d})_k - (q^{2r}, abq^{2r}; q^{2d})_k \equiv 0 \pmod{(1-a)(1-b)}.$$

Since $1-q^{\pm 2n} \equiv 0 \pmod{\Phi_n(q^2)}$, putting $a = q^{-2\alpha n}$ and $b = q^{2\alpha n}$ in the above congruence, we immediately get the desired q-congruence (2.1). □

We will make use of a wonderful transformation formula due to Andrews [1, Theorem 4], which can be written as follows:

$$\begin{aligned}
&\sum_{k\geqslant 0} \frac{(a, q\sqrt{a}, -q\sqrt{a}, b_1, c_1, \dots, b_m, c_m, q^{-N}; q)_k}{(q, \sqrt{a}, -\sqrt{a}, aq/b_1, aq/c_1, \dots, aq/b_m, aq/c_m, aq^{N+1}; q)_k} \left(\frac{a^m q^{m+N}}{b_1c_1\cdots b_mc_m}\right)^k \\
&\quad = \frac{(aq, aq/b_mc_m; q)_N}{(aq/b_m, aq/c_m; q)_N} \sum_{j_1,\dots,j_{m-1}\geqslant 0} \frac{(aq/b_1c_1; q)_{j_1}\cdots(aq/b_{m-1}c_{m-1}; q)_{j_{m-1}}}{(q;q)_{j_1}\cdots(q;q)_{j_{m-1}}} \\
&\qquad \times \frac{(b_2, c_2; q)_{j_1}\dots(b_m, c_m; q)_{j_1+\cdots+j_{m-1}}}{(aq/b_1, aq/c_1; q)_{j_1}\dots(aq/b_{m-1}, aq/c_{m-1}; q)_{j_1+\cdots+j_{m-1}}} \\
&\qquad \times \frac{(q^{-N}; q)_{j_1+\cdots+j_{m-1}}}{(b_mc_mq^{-N}/a; q)_{j_1+\cdots+j_{m-1}}} \frac{(aq)^{j_{m-2}+\cdots+(m-2)j_1} q^{j_1+\cdots+j_{m-1}}}{(b_2c_2)^{j_1}\cdots(b_{m-1}c_{m-1})^{j_1+\cdots+j_{m-2}}}.
\end{aligned} \tag{2.2}$$

This transformation can be deemed a multi-series generalization of Watson's ${}_8\phi_7$ transformation formula (see [3, Appendix (III.18)]) which we state below in standard notation for q-series [3, Section 1] (n being a non-negative integer):

$$
\begin{aligned}
&{}_8\phi_7\left[\begin{matrix} a, & qa^{\frac12}, & -qa^{\frac12}, & b, & c, & d, & e, & q^{-n} \\ & a^{\frac12}, & -a^{\frac12}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1}\end{matrix};q,\ \frac{a^2q^{n+2}}{bcde}\right] \\
&\quad = \frac{(aq,aq/de;q)_n}{(aq/d,aq/e;q)_n}\,{}_4\phi_3\left[\begin{matrix} aq/bc, & d, & e, & q^{-n} \\ aq/b, & aq/c, & deq^{-n}/a \end{matrix};q,\ q\right]. \qquad (2.3)
\end{aligned}
$$

Next, we require the following very-well-poised Karlsson–Minton type summation due to Gasper [2, Eq. (5.13)] (see also [3, Ex. 2.33 (i)]):

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\frac{(a,q\sqrt{a},-q\sqrt{a},b,a/b,d,e_1,aq^{n_1+1}/e_1,\dots,e_m,aq^{n_m+1}/e_m;q)_k}{(q,\sqrt{a},-\sqrt{a},aq/b,bq,aq/d,aq/e_1,e_1q^{-n_1},\dots,aq/e_m,e_mq^{-n_m};q)_k}\left(\frac{q^{1-\nu}}{d}\right)^k \\
&\qquad = \frac{(q,aq,aq/bd,bq/d;q)_\infty}{(bq,aq/b,aq/d,q/d;q)_\infty}\prod_{j=1}^{m}\frac{(aq/be_j,bq/e_j;q)_{n_j}}{(aq/e_j,q/e_j;q)_{n_j}}, \qquad (2.4)
\end{aligned}
$$

where $n_1,\dots,n_m$ are non-negative integers, $\nu=n_1+\dots+n_m$, and $|q^{1-\nu}/d|<1$ if the series does not terminate. An elliptic extension of the terminating $d=q^{-\nu}$ case of (2.4) was given by Rosengren and the second author [22, Eq. (1.7)].

