Some congruences related to hypergeometric polynomials

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Abstract. We prove that, for any odd prime p, the following congruence holds modulo p:

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{4^k}d_k\left(-\frac{1}{4}\right)^2\equiv\begin{cases}2(-1)^{\frac{p-1}{4}}x, & \text{if } p=x^2+y^2 \text{ with } x\equiv 1 \pmod 4,\\ 0, & \text{if } p\equiv 3 \pmod 4,\end{cases}$$

where $d_n(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k}2^k$. The $p\equiv 3 \pmod 4$ case confirms a conjecture of Z.-W. Sun. We also give three similar congruences, including a special case of another conjecture of Z.-W. Sun.

Keywords: Delannoy number; congruence; prime; Fermat's little theorem

MR Subject Classifications: 33C05, 11A07

1 Introduction

It is well known that the *Delannoy number*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}2^k=\sum_{k=0}^{n}\binom{n}{k}\binom{n+m-k}{n}$$

counts lattice paths from $(0,0)$ to (m,n) using only single steps east $(1,0)$, north $(0,1)$, or northeast $(1,1)$. Recently, Z.-W. Sun [1] introduced the following polynomials

$$d_n(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k}2^k.$$

and established some interesting supercongruences involving $d_n(x)$, such as

$$\sum_{k=0}^{p-1}(-1)^k d_k(x)^2\equiv(-1)^{\langle x\rangle_p} \pmod{p^2},$$

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where p is an odd prime and $\langle x\rangle_p$ denotes the least non-negative integer r with $r\equiv x \pmod p$. He also made several interesting conjectures on congruences involving $d_n(x)$, such as (see [1, Conecture 6.2])

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{4^k}d_k\left(-\frac{1}{6}\right)^2\equiv\frac{p}{3}\left(\frac{p}{3}\right)\left(4\left(\frac{-2}{p}\right)-1\right)\pmod{p^2}, \tag{1.1}$$

where $p>3$ is a prime and $(\frac{\cdot}{p})$ is the Legendre symbol.

In this paper, we shall prove the following result.

Theorem 1.1 *Let p be an odd prime. Then modulo p,*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{4^k}d_k\left(-\frac{1}{4}\right)^2\equiv\begin{cases}2(-1)^{\frac{p-1}{4}}x, & \text{if } p=x^2+y^2 \text{ with } x\equiv 1 \pmod 4,\\ 0, & \text{if } p\equiv 3 \pmod 4,\end{cases} \tag{1.2}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{4^k}d_k\left(-\frac{1}{6}\right)^2\equiv 0,\quad \text{if } p>3, \tag{1.3}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{4^k}d_k\left(\frac{1}{4}\right)^2\equiv\begin{cases}0, & \text{if } p\equiv 1 \pmod 4,\\ (-1)^{\frac{p+1}{4}}\binom{\frac{p-1}{2}}{\frac{p-3}{4}}, & \text{if } p\equiv 3 \pmod 4.\end{cases} \tag{1.4}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{4^k}d_k\left(\frac{1}{6}\right)^2\equiv 0,\quad \text{if } p>5. \tag{1.5}$$

The $p\equiv 3$ case of (1.2) was originally conjectured by Z.-W. Sun (see [1, Conjecture 6.3]), and the congruence (1.3) confirms the congruence (1.1) modulo p.

2 Proof of Theorem 1.1

In a previous paper, the second author [2, Lemma 3.1] gives the following identity:

$$d_n(x)^2=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{x}{k}\binom{x+k}{k}4^k,$$

which is a special case of [3, p. 80, (2.5.32)] by noticing [3, p. 31, (1.7.1.3)] (pointed out by Wadim Zudilin).

For any odd prime p and $0\leqslant k\leqslant p-1$, it is easy to see that

$$\frac{\binom{2k}{k}}{4^k}\equiv\begin{cases}(-1)^k\binom{\frac{p-1}{2}}{k} \pmod p, & \text{if } 0\leqslant k\leqslant\frac{p-1}{2},\\ 0 \pmod p, & \text{if } \frac{p+1}{2}\leqslant k\leqslant p-1.\end{cases}$$

It follows that, for any p-adic integer x,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{4^k}d_k(x)^2 \equiv \sum_{k=0}^{\frac{p-1}{2}}(-1)^k\binom{\frac{p-1}{2}}{k}\sum_{j=0}^{k}\binom{k+j}{2j}\binom{x}{j}\binom{x+j}{j}4^j$$
$$= (-1)^{\frac{p-1}{2}}\sum_{j=0}^{\frac{p-1}{2}}\binom{x}{j}\binom{x+j}{j}\binom{j}{\frac{p-1}{2}-j}4^j \pmod p \tag{2.1}$$

by noticing the Chu-Vandermonde identity

$$\sum_{k=j}^{\frac{p-1}{2}}(-1)^k\binom{\frac{p-1}{2}}{k}\binom{k+j}{2j} = (-1)^{\frac{p-1}{2}}\binom{j}{\frac{p-1}{2}-j}.$$

