Proof of a conjecture involving Sun polynomials

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Abstract. The Sun polynomials $g_n(x)$ are defined by

$$g_n(x)=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}x^k.$$

We prove that, for any positive integer n, there hold

$$\frac{1}{n}\sum_{k=0}^{n-1}(4k+3)g_k(x)\in\mathbb{Z}[x],\quad\text{and}$$
$$\sum_{k=0}^{n-1}(8k^2+12k+5)g_k(-1)\equiv 0\pmod{n}.$$

The first one confirms a recent conjecture of Z.-W. Sun, while the second one partially answers another conjecture of Z.-W. Sun. We give three different proofs of the former. One of them depends on the following congruence:

$$\binom{m+n-2}{m-1}\binom{n}{m}\binom{2n}{n}\equiv 0\pmod{m+n}\quad\text{for } m,n\geqslant 1.$$

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1 Introduction

Recently, Z.-W. Sun [20] introduced the polynomials

$$g_n(x)=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}x^k,$$

which we call *Sun polynomials* here, and proved many interesting identities and congruences involving $g_n(x)$, such as

$$\frac{1}{3n^2}\sum_{k=0}^{n-1}(4k+3)g_k=\sum_{k=0}^{n-1}\frac{1}{k+1}\binom{2k}{k}\binom{n-1}{k}^2,$$

$$\sum_{k=0}^{p-1}kg_k\equiv-\frac{3}{4}\pmod{p^2},$$

where p is an odd prime and $g_k=g_k(1)$. Z.-W. Sun [19] also conjectured that

$$\sum_{k=0}^{\infty}\frac{16k+5}{324^k}\binom{2k}{k}g_k(-20)=\frac{189}{25\pi}.$$

Some other congruences involving g_n can be found in [10, 18, 19].

The Sun polynomials also satisfy the following identities [20, (2.7),(2.11)]:

$$\sum_{k=0}^{n}\binom{n}{k}f_k(x)=g_n(x),$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}g_k(x)=A_n(x),$$

where $f_n(x)$ and $A_n(x)$ are respectively the Franel polynomials and Apéry polynomials [17] defined as

$$f_n(x)=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}x^k,$$

$$A_n(x)=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2x^k.$$

The objective of this paper is to prove the following result, which was originally conjectured by Z.-W. Sun (see [20, Conjecture 4.1(ii)]).

Theorem 1.1 *Let n be a positive integer. Then*

$$\frac{1}{n}\sum_{k=0}^{n-1}(4k+3)g_k(x)\in\mathbb{Z}[x], \tag{1.1}$$

$$\sum_{k=0}^{n-1}(8k^2+12k+5)g_k(-1)\equiv 0\pmod{n},. \tag{1.2}$$

Remark. For the congruence (1.2), Z.-W. Sun [20] made the following stronger conjecture:

$$\sum_{k=0}^{n-1}(8k^2+12k+5)g_k(-1)\equiv n^2 \pmod{2n^2}, \tag{1.3}$$
$$\sum_{k=0}^{p-1}(8k^2+12k+5)g_k(-1)\equiv 3p^2 \pmod{p^3}, \tag{1.4}$$

where p is a prime.

In order to prove Theorem 1.1, we need to establish some preliminary results in Section 3. However, since the following result is interesting in its own right, we label it as a theorem here.

Theorem 1.2 *Let m and n be positive integers. Then*

$$\binom{m+n-2}{m-1}\binom{n}{m}\binom{2n}{n}\equiv 0 \pmod{m+n}. \tag{1.5}$$

It is worth mentioning that Gessel [5, Section 7] proved a similar result as follows:

$$\frac{m}{2}\binom{2m}{m}\binom{2n}{n}\equiv 0 \pmod{m+n},$$

of which a generalization was given by the author [6, Theorem 1.4].

The paper is organized as follows. Applying the same techniques in [6, 7], we shall prove a q-analogue of Theorem 1.2 in the next section. In Section 3, we give three lemmas, one of which is closely related to Theorem 1.2. Two proofs of (1.1) and a proof of (1.2) will be given in Section 4. The second proof of (1.1) is motivated by Sun [20, Lemma 3.4] and its proof. We shall also give a q-analogue (the third proof) of (1.1) in Section 5. We end the paper in Section 6 with a related conjecture.

