Some congruences related to hypergeometric polynomials

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Abstract. We prove that, for any odd prime $p$, the following congruence holds modulo $p$:

$$\sum_{k=0}^{p-1} \binom{2k}{k} d_k \left(-\frac{1}{4}\right)^k \equiv \begin{cases} 2(-1)^{p-1} x, & \text{if } p = x^2 + y^2 \text{ with } x \equiv 1 \pmod{4}, \\ 0, & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

where $d_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{x}{k} 2^k$. The $p \equiv 3 \pmod{4}$ case confirms a conjecture of Z.-W. Sun. We also give three similar congruences, including a special case of another conjecture of Z.-W. Sun.

Keywords: Delannoy number; congruence; prime; Fermat’s little theorem

MR Subject Classifications: 33C05, 11A07

1 Introduction

It is well known that the Delannoy number

$$\sum_{k=0}^{n} \binom{n}{k} \binom{m}{k} 2^k = \sum_{k=0}^{n} \binom{n}{k} \binom{n + m - k}{n}$$

counts lattice paths from $(0, 0)$ to $(m, n)$ using only single steps east $(1, 0)$, north $(0, 1)$, or northeast $(1, 1)$. Recently, Z.-W. Sun [1] introduced the following polynomials

$$d_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{x}{k} 2^k.$$

and established some interesting supercongruences involving $d_n(x)$, such as

$$\sum_{k=0}^{p-1} (-1)^k d_k(x)^2 \equiv (-1)^{(x)_p} (\text{mod } p^2),$$

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where \( p \) is an odd prime and \( \langle x \rangle_p \) denotes the least non-negative integer \( r \) with \( r \equiv x \pmod{p} \). He also made several interesting conjectures on congruences involving \( d_n(x) \), such as (see [1, Conjecture 6.2])

\[
\sum_{k=0}^{p-1} \frac{(2k)}{4^k} d_k \left( -\frac{1}{6} \right)^2 \equiv \frac{p}{3} \left( \frac{p}{3} \right) \left( 4 \left( -\frac{2}{p} \right) - 1 \right) \pmod{p^2},
\]

(1.1)

where \( p > 3 \) is a prime and \( \left( \cdot \right)_p \) is the Legendre symbol.

In this paper, we shall prove the following result.

**Theorem 1.1** Let \( p \) be an odd prime. Then modulo \( p \),

\[
\sum_{k=0}^{p-1} \frac{(2k)}{4^k} d_k \left( -\frac{1}{4} \right)^2 \equiv \begin{cases} 2(-1)^{\frac{p+1}{2}} x, & \text{if } p = x^2 + y^2 \text{ with } x \equiv 1 \pmod{4}, \\ 0, & \text{if } p \equiv 3 \pmod{4}, \end{cases}
\]

(1.2)

\[
\sum_{k=0}^{p-1} \frac{(2k)}{4^k} d_k \left( -\frac{1}{6} \right)^2 \equiv 0, \quad \text{if } p > 3,
\]

(1.3)

\[
\sum_{k=0}^{p-1} \frac{(2k)}{4^k} d_k \left( \frac{1}{4} \right)^2 \equiv \begin{cases} 0, & \text{if } p \equiv 1 \pmod{4}, \\ (-1)^{p+1} \left( \frac{p+1}{p-1} \right), & \text{if } p \equiv 3 \pmod{4}. \end{cases}
\]

(1.4)

\[
\sum_{k=0}^{p-1} \frac{(2k)}{4^k} d_k \left( \frac{1}{6} \right)^2 \equiv 0, \quad \text{if } p > 5.
\]

(1.5)

The \( p \equiv 3 \) case of (1.2) was originally conjectured by Z.-W. Sun (see [1, Conjecture 6.3]), and the congruence (1.3) confirms the congruence (1.1) modulo \( p \).

## 2 Proof of Theorem 1.1

In a previous paper, the second author [2, Lemma 3.1] gives the following identity:

\[
d_n(x)^2 = \sum_{k=0}^{n} \binom{n+k}{2k} \frac{x(x+k)}{k} 4^k,
\]

which is a special case of [3, p. 80, (2.5.32)] by noticing [3, p. 31, (1.7.1.3)] (pointed out by Wadim Zudilin).