It is worth mentioning that the right-hand side of (2.4) vanishes for $d=bq$. Additionally, putting $b=q^{-N}$ we get the following terminating summation:

$$
\sum_{k=0}^{N}\frac{(a,q\sqrt{a},-q\sqrt{a},e_1,aq^{n_1+1}/e_1,\dots,e_m,aq^{n_m+1}/e_m,q^{-N};q)_k}{(q,\sqrt{a},-\sqrt{a},aq/e_1,e_1q^{-n_1},\dots,aq/e_m,e_mq^{-n_m},aq^{N+1};q)_k}q^{(N-\nu)k}=0, \qquad (2.5)
$$

which is valid for $N>\nu=n_1+\dots+n_m$.

By properly combining (2.2) with (2.5), we can establish the following multi-series summation formula, which is a generalization of [13, Lemma 2].

Lemma 2.2. *Let $m\geqslant 2$ and $0\leqslant s\leqslant m-1$. Let q, a and $e_{s+1},\dots,e_{m+1}$ be arbitrary parameters with $e_{m+1}=e_{s+1}$, and let $n_1,\dots,n_m$ and N be non-negative integers satisfying $N>n_1+\dots+n_m$. Then*

$$
\begin{aligned}
0 = &\sum_{j_1,\dots,j_{m-1}\geqslant 0}\frac{(q^{-n_1};q)_{j_1}\cdots(q^{-n_s};q)_{j_s}(e_{s+1}q^{-n_{s+1}}/e_{s+2};q)_{j_{s+1}}\cdots(e_{m-1}q^{-n_{m-1}}/e_m;q)_{j_{m-1}}}{(q;q)_{j_1}\cdots(q;q)_{j_{m-1}}} \\
&\times\frac{(\sqrt{a}q^{\frac{n_2+1}{2}},\sqrt{a}q^{\frac{n_2+1}{2}};q)_{j_1}\cdots(\sqrt{a}q^{\frac{n_s+1}{2}},\sqrt{a}q^{\frac{n_s+1}{2}};q)_{j_1+\dots+j_{s-1}}}{(\sqrt{a}q^{\frac{1-n_1}{2}},\sqrt{a}q^{\frac{1-n_1}{2}};q)_{j_1}\cdots(\sqrt{a}q^{\frac{1-n_{s-1}}{2}},\sqrt{a}q^{\frac{1-n_{s-1}}{2}};q)_{j_1+\dots+j_{s-1}}} \\
&\times\frac{(aq^{n_{s+1}+1}/e_{s+1},e_{s+2};q)_{j_1+\dots+j_s}\dots(aq^{n_m+1}/e_m,e_{m+1};q)_{j_1+\dots+j_{m-1}}}{(e_sq^{-n_s},aq/e_{s+1};q)_{j_1+\dots+j_s}\dots(e_{m-1}q^{-n_{m-1}},aq/e_m;q)_{j_1+\dots+j_{m-1}}} \\
&\times\frac{(q^{-N};q)_{j_1+\dots+j_{m-1}}}{(e_1q^{n_m-N+1}/e_m;q)_{j_1+\dots+j_{m-1}}(aq^{n_2})^{j_1}\cdots(aq^{n_s})^{j_1+\dots+j_{s-1}}}
\end{aligned}
$$

$$\times\frac{(aq)^{j_{m-2}+\cdots+(m-2)j_1}q^{j_1+\cdots+j_{m-1}}}{(aq^{n_{s+1}+1}e_{s+2}/e_{s+1})^{j_1+\cdots+j_s}\cdots(aq^{n_{m-1}+1}e_m/e_{m-1})^{j_1+\cdots+j_{m-2}}}. \tag{2.6}$$