2 A q-analogue of Theorem 1.2

Recall that the *q-binomial coefficients* are defined by

$$\begin{bmatrix}n\\k\end{bmatrix}_q=\begin{cases}\displaystyle\prod_{i=1}^{k}\frac{1-q^{n-k+i}}{1-q^i}, & \text{if } 0\leqslant k\leqslant n,\\ 0, & \text{otherwise.}\end{cases}$$

We now state the announced strengthening of Theorem 1.2.

Theorem 2.1 *Let m and n be positive integers. Then*

$$\frac{1-q}{1-q^{m+n}}\begin{bmatrix}m+n-2\\m-1\end{bmatrix}_q\begin{bmatrix}n\\m\end{bmatrix}_q\begin{bmatrix}2n\\n\end{bmatrix}_q \tag{2.1}$$

is a polynomial in q with non-negative integer coefficients.

It is clear that Theorem 1.2 can be deduced from Theorem 2.1 by letting $q \to 1$.

A polynomial $A(q) = \sum_{i=0}^{d} a_i q^i$ in q of degree d is called *reciprocal* if $a_i = a_{d-i}$ for all i, and that it is called *unimodal* if there is an index r such that $0 \leqslant a_0 \leqslant \cdots \leqslant a_r \geqslant \cdots \geqslant a_d \geqslant 0$. The following is an elementary but crucial property of reciprocal and unimodal polynomials (see, for example, [1] or [15, Proposition 1]).

Lemma 2.2 *If $A(q)$ and $B(q)$ are reciprocal and unimodal polynomials, then so is their product $A(q)B(q)$.*

Similarly to the proof of [7, Theorem 3.1], we also need the following result. We refer the reader to [13, Proposition 10.1.(iii)] and [3, Proof of Theorem 2] for similar mathematical ideas.

Lemma 2.3 [7, Lemma 5.1] *Let $P(q)$ be a reciprocal and unimodal polynomial and m and n positive integers with $m \leqslant n$. Furthermore, assume that $\frac{1-q^m}{1-q^n}P(q)$ is a polynomial in q. Then $\frac{1-q^m}{1-q^n}P(q)$ has non-negative coefficients.*

Proof of Theorem 2.1. It is well known that the q-binomial coefficients are reciprocal and unimodal polynomials in q (see, for example, [16, Ex. 7.75.d]). By Lemma 2.2, so is the product of three q-binomial coefficients. In view of Lemma 2.3, to prove Theorem 2.1, it suffices to show that the expression (2.1) is a polynomial in q. We shall accomplish this by considering a count of cyclotomic polynomials.

Recall that

$$q^n - 1 = \prod_{d|n} \Phi_d(q),$$

where $\Phi_d(q)$ denotes the d-th cyclotomic polynomial in q. Therefore,

$$\frac{1-q}{1-q^{m+n}} {m+n-2 \brack m-1}_q {n \brack m}_q {2n \brack n}_q = \prod_{d=2}^{2n} \Phi_d(q)^{e_d},$$

with

$$\begin{aligned} e_d = & -\chi(d \mid m+n) + \left\lfloor \frac{m+n-2}{d} \right\rfloor + \left\lfloor \frac{2n}{d} \right\rfloor \\ & - \left\lfloor \frac{m-1}{d} \right\rfloor - \left\lfloor \frac{n-1}{d} \right\rfloor - \left\lfloor \frac{m}{d} \right\rfloor - \left\lfloor \frac{n}{d} \right\rfloor - \left\lfloor \frac{n-m}{d} \right\rfloor, \end{aligned}$$

where $\chi(\mathcal{S}) = 1$ if $\mathcal{S}$ is true and $\chi(\mathcal{S}) = 0$ otherwise. The number e_d is obviously non-negative, unless $d \mid m+n$.

So, let us assume that $d \mid m+n$ and $d \geqslant 2$. We consider two cases: If $d \mid m$, then $d \mid n$, and so

$$\left\lfloor \frac{m+n-2}{d} \right\rfloor - \left\lfloor \frac{m-1}{d} \right\rfloor - \left\lfloor \frac{n-1}{d} \right\rfloor = \frac{m+n-d}{d} - \frac{m-d}{d} - \frac{n-d}{d} = 1.$$

Namely, $e_d = 0$ is non-negative; If $d \nmid m$, then

$$\left\lfloor \frac{2n}{d} \right\rfloor - \left\lfloor \frac{m}{d} \right\rfloor - \left\lfloor \frac{n}{d} \right\rfloor - \left\lfloor \frac{n-m}{d} \right\rfloor = \frac{m+n}{d} - \left\lfloor \frac{m}{d} \right\rfloor - \left\lfloor \frac{n}{d} \right\rfloor = 1.$$