It is not difficult to see that

$$\binom{-\frac14}{j}\binom{-\frac14+j}{j} = (-1)^j\frac{\binom{4j}{2j}\binom{2j}{j}}{64^j} \equiv 0 \quad \text{for } \frac{p}{4}\leqslant j\leqslant\frac{p-1}{2},$$
$$\binom{-\frac16}{j}\binom{-\frac16+j}{j} = (-1)^j\frac{\binom{6j}{3j}\binom{3j}{j}}{432^j} \equiv 0 \quad \text{for } \frac{p}{6}\leqslant j\leqslant\frac{p-1}{2},$$
$$\binom{\frac14}{j}\binom{\frac14+j}{j} = (-1)^{j-1}\frac{(4j+1)\binom{4j}{2j}\binom{2j}{j}}{(4j-1)64^j} \equiv 0 \quad \text{for } j=\frac{p-1}{4} \text{ or } \frac{p+3}{4}\leqslant j\leqslant\frac{p-1}{2},$$
$$\binom{\frac16}{j}\binom{\frac16+j}{j} = (-1)^{j-1}\frac{(6j+1)\binom{6j}{3j}\binom{3j}{j}}{(6j-1)432^j} \equiv 0 \quad \text{for } \frac{p+3}{6}\leqslant j\leqslant\frac{p-1}{2},$$

and $\binom{j}{\frac{p-1}{2}-j}=0$ for $0\leqslant j<\frac{p-1}{4}$. Letting $x=\mp\frac16$ in (2.1), we immediately obtain (1.3) and (1.5). Letting $x=\mp\frac14$ in (2.1), we get

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{4^k}d_k\left(-\frac14\right)^2 \equiv \begin{cases}(-1)^{\frac{p-1}{4}}\dfrac{\binom{p-1}{\frac{p-1}{2}}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}}{16^{\frac{p-1}{4}}} \pmod p, & \text{if } p\equiv 1 \pmod 4,\\ 0 \pmod p, & \text{if } p\equiv 3 \pmod 4,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{4^k}d_k\left(\frac14\right)^2 \equiv \begin{cases}0 \pmod p, & \text{if } p\equiv 1 \pmod 4,\\ (-1)^{\frac{p+1}{4}}\dfrac{(4p+2)\binom{p+1}{\frac{p+1}{2}}\binom{\frac{p+1}{2}}{\frac{p+1}{4}}}{4p\cdot 16^{\frac{p+1}{4}}} \pmod p, & \text{if } p\equiv 3 \pmod 4,\end{cases}$$

We now suppose that p is a prime such that $p\equiv 1 \pmod 4$ and $p=x^2+y^2$ with $x\equiv 1 \pmod 4$. Then by the Beukers-Chowla-Dwork-Evans congruence [4, 5]:

$$\binom{\frac{p-1}{2}}{\frac{p-1}{4}} \equiv \frac{2^{p-1}+1}{2}\left(2x-\frac{p}{2x}\right) \pmod{p^2},$$

and Fermat's little theorem, we have $\binom{\frac{p-1}{2}}{\frac{p-1}{4}} \equiv 2x \pmod{p}$. Moreover, we have

$$\binom{p-1}{\frac{p-1}{2}} \equiv 16^{\frac{p-1}{4}} \equiv 1 \pmod{p}.$$

This proves (1.2). Finally, suppose that p is a prime with $p \equiv 3 \pmod{4}$. Then

$$\frac{(4p+2)\binom{p+1}{\frac{p+1}{2}}\binom{\frac{p+1}{2}}{\frac{p+1}{4}}}{4p \cdot 16^{\frac{p+1}{4}}} \equiv \binom{\frac{p-1}{2}}{\frac{p-3}{4}} \pmod{p}.$$

This proves (1.4).

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References

[1] Sun Z-W. Supercongruences involving dual sequences. ArXiv Preprint. Available from: http://arxiv.org/pdf/1512.00712v3.pdf.

[2] Guo VJW. Proof of Sun's conjectures on integer-valued polynomials. J Math Anal Appl. 2016;444:182–191.

[3] Slater LJ. Generalized hypergeometric functions. Cambridge: Cambridge University Press; 1966.

[4] Chowla S, Dwork B, Evans R. On the mod p^2 determination of $\binom{(p-1)/2}{(p-1)/4}$. J Number Theory 1986;24:188–196.

[5] Pan H. An elementary approach to $\binom{(p-1)/2}{(p-1)/4}$ modulo p^2. Taiwanese J Math. 2012;16:2197–2202.