Proof. Performing the parameter substitutions $b_i=c_i=\sqrt{a}q^{\frac{1+n_i}{2}}$ for $1\leqslant i\leqslant s$, and $b_i\mapsto aq^{n_i+1}/e_i$, $c_i\mapsto e_{i+1}$, for $s+1\leqslant i\leqslant m$ (where $e_{m+1}=e_{s+1}$) in the multisum transformation (2.2), and dividing both sides by the prefactor of the multisum, we see that the series on the right-hand side of (2.6) is equal to

$$\frac{(e_mq^{-n_m},aq/e_{s+1};q)_N}{(aq,e_mq^{-n_m}/e_{s+1};q)_N}\times\sum_{k=0}^{N}\frac{(a,q\sqrt{a},-q\sqrt{a},e_1,aq^{n_1+1}/e_1,\ldots,e_m,aq^{n_m+1}/e_m,q^{-N};q)_k}{(q,\sqrt{a},-\sqrt{a},aq/e_1,e_1q^{-n_1},\ldots,aq/e_m,e_mq^{-n_m},aq^{N+1};q)_k}q^{(N-\nu)k},$$

with $e_i=\sqrt{a}q^{\frac{1+n_i}{2}}$ for $1\leqslant i\leqslant s$ and $\nu=n_1+\cdots+n_m$. Now the last sum vanishes by the special case (2.5) of Gasper's summation. $\square$

We have gathered enough ingredients and are able to prove the q-congruence (1.5) modulo $\Phi_n(q^2)^4$.

Proof of (1.5) *modulo* $\Phi_n(q^2)^4$. For $M=(dn-n-r)/d$, the left-hand side of (1.5) can be written as the following multiple of a terminating ${}_{2d+2s+4}\phi_{2d+2s+3}$ series:

$$[2r]\sum_{k=0}^{(dn-n-r)/d}\Bigg(\frac{(q^{2r},q^{2d+r},-q^{2d+r},q^{2d+r},\ldots,q^{2d+r},q^{2r},\ldots,q^{2r};q^{2d})_k}{(q^{2d},q^r,-q^r,q^r,\ldots,q^r,q^{2d},\ldots,q^{2d};q^{2d})_k}\times\frac{(q^{2d+2(d-1)n},q^{2r-2(d-1)n};q^{2d})_k}{(q^{2r-2(d-1)n},q^{2d+2(d-1)n};q^{2d})_k}q^{2d(d-r-s-1)k}\Bigg).$$

Here, the $q^{2d+r},\ldots,q^{2d+r}$ in the numerator means $2s$ instances of q^{2d+r}, the $q^{2r},\ldots,q^{2r}$ in the numerator refers to $2d-1$ instances of q^{2r}. Similarly, the $q^r,\ldots,q^r$ in the denominator means $2s$ instances of q^r, and the $q^{2d},\ldots,q^{2d}$ in the denominator refers to $2d-1$ instances of q^{2d}.

Now, by the $m=d+s$ case of Andrews' transformation (2.2), we can write the above expression as

$$[2r]\frac{(q^{2(d+r)},q^{-2(d-1)n};q^{2d})_{(dn-n-r)/d}}{(q^{2d},q^{2r-2(d-1)n};q^{2d})_{(dn-n-r)/d}}\times\sum_{j_1,\ldots,j_{d+s-1}\geqslant 0}\frac{(q^{-2d};q^{2d})_{j_1}\cdots(q^{-2d};q^{2d})_{j_s}(q^{2(d-r)};q^{2d})_{j_{s+1}}\cdots(q^{2(d-r)};q^{2d})_{j_{d+s-1}}}{(q^{2d};q^{2d})_{j_1}\cdots(q^{2d};q^{2d})_{j_{d+s-1}}}$$

$$\times\frac{(q^{2d+r},q^{2d+r};q^{2d})_{j_1}\cdots(q^{2d+r},q^{2d+r};q^{2d})_{j_1+\cdots+j_{s-1}}}{(q^r,q^r;q^{2d})_{j_1}\cdots(q^r,q^r;q^{2d})_{j_1+\cdots+j_{s-1}}}$$