That is, $e_d = 0$ is still non-negative. This completes the proof of polynomiality of (2.1). □

3 Some preliminary results

Lemma 3.1 *Let n be a non-negative integer. Then*

$$\binom{x}{n}^2 = \sum_{k=0}^{n} \binom{x}{n+k} \binom{n+k}{k} \binom{n}{k}. \tag{3.1}$$

Proof. Applying Chu-Vandermonde's identity (see, for example, [9, p. 32])

$$\binom{x}{n} = \sum_{k=0}^{n} \binom{x-n}{k} \binom{n}{k},$$

and noticing that $\binom{x}{n}\binom{x-n}{k} = \binom{x}{n+k}\binom{n+k}{k}$, we obtain (3.1). In fact, Eq. (3.1) is a special case of [14, p. 15, Eq. (9)]. □

Lemma 3.2 *Let n be a positive integer and let $0 \leqslant k \leqslant n$. Then*

$$\sum_{m=k}^{n-1} (4m+3) \binom{m}{k} = (4n-1) \binom{n}{k+1} - 4 \binom{n}{k+2},$$

$$\sum_{m=k}^{n-1} (8m^2+12m+5) \binom{m}{k} = (8n^2-4n+1) \binom{n}{k+1} - (16n-12) \binom{n}{k+2} + 16 \binom{n}{k+3}.$$

Proof. Proceed by induction on n. □

Lemma 3.3 *Let m and n be non-negative integers. Then*

$$\binom{m+n}{m} \binom{n+1}{m} \binom{2n}{n} \frac{3m^2+n^2+m+n}{(m+n)(n+1)} \equiv 0 \pmod{m+n+1}.$$

Proof. Note that

$$\frac{3m^2+n^2+m+n}{(m+n)(n+1)} = \frac{3m(m+n+1)}{(m+n)(n+1)} + \frac{n(m+n+1)}{(m+n)(n+1)} - \frac{2m(2n+1)}{(m+n)(n+1)}.$$

Since $\frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}$ is an integer (the n-th Catalan number), we see that

$$\binom{m+n}{m}\binom{2n}{n}\frac{m}{(m+n)(n+1)} = \binom{m+n-1}{m-1}\binom{2n}{n}\frac{1}{n+1},$$

and

$$\binom{m+n}{m}\binom{2n}{n}\frac{n}{(m+n)(n+1)} = \binom{m+n-1}{m}\binom{2n}{n}\frac{1}{n+1},$$

are both integers. It remains to show that

$$\binom{m+n}{m}\binom{n+1}{m}\binom{2n}{n}\frac{2m(2n+1)}{(m+n)(n+1)} \equiv 0 \pmod{m+n+1},$$

i.e.,

$$\binom{m+n-1}{m-1}\binom{n+1}{m}\binom{2n+2}{n+1} \equiv 0 \pmod{m+n+1}.$$

But this is just the $n \to n+1$ case of the congruence (1.5). □

Lemma 3.4 *Let*

$$S_n = f_{n-3}(-1) = \sum_{k=0}^{n-3}(-1)^k\binom{2k}{k}\binom{n-3}{k}\binom{k}{n-k-3}.$$

Then there hold the following congruences:

$$S_{3n} \equiv S_{3n+1} \equiv -S_{3n+2} \pmod 3, \tag{3.2}$$
$$S_{4n+2} \equiv 0 \pmod 4, \tag{3.3}$$
$$S_{n+2} + 12S_{n+1} + 16S_n \equiv 0 \pmod n. \tag{3.4}$$

Proof. Zeilberger's algorithm [9, 12] gives the following recurrence relation for S_n:

$$\begin{aligned}&(5n^3 - 8n^2)S_{n+3} + (45n^3 - 117n^2 + 90n - 24)S_{n+2}\\ &\quad + (200n^3 - 720n^2 + 824n - 288)S_{n+1} + (160n^3 - 736n^2 + 1024n - 384)S_n = 0.\end{aligned} \tag{3.5}$$

Replacing n by $3n-1$ in (3.5), we obtain

$$-S_{3n+2} - S_{3n} \equiv 0 \pmod 3,$$

while replacing n by $3n+1$ in (3.5), we get

$$S_{3n+2} + S_{3n+1} \equiv 0 \pmod 3.$$

This proves (3.2). Similarly, replacing n by $4n-1$ in (3.5), we are led to (3.3).