For any odd prime \( p \) and \( 0 \leq k \leq p - 1 \), it is easy to see that

\[
\frac{(2k)}{4^k} \equiv \begin{cases} (-1)^k \left( \frac{p-1}{2} \right) \pmod{p}, & \text{if } 0 \leq k \leq \frac{p-1}{2}, \\ 0 \pmod{p}, & \text{if } \frac{p+1}{2} \leq k \leq p - 1. \end{cases}
\]
It follows that, for any $p$-adic integer $x$,

$$
\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{4^k} d_k(x)^2 \equiv \sum_{k=0}^{\frac{p-1}{2}} (-1)^k \binom{\frac{p-1}{2}}{k} \sum_{j=0}^{k} \binom{k+j}{2j} \binom{x}{k} \binom{x+j}{j} 4^j \equiv (-1)^{\frac{p-1}{2}} \sum_{j=0}^{\frac{p-1}{2}} \binom{x}{j} \binom{x+j}{j} \binom{p-1}{2} - j \binom{p-1}{2} 4^j \pmod{p} \tag{2.1}
$$

by noticing the Chu-Vandermonde identity

$$
\sum_{k=j}^{\frac{p-1}{2}} (-1)^k \binom{\frac{p-1}{2}}{k} \binom{k+j}{2j} = (-1)^{\frac{p-1}{2}} \binom{p-1}{2} - j \binom{p-1}{2}.
$$

It is not difficult to see that

$$\binom{-\frac{1}{3}}{j} \binom{-\frac{1}{4} + j}{j} = (-1)^j \frac{\binom{4j}{2j} \binom{2j}{j}}{64^j} \equiv 0 \text{ for } \frac{p}{4} \leq j \leq \frac{p-1}{2},$$

$$\binom{-\frac{1}{6}}{j} \binom{-\frac{1}{4} + j}{j} = (-1)^j \frac{\binom{6j}{3j} \binom{3j}{j}}{432^j} \equiv 0 \text{ for } \frac{p}{6} \leq j \leq \frac{p-1}{2},$$

$$\binom{\frac{1}{3}}{j} \binom{\frac{1}{4} + j}{j} = (-1)^{j-1} \frac{\binom{4j+1}{2j} \binom{4j}{2j}}{(4j-1)64^j} \equiv 0 \text{ for } j = \frac{p-1}{4} \text{ or } \frac{p+3}{4} \leq j \leq \frac{p-1}{2},$$

$$\binom{\frac{1}{6}}{j} \binom{\frac{1}{4} + j}{j} = (-1)^{j-1} \frac{\binom{6j+1}{3j} \binom{3j}{j}}{(6j-1)432^j} \equiv 0 \text{ for } \frac{p+3}{6} \leq j \leq \frac{p-1}{2},$$

and $\binom{-\frac{1}{2} - j}{j} = 0$ for $0 \leq j < \frac{p-1}{4}$. Letting $x = \pm \frac{1}{6}$ in (2.1), we immediately obtain (1.3) and (1.5). Letting $x = \pm \frac{1}{4}$ in (2.1), we get

$$
\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{4^k} d_k \left(-\frac{1}{4}\right)^2 \equiv \begin{cases} (-1)^{\frac{p-1}{2}} \frac{\binom{p-1}{2}}{4p+1} & \text{ (mod p), if } p \equiv 1 \pmod{4}, \\
0 & \text{ (mod p), if } p \equiv 3 \pmod{4}, \end{cases}
$$

$$
\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{4^k} d_k \left(\frac{1}{4}\right)^2 \equiv \begin{cases} 0 & \text{ (mod p), if } p \equiv 1 \pmod{4}, \\
(-1)^{\frac{p+1}{4}} \frac{\binom{p+1}{4} \binom{p+1}{2}}{4p+1} & \text{ (mod p), if } p \equiv 3 \pmod{4}. \end{cases}
$$

We now suppose that $p$ is a prime such that $p \equiv 1 \pmod{4}$ and $p = x^2 + y^2$ with $x \equiv 1 \pmod{4}$. Then by the Beukers-Chowla-Dwork-Evans congruence [4,5],

$$
\binom{p-1}{2} \equiv \frac{2^{p-1} + 1}{2} \left(2x - \frac{p}{2x}\right) \pmod{p^2},
$$
and Fermat’s little theorem, we have \( \left( \frac{p-1}{2} \right) \equiv 2x \pmod{p} \). Moreover, we have

\[
\left( \frac{p-1}{2} \right) \equiv 16^{\frac{p-1}{4}} \equiv 1 \pmod{p}.
\]

This proves (1.2). Finally, suppose that \( p \) is a prime with \( p \equiv 3 \pmod{4} \). Then

\[
\frac{(4p+2)(\frac{p+1}{2})(\frac{p+1}{4})}{4p \cdot 16^{\frac{p+1}{4}}} \equiv \left( \frac{p-1}{2} \right) \pmod{p}.
\]

This proves (1.4).

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References

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