$$\times \frac{(q^{2r},q^{2r};q^{d})_{j_1+\cdots+j_s}\cdots(q^{2r},q^{2r};q^{2d})_{j_1+\cdots+j_{d+s-2}}(q^{2r},q^{2d+2(d-1)n};q^{2d})_{j_1+\cdots+j_{d+s-1}}}{(q^{2d},q^{2d};q^{2d})_{j_1+\cdots+j_s}\cdots(q^{2d},q^{2d};q^{2d})_{j_1+\cdots+j_{d+s-1}}}$$

$$\times \frac{(q^{2r-2(d-1)n};q^{2d})_{j_1+\cdots+j_{d+s-1}}}{(q^{2(d+r)};q^{2d})_{j_1+\cdots+j_{d+s-1}}}$$

$$\times \frac{q^{2(d+r)(j_{d+s-2}+\cdots+(d+s-2)j_1)}q^{2d(j_1+\cdots+j_{d+s-1})}}{q^{2(2d+r)j_1}\cdots q^{2(2d+r)(j_1+\cdots+j_s)}q^{4r(j_1+\cdots+j_{s+1})}\cdots q^{4r(j_1+\cdots+j_{d+s-2})}}. \tag{2.7}$$

It is not difficult to see that the q-shifted factorial $(q^{2(d+r)};q^{2d})_{(dn-n-r)/d}$ (appearing in the prefactor of the multi-sum) has the factor $1-q^{2(d-1)n}$ which is divisible by $\Phi_n(q^2)$. Similarly, the q-shifted factorial $(q^{-2(d-1)n};q^{2d})_{(dn-n-r)/d}$ has the factor $1-q^{-2(d-1)n}$ (being congruent to 0 modulo $\Phi_n(q^2)$) since $(dn-n-r)/d\geqslant 1$ holds in view of the conditions $d\geqslant 2$, $r\leqslant d-2$, and $n\geqslant d-r$. This shows that the q-factorial $(q^{2(d+r)},q^{-2(d-1)n};q^{2d})_{(dn-n-r)/d}$ appearing in the numerator of the prefactor of the multi-sum is congruent to 0 modulo $\Phi_n(q^2)^2$. Moreover, it is clear that the q-factorial $(q^{2d},q^{2r-2(d-1)n};q^{2d})_{(dn-n-r)/d}$ in the denominator is coprime with $\Phi_n(q^2)$.

Notice that the non-zero terms in the multi-sum in (2.7) are those with multi-index $(j_1,\dots,j_{d+s-1})$ satisfying the conditions $j_1+\cdots+j_{d+s-1}\leqslant (dn-n-r)/d$ and $j_1,\dots,j_s\leqslant 1$ because the product $(q^{-2d};q^{2d})_{j_1}\cdots(q^{-2d};q^{2d})_{j_s}(q^{2r-2(d-1)n};q^{2d})_{j_1+\cdots+j_{d+s-1}}$ appears as a factor in the numerator. None of the factors appearing in the denominator of the multi-sum of (2.7) incorporate a factor of the form $1-q^{2\alpha n}$ (and are consequently coprime with $\Phi_n(q^2)$), except for $(q^{2(d+r)};q^{2d})_{j_1+\cdots+j_{d+s-1}}$ when $j_1+\cdots+j_{d+s-1}=(dn-n-r)/d$. Let $n=ad-r$ (with $a\geqslant 1$). Consider the case $j_1+\cdots+j_{d+s-1}=(dn-n-r)/d=a(d-1)-r$ and $j_1,\dots,j_s\leqslant 1$. Then $j_{s+1}+\cdots+j_{d+s-1}\geqslant a(d-1)-r-s$. Since $r\leqslant d-s-2$, there must exist an i $(s+1\leqslant i\leqslant d+s-1)$ such that $j_i\geqslant a$. Then $(q^{2(d-r)};q^{2d})_{j_i}$ contains the factor $1-q^{2(d-r)+2d(a-1)}=1-q^{2n}$ which is a multiple of $\Phi_n(q^2)$. Hence, the reduced denominator of the multi-sum in (2.7) is coprime with $\Phi_n(q^2)$. It remains to prove that the multi-sum in (2.7), ignoring the prefactor, is congruent to 0 modulo $\Phi_n(q^2)^2$.