In order to prove (3.4), we need to consider four cases:

- If $\gcd(n, 24) = 1$, then (3.5) means that $-24(S_{n+2} + 12S_{n+1} + 16S_n) \equiv 0 \pmod{n}$, i.e., the congruence (3.4) holds.
- If $\gcd(n, 24) = 2, 4, 8$, then (3.5) means that

$$\pm 2nS_{n+2} - 24(S_{n+2} + 12S_{n+1} + 16S_n) \equiv 0 \pmod{8n}.$$

  By (3.3), we have $2nS_{n+2} \equiv 0 \pmod{8n}$ in this case, and so the congruence (3.4) holds.
- If $\gcd(n, 24) = 3$, then (3.5) means that

$$2nS_{n+1} + nS_n - 24(S_{n+2} + 12S_{n+1} + 16S_n) \equiv 0 \pmod{3n}.$$

  By (3.2), we have $2nS_{n+1} + nS_n \equiv 0 \pmod{3n}$ in this case, and so the congruence (3.4) holds.
- If $\gcd(n, 24) = 6, 12, 24$, then (3.5) means that

$$30nS_{n+2} + 8nS_{n+1} + 16nS_n - 24(S_{n+2} + 12S_{n+1} + 16S_n) \equiv 0 \pmod{24n}, \quad \text{or}$$

$$18nS_{n+2} + 8nS_{n+1} + 16nS_n - 24(S_{n+2} + 12S_{n+1} + 16S_n) \equiv 0 \pmod{24n}.$$

  By (3.3), we have $30nS_{n+2} \equiv 18nS_{n+2} \equiv 0 \pmod{24n}$ and $8nS_{n+1} + 16nS_n \equiv 0 \pmod{24n}$ in this case, and so the congruence (3.4) still holds.

□

4 Proof of Theorem 1.1

First Proof of (1.1). By Lemmas 3.1 and 3.2, we have

$$\begin{aligned}
&\sum_{m=0}^{n-1}(4m+3)g_m(x) \\
&= \sum_{m=0}^{n-1}(4m+3)\sum_{k=0}^{m}\binom{m}{k}^2\binom{2k}{k}x^k \\
&= \sum_{m=0}^{n-1}(4m+3)\sum_{k=0}^{m}\binom{2k}{k}x^k\sum_{i=0}^{k}\binom{m}{k+i}\binom{k+i}{i}\binom{k}{i} \\
&= \sum_{k=0}^{n-1}\binom{2k}{k}x^k\sum_{i=0}^{k}\binom{k+i}{i}\binom{k}{i}\sum_{m=k+i}^{n-1}(4m+3)\binom{m}{k+i} \\
&= \sum_{k=0}^{n-1}\binom{2k}{k}x^k\sum_{i=0}^{k}\left((4n-1)\binom{n}{k+i+1} - 4\binom{n}{k+i+2}\right)\binom{k+i}{i}\binom{k}{i}. \qquad (4.1)
\end{aligned}$$

For any non-negative integer $k \leqslant n-1$, to prove that the coefficient of x^k in the right-hand side of (4.1) is a multiple of n, it suffices to show that

$$\binom{2k}{k}\sum_{i=0}^{k}\left(\binom{n}{k+i+1}+4\binom{n}{k+i+2}\right)\binom{k+i}{i}\binom{k}{i} \equiv 0 \pmod{n}. \tag{4.2}$$

We shall accomplish the proof of (4.2) by using a minor trick. Rewrite the left-hand side of (4.2) as

$$\begin{aligned}\binom{2k}{k}&\sum_{i=0}^{k+1}\binom{n}{k+i+1}\left(\binom{k+i}{i}\binom{k}{i}+4\binom{k+i-1}{i-1}\binom{k}{i-1}\right)\\ &=\sum_{i=0}^{k+1}\binom{n}{k+i+1}\binom{k+i}{i}\binom{k+1}{i}\binom{2k}{k}\frac{k^2+3i^2+k+i}{(k+i)(k+1)}.\end{aligned}$$

Then, by Lemma 3.3, for each $i \leqslant k+1$, the expression

$$\binom{k+i}{i}\binom{k+1}{i}\binom{2k}{k}\frac{k^2+3i^2+k+i}{(k+i)(k+1)}.$$

is a multiple of $k+i+1$. Finally, noticing that

$$\binom{n}{k+i+1}(k+i+1)=n\binom{n-1}{k+i} \equiv 0 \pmod{n},$$

we complete the proof. □

Second Proof of (1.1). This proof is motivated by [20, Lemma 3.4 and its proof]. It is clear that (1.1) is equivalent to the following congruence:

$$\binom{2j}{j}\sum_{k=j}^{n-1}(4k+3)\binom{k}{j}^2 \equiv 0 \pmod{n}. \tag{4.3}$$

Denote the left-hand side of (4.3) by u_j. Then by Zeilberger's algorithm [12], we have

$$\begin{aligned}u_{j+1}-u_j &= -\binom{2j}{j}\binom{n-1}{j}^2\frac{(9j+6)(j+1)n^2+(12j^2-8jn-4n+14j+4)n^3}{(j+1)^3(j+2)}\\ &= -\binom{2j}{j}\binom{n-1}{j}\binom{n+1}{j+2}\frac{(9j+6)n}{(j+1)(n+1)}\\ &\quad -\binom{2j}{j}\binom{n}{j+1}\binom{n+1}{j+2}\frac{(12j^2-8jn-4n+14j+4)n}{(j+1)(n+1)}.\end{aligned} \tag{4.4}$$

Noticing that $\frac{1}{j+1}\binom{2j}{j}$ is an integer and $n+1$ is relatively prime to n, from (4.4) we immediately get

$$u_{j+1}-u_j \equiv 0 \pmod{n}.$$

Since $u_0 = 2n^2 + n \equiv 0 \pmod{n}$, we conclude that $u_j \equiv 0 \pmod{n}$ for all j. This proves (4.3).

Proof of (1.2). By Lemmas 3.1 and 3.2, similarly to (4.1), we have

$$\begin{aligned}
&\sum_{m=0}^{n-1}(8m^2+12m+5)g_m(-1)\\
&\quad=\sum_{k=0}^{n-1}\binom{2k}{k}(-1)^k\sum_{i=0}^{k}\left((8n^2-4n+1)\binom{n}{k+i+1}-(16n-12)\binom{n}{k+i+2}\right.\\
&\qquad\left.+16\binom{n}{k+i+3}\right)\binom{k+i}{i}\binom{k}{i}. \qquad (4.5)
\end{aligned}$$

In view of (4.2), it follows from (4.5) that

$$\begin{aligned}
&\sum_{m=0}^{n-1}(8m^2+12m+5)g_m(-1)\\
&\quad\equiv\sum_{k=0}^{n-1}\binom{2k}{k}(-1)^k\sum_{i=0}^{k}\left(\binom{n}{k+i+1}+12\binom{n}{k+i+2}+16\binom{n}{k+i+3}\right)\\
&\qquad\times\binom{k+i}{i}\binom{k}{i}\\
&\quad=\sum_{m=0}^{n-1}\left(\binom{n}{m+1}+12\binom{n}{m+2}+16\binom{n}{m+3}\right)\\
&\qquad\times\sum_{k=0}^{m}(-1)^k\binom{2k}{k}\binom{m}{k}\binom{k}{m-k} \pmod{2n^2}. \qquad (4.6)
\end{aligned}$$

Note that the right-hand side of (4.6) may be written as

$$\begin{aligned}
&\sum_{m=0}^{n-1}\left(\binom{n}{m+1}+12\binom{n}{m+2}+16\binom{n}{m+3}\right)S_{m+3}\\
&\quad=\sum_{m=1}^{n}\binom{n}{m}(S_{m+2}+12S_{m+1}+16S_m), \qquad (4.7)
\end{aligned}$$

which is clearly congruent to 0 modulo n by (3.4) and the fact that $m\binom{n}{m} = n\binom{n-1}{m-1}$. □

5 A q-analogue of (1.1)

Define the q-analogue of Sun polynomials as follows:

$$g_n(x;q)=\sum_{k=0}^{n}{\begin{bmatrix}n\\k\end{bmatrix}}_q^2{\begin{bmatrix}2k\\k\end{bmatrix}}_q x^k.$$

We have the following congruences related to $g_n(x;q)$.