By repeatedly applying Lemma 2.1, the multi-sum in (2.7), without the prefactor, modulo $\Phi_n(q^2)^2$, is congruent to

$$\sum_{j_1,\dots,j_{d+s-1}\geqslant 0}\frac{(q^{-2d};q^{2d})_{j_1}\cdots(q^{-2d};q^{2d})_{j_s}(q^{2(d-r)};q^{2d})_{j_{s+1}}\cdots(q^{2(d-r)};q^{2d})_{j_{d+s-1}}}{(q^{2d};q^{2d})_{j_1}\cdots(q^{2d};q^{2d})_{j_{d+s-1}}}$$

$$\times \frac{(q^{2d+r},q^{2d+r};q^{2d})_{j_1}\cdots(q^{2d+r},q^{2d+r};q^{2d})_{j_1+\cdots+j_{s-1}}}{(q^{r},q^{r};q^{2d})_{j_1}\cdots(q^{r},q^{r};q^{2d})_{j_1+\cdots+j_{s-1}}}$$

$$\times \frac{(q^{2r-2(d-1)n},q^{2r+2(d-1)n};q^{2d})_{j_1+\cdots+j_s}\cdots(q^{2(r-n)},q^{2(r+n)};q^{2d})_{j_1+\cdots+j_{d+s-2}}}{(q^{2d+2dn},q^{2d-2dn};q^{2d})_{j_1+\cdots+j_s}\cdots(q^{2d+4n},q^{2d-4n};q^{2d})_{j_1+\cdots+j_{d+s-2}}}$$

$$\times \frac{(q^{2r},q^{2d+2(d-1)n};q^{2d})_{j_1+\cdots+j_{d+s-1}}}{(q^{2d+2n},q^{2d-2n};q^{2d})_{j_1+\cdots+j_{d+s-1}}}\frac{(q^{2r-2(d-1)n};q^{2d})_{j_1+\cdots+j_{d+s-1}}}{(q^{2(d+r)};q^{2d})_{j_1+\cdots+j_{d+s-1}}}$$

$$\times \frac{q^{2(d+r)(j_{d+s-2}+\cdots+(d+s-2)j_1)}q^{2d(j_1+\cdots+j_{d+s-1})}}{q^{2(2d+r)j_1}\cdots q^{2(2d+r)(j_1+\cdots+j_s)}q^{4r(j_1+\cdots+j_{s+1})}\cdots q^{4r(j_1+\cdots+j_{d+s-2})}}.$$

However, this sum vanishes in light of the $m=d+s$, $q\mapsto q^{2d}$, $a\mapsto q^{2r}$, $e_{s+1}=q^{2d+2(d-1)n}$, $e_{s+i}\mapsto q^{2r+2(d-i+1)n}$ $(2\leqslant i\leqslant d)$, $n_1=\cdots=n_s=1$, $n_{s+1}=0$, $n_{s+j}\mapsto(n+r-d)/d$ $(2\leqslant j\leqslant d)$, $N=(dn-n-r)/d$, case of Lemma 2.2.

This proves that the q-hypergeometric congruence (1.5) is true modulo $\Phi_n(q^2)^4$ for $M=(dn-n-r)/d$. Since $n\geqslant d-r$, we have $(dn-n-r)/d\leqslant n-1$. It is clear that $(q^{2r};q^{2d})_k/(q^{2d};q^{2d})_k\equiv 0 \pmod{\Phi_n(q^2)}$ for $(dn-n-r)/d<k\leqslant n-1$. Thus, we see that (1.5) is also true modulo $\Phi_n(q^2)^4$ for $M=n-1$. □

3. Proof of (1.5) modulo $[n]_{q^2}$

We need the following easily proved result, which first appeared in [14, Lemma 2.1].

Lemma 3.1. *Let d, m and n be positive integers with $m\leqslant n-1$. Let r be an integer satisfying $dm\equiv -r \pmod{n}$. Then, for $0\leqslant k\leqslant m$,*

$$\frac{(aq^r;q^d)_{m-k}}{(q^d/a;q^d)_{m-k}}\equiv(-a)^{m-2k}\frac{(aq^r;q^d)_k}{(q^d/a;q^d)_k}q^{m(dm-d+2r)/2+(d-r)k} \pmod{\Phi_n(q)}.$$

In order to prove that (1.5) is true modulo $[n]_{q^2}$, we shall establish the following more general result. It is easy to see that the condition $n\equiv -r \pmod{d}$ guarantees that $m=(dn-n-r)/d$ is an integer.