Theorem 5.1 *Let n be a positive integer. Then*

$$(1+q)^2\sum_{k=0}^{n-1} q^{2k}[k+1]_{q^2} g_k(x;q^2) \equiv \sum_{k=0}^{n-1} q^{2k} g_k(x;q^2) \pmod{\prod_{\substack{d\mid n\\ d>1 \text{ is odd}}} \Phi_d(q)}, \tag{5.1}$$

$$\sum_{k=0}^{n-1} q^k g_k(x;q) \equiv 0 \pmod{\prod_{\substack{d\mid n\\ d \text{ is even}}} \Phi_d(q)}, \tag{5.2}$$

$$\sum_{j=0}^{n-1} x^j [j+1]_{q^2} {2j \brack j}_q \sum_{k=j}^{n-1} q^k {k \brack j}_q {k+1 \brack j+1}_q \equiv 0 \pmod{\prod_{\substack{d\mid n\\ d>2 \text{ is even}}} \Phi_d(q)}, \tag{5.3}$$

where $[n]_q = \frac{1-q^n}{1-q}$ denotes a q-integer.

Proof. It is clear that

$$\begin{aligned}\sum_{k=0}^{n-1} q^{2k}[k+1]_{q^2} g_k(x;q^2) &= \sum_{k=0}^{n-1} q^{2k}[k+1]_{q^2} \sum_{j=0}^{k} {k \brack j}_{q^2}^2 {2j \brack j}_{q^2} x^j \\ &= \sum_{j=0}^{n-1} [j+1]_{q^2} {2j \brack j}_{q^2} x^j \sum_{k=j}^{n-1} q^{2k} {k+1 \brack j+1}_{q^2} {k \brack j}_{q^2}.\end{aligned}$$

Suppose that $d \mid n$ and d is odd. It is easy to see that $\Phi_d(q)$ divides $\Phi_d(q^2)$. Write $j = \gamma d + \delta$, where $0 \leqslant \delta \leqslant d-1$. If $d \leqslant 2\delta$, then by the q-Lucas theorem (see Olive [11], Désarménien [4, Proposition 2.2] or Guo and Zeng [8, Proposition 2.1]),

$${2j \brack j}_{q^2} \equiv \binom{2\gamma+1}{\gamma} {2\delta - d \brack \delta}_{q^2} = 0 \pmod{\Phi_d(q)}.$$

Now assume that $\delta \leqslant \frac{d-1}{2}$. Then applying the q-Lucas theorem, we have

$$\begin{aligned}\sum_{k=j}^{n-1} q^{2k} {k+1 \brack j+1}_{q^2} {k \brack j}_{q^2} &= \sum_{\alpha=0}^{\frac{n}{d}-1} \sum_{\beta=0}^{d-1} q^{2(\alpha d+\beta)} {\alpha d+\beta+1 \brack \gamma d+\delta+1}_{q^2} {\alpha d+\beta \brack \gamma d+\delta}_{q^2} \\ &\equiv \sum_{\alpha=0}^{\frac{n}{d}-1} \binom{\alpha}{\gamma}^2 \sum_{\beta=0}^{d-1} q^{2\beta} {\beta+1 \brack \delta+1}_{q^2} {\beta \brack \delta}_{q^2} \pmod{\Phi_d(q)}.\end{aligned} \tag{5.4}$$

It is easy to see that

$$\begin{aligned}\sum_{\beta=0}^{d-1} q^{2\beta}\begin{bmatrix}\beta+1\\ \delta+1\end{bmatrix}_{q^2}\begin{bmatrix}\beta\\ \delta\end{bmatrix}_{q^2} &= \sum_{r=0}^{d-1-\delta} q^{2r+2\delta}\begin{bmatrix}r+\delta+1\\ \delta+1\end{bmatrix}_{q^2}\begin{bmatrix}r+\delta\\ \delta\end{bmatrix}_{q^2}\\ &= \sum_{r=0}^{d-1-\delta} q^{2\delta+6r+4r\delta+2r^2}\begin{bmatrix}-\delta-2\\ r\end{bmatrix}_{q^2}\begin{bmatrix}-\delta-1\\ r\end{bmatrix}_{q^2}\\ &\equiv \sum_{r=0}^{d-1-\delta} q^{2(d-2-\delta-r)(d-1-\delta-r)-4\delta-2\delta^2-4}\begin{bmatrix}d-\delta-2\\ r\end{bmatrix}_{q^2}\begin{bmatrix}d-\delta-1\\ d-\delta-1-r\end{bmatrix}_{q^2}\\ &= q^{-4\delta-2\delta^2-4}\begin{bmatrix}2d-2\delta-3\\ d-1-\delta\end{bmatrix}_{q^2} \pmod{\Phi_d(q)}, \qquad (5.5)\end{aligned}$$

where we have used the q-Chu-Vandemonde identity (see [2, (3.3.10)]) in the last step. Furthermore, we have

$$\begin{bmatrix}2d-2\delta-3\\ d-1-\delta\end{bmatrix}_{q^2} \equiv 0 \pmod{\Phi_d(q)}$$

for $\delta \leqslant \frac{d-3}{2}$. This proves that

$$\sum_{k=j}^{n-1} q^{2k}\begin{bmatrix}k+1\\ j+1\end{bmatrix}_{q^2}\begin{bmatrix}k\\ j\end{bmatrix}_{q^2} \equiv \begin{cases}\displaystyle\sum_{\alpha=0}^{\frac{n}{d}-1}\binom{\alpha}{\gamma}^2 q^{\frac{d-5}{2}}\begin{bmatrix}d-2\\ \frac{d-1}{2}\end{bmatrix}_{q^2}, & \text{if } j\equiv \frac{d-1}{2} \pmod{d},\\ 0, & \text{otherwise.}\end{cases} \pmod{\Phi_d(q)}.$$

Hence, writing $j=\gamma d+\frac{d-1}{2}$ and applying the q-Lucas theorem, we obtain

$$\begin{aligned}&\sum_{j=0}^{n-1}[j+1]_{q^2}\begin{bmatrix}2j\\ j\end{bmatrix}_{q^2} x^j\sum_{k=j}^{n-1} q^{2k}\begin{bmatrix}k+1\\ j+1\end{bmatrix}_{q^2}\begin{bmatrix}k\\ j\end{bmatrix}_{q^2}\\ &\equiv \sum_{\gamma=0}^{\frac{n}{d}-1}\sum_{\alpha=0}^{\frac{n}{d}-1}\binom{2\gamma}{\gamma}\binom{\alpha}{\gamma}^2 x^{\gamma d+\frac{d-1}{2}} q^{\frac{d-5}{2}}\begin{bmatrix}\frac{d+1}{2}\end{bmatrix}_{q^2}\begin{bmatrix}d-1\\ \frac{d-1}{2}\end{bmatrix}_{q^2}\begin{bmatrix}d-2\\ \frac{d-1}{2}\end{bmatrix}_{q^2}\\ &\equiv \sum_{\gamma=0}^{\frac{n}{d}-1}\sum_{\alpha=0}^{\frac{n}{d}-1}\binom{\alpha}{\gamma}^2\frac{x^{\gamma d+\frac{d-1}{2}}}{q(1+q)^2} \pmod{\Phi_d(q)},\end{aligned}$$

where we have used the congruence

$$\begin{bmatrix}d-1\\ k\end{bmatrix}_{q^2} \equiv (-1)^k q^{-k(k+1)} \pmod{\Phi_d(q)} \quad \text{for } 0\leqslant k\leqslant d-1.$$

On the other hand, we have

$$\sum_{k=0}^{n-1} q^{2k} g_k(x;q^2) = \sum_{j=0}^{n-1} {2j \brack j}_{q^2} x^j \sum_{k=j}^{n-1} q^{2k} {k \brack j}_{q^2}^2.$$

Similarly as before, if $j=\gamma d+\delta$ and $\delta \leqslant \frac{d-1}{2}$, then

$$\begin{aligned}
\sum_{k=j}^{n-1} q^{2k} {k \brack j}_{q^2}^2 &\equiv \sum_{\alpha=0}^{\frac{n}{d}-1} \binom{\alpha}{\gamma}^2 \sum_{\beta=0}^{d-1} q^{2\beta} {\beta \brack \delta}_{q^2}^2 \\
&= \sum_{\alpha=0}^{\frac{n}{d}-1} \binom{\alpha}{\gamma}^2 \sum_{r=0}^{d-\delta-1} q^{2\delta+4r+4r\delta+2r^2} {-\delta-1 \brack r}_{q^2}^2 \\
&\equiv \sum_{\alpha=0}^{\frac{n}{d}-1} \binom{\alpha}{\gamma}^2 q^{-2\delta-2\delta^2-2} {2d-2\delta-2 \brack d-1-\delta}_{q^2}^2 \pmod{\Phi_d(q)}.
\end{aligned} \tag{5.6}$$

It is obvious that the right-hand side of (5.6) divisible by $\Phi_d(q)$ for $\delta \leqslant \frac{d-3}{2}$, which means that

$$\begin{aligned}
\sum_{k=0}^{n-1} q^{2k} g_k(x;q^2) &\equiv \sum_{\gamma=0}^{\frac{n}{d}-1} \sum_{\alpha=0}^{\frac{n}{d}-1} \binom{2\gamma}{\gamma} \binom{\alpha}{\gamma}^2 x^{\gamma d+\frac{d-1}{2}} q^{\frac{d-3}{2}} {d-1 \brack \frac{d-1}{2}}_{q^2}^2 \\
&\equiv \sum_{\gamma=0}^{\frac{n}{d}-1} \sum_{\alpha=0}^{\frac{n}{d}-1} \binom{2\gamma}{\gamma} \binom{\alpha}{\gamma}^2 \frac{x^{\gamma d+\frac{d-1}{2}}}{q} \pmod{\Phi_d(q)}.
\end{aligned}$$

This proves (5.1).