Theorem 3.2. *Let d,n be positive integers with $\gcd(d,n)=1$. Let r,s be integers with $s\geqslant 0$. Then*

$$\sum_{k=0}^{m}[2dk+r]_{q^2}[2dk+r]^{2s}\frac{(q^{2r};q^{2d})_k^{2d}}{(q^{2d};q^{2d})_k^{2d}}q^{2d(d-r-s-1)k}\equiv 0 \pmod{[n]_{q^2}},\tag{3.1}$$

and

$$\sum_{k=0}^{n-1}[2dk+r]_{q^2}[2dk+r]^{2s}\frac{(q^{2r};q^{2d})_k^{2d}}{(q^{2d};q^{2d})_k^{2d}}q^{2d(d-r-s-1)k}\equiv 0 \pmod{[n]_{q^2}},\tag{3.2}$$

where $0\leqslant m\leqslant n-1$ and $dm\equiv -r \pmod{n}$.

Proof. Since $\gcd(d,n)=1$, there exists a non-negative integer $m\leqslant n-1$ such that $dm\equiv -r \pmod{n}$. By the $a=1$ and $q\mapsto q^2$ case of Lemma 3.1, we can easily verify that, for $0\leqslant k\leqslant m$, the k-th and $(m-k)$-th terms on the left-hand side of (3.1) cancel each other modulo $\Phi_n(q)$, i.e.,

$$\begin{aligned}&[2d(m-k)+r]_{q^2}[2d(m-k)+r]^{2s}\frac{(q^{2r};q^{2d})_{m-k}^{2d}}{(q^{2d};q^{2d})_{m-k}^{2d}}q^{2d(d-r-s-1)(m-k)}\\&\quad\equiv -[2dk+r]_{q^2}[2dk+r]^{2s}\frac{(q^{2r};q^{2d})_k^{2d}}{(q^{2d};q^{2d})_k^{2d}}q^{2d(d-r-s-1)k} \pmod{\Phi_n(q^2)}.\end{aligned}$$

This immediately leads to the q-congruence (3.1) modulo $\Phi_n(q^2)$. Furthermore, since $dm \equiv -r \pmod{n}$, for $m < k \leqslant n-1$ the q-shifted factorial $(q^{2r};q^{2d})_k$ contains the factor $1-q^{2r+2dm} = 1-q^{2an}$ (a is an integer) and is therefore congruent to 0 modulo $\Phi_n(q^2)$. At the same time, the q-shifted factorial $(q^{2d};q^{2d})_k$ is coprime with $\Phi_n(q^2)$ for $m < k \leqslant n-1$. Thus, each summand in (3.2) with k in the range $m < k \leqslant n-1$ is congruent to 0 modulo $\Phi_n(q^2)$. This, along with (3.1), establishes the q-congruence (3.2) modulo $\Phi_n(q^2)$.

Let $\zeta \neq 1$ be an n-th root of unity. That is, ζ is a primitive root of unity of odd degree n_1 such that $n_1 \mid n$. Let $c_q(k)$ stand for the k-th term on the left-hand side of (3.2). Specifically,

$$c_q(k) = [2dk+r]_{q^2}[2dk+r]^{2s}\frac{(q^{2r};q^{2d})_k^{2d}}{(q^{2d};q^{2d})_k^{2d}}q^{2d(d-r-s-1)k}.$$

Now the q-congruences (3.1) and (3.2) modulo $\Phi_n(q^2)$ with $n = n_1$ imply that

$$\sum_{k=0}^{m_1} c_\zeta(k) = \sum_{k=0}^{n_1-1} c_\zeta(k) = 0,$$

and

$$\sum_{k=0}^{m_1} c_{-\zeta}(k) = \sum_{k=0}^{n_1-1} c_{-\zeta}(k) = 0,$$

where $0 \leqslant m_1 \leqslant n_1-1$ and $dm_1 \equiv -r \pmod{n_1}$. Noticing that, for all non-negative integers ℓ and k,