Now assume that d is an even divisor of n. Similarly to (5.6), we have

$$\sum_{k=j}^{n-1} q^k {k \brack j}_q^2 \equiv 0 \pmod{\Phi_d(q)} \quad \text{for } j=\gamma d+\delta \text{ and } 0 \leqslant \delta \leqslant \frac{d}{2}-1.$$

On the other hand, if $j=\gamma d+\delta$ with $\frac{d}{2} \leqslant \delta \leqslant d-1$, then by the q-Lucas theorem, we obtain

$$ {2j \brack j}_q = {2\gamma d+2\delta \brack \gamma d+\delta}_q \equiv 0 \pmod{\Phi_d(q)}.$$

This proves (5.2).

Suppose that $d>2$ is even and $d \mid n$. Similarly to (5.4) and (5.5), we get

$$\sum_{k=j}^{n-1} q^k {k \brack j}_q {k+1 \brack j+1}_q \equiv \sum_{\alpha=0}^{\frac{n}{d}-1} \binom{\alpha}{\gamma}^2 q^{-2\delta-\delta^2-2} {2d-2\delta-3 \brack d-1-\delta}_q \equiv 0 \pmod{\Phi_d(q)}$$

for $j=\gamma d+\delta$ and $0\leqslant\delta\leqslant\frac{d}{2}-2$. On the other hand, if $j=\gamma d+\delta$ with $\frac{d}{2}\leqslant\delta\leqslant d-1$, then

$$\begin{bmatrix}2j\\ j\end{bmatrix}_q\equiv 0\pmod{\Phi_d(q)}.$$

while if $j=\gamma d+\frac{d}{2}-1$, then

$$[j+1]_{q^2}\equiv\frac{1-q^d}{1-q^2}\equiv 0\pmod{\Phi_d(q)}.$$

This proves (5.3). □

Recall that for $d>1$, we have

$$\Phi_d(1)=\begin{cases}p, & \text{if } d=p^\alpha \text{ is a prime power,}\\ 1 & \text{otherwise.}\end{cases}$$

Write $n=2^r n_1$, where n_1 is an odd integer. Then

$$\prod_{\substack{d\mid n\\ d>1\text{ is odd}}}\Phi_d(1)=n_1,\quad\text{and}\quad\prod_{\substack{d\mid n\\ d\text{ is even}}}\Phi_d(1)=2^r.$$

Letting $q=1$ in (5.1)–(5.3), we immediately get

$$\sum_{k=0}^{n-1}(4k+3)g_k(x)\equiv 0\pmod{n_1},\tag{5.7}$$

and

$$2\sum_{k=0}^{n-1}kg_k(x)\equiv\sum_{k=0}^{n-1}g_k(x)\equiv 0\pmod{2^r}.\tag{5.8}$$

It is clear that (1.1) follows from (5.7) and (5.8). Therefore, Theorem 5.1 may be deemed a q-analogue of (1.1).

6 An open problem

Numerical calculation suggests the following conjecture on congruences involving S_n.

Conjecture 6.1 *Let n be a positive integer and p a prime. Then*

$$\sum_{k=1}^{n}(-1)^k\frac{S_{k+2}+12S_{k+1}+16S_k}{k}\equiv 0\pmod{n},\tag{6.1}$$

$$\sum_{k=1}^{p}(-1)^k\frac{S_{k+2}+12S_{k+1}+16S_k}{k}\equiv 2p(-1)^{\frac{p+1}{2}}\pmod{p^2}.$$

By (4.6)–(4.7), it is easy to see that if the $n = p$ case of (6.1) is true, then we have

$$\sum_{k=0}^{p-1}(8k^2 + 12k + 5)g_k(-1) \equiv 0 \pmod{p^2}.$$

which is a special case of (1.3) and (1.4) conjectured by Z.-W. Sun.

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