$$\frac{c_\zeta(\ell n_1+k)}{c_\zeta(\ell n_1)} = \lim_{q\to\zeta}\frac{c_q(\ell n_1+k)}{c_q(\ell n_1)} = \frac{c_\zeta(k)}{[r]_{\zeta^2}[r]_\zeta^{2s}},$$

we have

$$\sum_{k=0}^{n-1} c_\zeta(k) = \sum_{\ell=0}^{n/n_1-1}\sum_{k=0}^{n_1-1} c_\zeta(\ell n_1+k) = \frac{1}{[r]_{\zeta^2}[r]_\zeta^{2s}}\sum_{\ell=0}^{n/n_1-1} c_\zeta(\ell n_1)\sum_{k=0}^{n_1-1} c_\zeta(k) = 0, \tag{3.3}$$

and

$$\sum_{k=0}^{m} c_\zeta(k) = \frac{1}{[r]_{\zeta^2}[r]_\zeta^{2s}}\sum_{\ell=0}^{(m-m_1)/n_1-1} c_\zeta(\ell n_1)\sum_{k=0}^{n_1-1} c_\zeta(k) + \frac{c_\zeta(m-m_1)}{[r]_{\zeta^2}[r]_\zeta^{2s}}\sum_{k=0}^{m_1} c_\zeta(k) = 0.$$

This proves that the two sums $\sum_{k=0}^{m} c_q(k)$ and $\sum_{k=0}^{n-1} c_q(k)$ are congruent to 0 modulo $\Phi_{n_1}(q)$. Along the same lines we may prove that they are also congruent to 0 modulo $\Phi_{n_1}(-q)$. Letting n_1 run over all divisors of n greater than 1, we deduce that these two sums are congruent to 0 modulo

$$\prod_{n_1\mid n,\, n_1>1} \Phi_{n_1}(q)\Phi_{n_1}(-q) = [n]_{q^2},$$

thus establishing (3.1) and (3.2). $\square$

4. Two open problems

It seems that, for $r=-1$ and $p\equiv 1 \pmod{d}$, the following stronger version of (1.6) is true.

Conjecture 4.1. *Let d,s be integers satisfying $d\geqslant 2$ and $0\leqslant s\leqslant d-1$. Let p be a prime with $p\equiv 1 \pmod{d}$. Then, for $m\geqslant 1$,*

$$\sum_{k=0}^{M}(2dk-1)^{2s+1}\frac{(-\frac{1}{d})_k^{2d}}{(1)_k^{2d}}\equiv 0 \pmod{p^{4m}}, \tag{4.1}$$

where $M=(dp^m-p^m+1)/d$ or $M=p^m-1$.

Furthermore, we believe the following q-analogue of (4.1) should also be true.

Conjecture 4.2. *Let d,s be integers satisfying $d\geqslant 2$ and $0\leqslant s\leqslant d-1$. Let $n\geqslant d+1$ be an integer with $n\equiv 1 \pmod{d}$. Then, for $m\geqslant 1$,*

$$\sum_{k=0}^{M}[2dk-1]_{q^2}[2dk-1]^{2s}\frac{(q^{-2};q^{2d})_k^{2d}}{(q^{2d};q^{2d})_k^{2d}}q^{2d(d-s)k}\equiv 0 \pmod{\prod_{i=1}^{m}\Phi_{n^i}(q)^4}, \tag{4.2}$$

where $M=(dn^m-n^m+1)/d$ or $M=n^m-1$.

Since $\Phi_{p^i}(1)=p$ for any prime p and positive integer i, the supercongruence (4.1) follows from (4.2) by taking $n=p$ and $q\to 1$. Applying Zeilberger's algorithm [20] and its q-analogue, the authors [10] have proved that, for any odd integer $n>1$,

$$\sum_{k=0}^{M}[4k-1]\frac{(q^{-1};q^2)_k^4}{(q^2;q^2)_k^4}q^{4k}\equiv -(1+3q+q^2)[n]^4 \pmod{[n]^4\Phi_n(q)},$$

where $M=(n+1)/2$ or $M=n-1$. Replacing n by n^m and q by q^2 in the above q-supercongruence, and noticing that $[n^m]_{q^2}$ contains the factor $\prod_{i=1}^{m}\Phi_{n^i}(q)$, we see that Conjecture 4.2 is true for $d=2$ and $s=0$.

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