Proof of a conjecture involving Sun polynomials

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Abstract. The Sun polynomials $g_n(x)$ are defined by

$$g_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} x^k.$$ 

We prove that, for any positive integer $n$, there hold

$$\frac{1}{n} \sum_{k=0}^{n-1} (4k + 3)g_k(x) \in \mathbb{Z}[x], \quad \text{and} \quad \sum_{k=0}^{n-1} (8k^2 + 12k + 5)g_k(-1) \equiv 0 \pmod{n}.$$ 

The first one confirms a recent conjecture of Z.-W. Sun, while the second one partially answers another conjecture of Z.-W. Sun. We give three different proofs of the former. One of them depends on the following congruence:

$$\binom{m+n-2}{m-1} \binom{n}{m} \binom{2n}{n} \equiv 0 \pmod{m+n} \quad \text{for } m, n \geq 1.$$ 

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1 Introduction

Recently, Z.-W. Sun [20] introduced the polynomials

$$g_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} x^k,$$
which we call *Sun polynomials* here, and proved many interesting identities and congruences involving \( g_n(x) \), such as

\[
\frac{1}{3n^2} \sum_{k=0}^{n-1} (4k + 3) g_k = \sum_{k=0}^{n-1} \frac{1}{k+1} \binom{2k}{k} \left( \frac{n-1}{k} \right)^2,
\]

\[
\sum_{k=0}^{p-1} k g_k \equiv -\frac{3}{4} \pmod{p^2},
\]

where \( p \) is an odd prime and \( g_k = g_k(1) \). Z.-W. Sun [19] also conjectured that

\[
\sum_{k=0}^{\infty} \frac{16k + 5}{324^k} \binom{2k}{k} g_k(-20) = \frac{189}{25\pi}.
\]

Some other congruences involving \( g_n \) can be found in [10,18,19].

The Sun polynomials also satisfy the following identities [20, (2.7),(2.11)]:

\[
\sum_{k=0}^{n} \binom{n}{k} f_k(x) = g_n(x),
\]

\[
\sum_{k=0}^{n} \binom{n}{k} \left( \binom{n+k}{k}(-1)^{n-k} g_k(x) = A_n(x),
\]

where \( f_n(x) \) and \( A_n(x) \) are respectively the Franel polynomials and Apéry polynomials [17] defined as

\[
f_n(x) = \sum_{k=0}^{n} \binom{n}{k} \bigg( \frac{2k}{n} \bigg)^x k,
\]

\[
A_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 \left( \binom{n+k}{k}^2 \right)^x.
\]

The objective of this paper is to prove the following result, which was originally conjectured by Z.-W. Sun (see [20, Conjecture 4.1(ii)]).

**Theorem 1.1** Let \( n \) be a positive integer. Then

\[
\frac{1}{n} \sum_{k=0}^{n-1} (4k + 3) g_k(x) \in \mathbb{Z}[x],
\]

\[
\sum_{k=0}^{n-1} (8k^2 + 12k + 5) g_k(-1) \equiv 0 \pmod{n},
\]

\[
\sum_{k=0}^{n-1} (8k^2 + 12k + 5) g_k(-1) \equiv 0 \pmod{n}.
\]
Remark. For the congruence (1.2), Z.-W. Sun [20] made the following stronger conjecture:

\[ \sum_{k=0}^{n-1} (8k^2 + 12k + 5)g_k(-1) \equiv n^2 \pmod{2n^2}, \]  

(1.3)

\[ \sum_{k=0}^{p-1} (8k^2 + 12k + 5)g_k(-1) \equiv 3p^2 \pmod{p^3}, \]  

(1.4)

where \( p \) is a prime.

In order to prove Theorem 1.1, we need to establish some preliminary results in Section 3. However, since the following result is interesting in its own right, we label it as a theorem here.

**Theorem 1.2** Let \( m \) and \( n \) be positive integers. Then

\[ \binom{m+n-2}{m-1} \binom{n}{m} \binom{2n}{n} \equiv 0 \pmod{m+n}. \]  

(1.5)

It is worth mentioning that Gessel [5, Section 7] proved a similar result as follows:

\[ \frac{m}{2} \binom{2m}{m} \binom{2n}{n} \equiv 0 \pmod{m+n}, \]

of which a generalization was given by the author [6, Theorem 1.4].

The paper is organized as follows. Applying the same techniques in [6, 7], we shall prove a \( q \)-analogue of Theorem 1.2 in the next section. In Section 3, we give three lemmas, one of which is closely related to Theorem 1.2. Two proofs of (1.1) and a proof of (1.2) will be given in Section 4. The second proof of (1.1) is motivated by Sun [20, Lemma 3.4] and its proof. We shall also give a \( q \)-analogue (the third proof) of (1.1) in Section 5. We end the paper in Section 6 with a related conjecture.

## 2 A \( q \)-analogue of Theorem 1.2

Recall that the \( q \)-binomial coefficients are defined by

\[ \left[ \begin{array}{c} n \\ k \end{array} \right]_q = \begin{cases} \prod_{i=1}^{k} \frac{1-q^{n-k+i}}{1-q^i}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases} \]

We now state the announced strengthening of Theorem 1.2.

**Theorem 2.1** Let \( m \) and \( n \) be positive integers. Then

\[ \frac{1-q}{1-q^{m+n}} \left[ \begin{array}{c} m+n-2 \\ m-1 \end{array} \right]_q \left[ \begin{array}{c} n \\ m \end{array} \right]_q \left[ \begin{array}{c} 2n \\ n \end{array} \right]_q \]

is a polynomial in \( q \) with non-negative integer coefficients.
It is clear that Theorem 1.2 can be deduced from Theorem 2.1 by letting $q \to 1$.

A polynomial $A(q) = \sum_{i=0}^d a_i q^i$ in $q$ of degree $d$ is called reciprocal if $a_i = a_{d-i}$ for all $i$, and that it is called unimodal if there is an index $r$ such that $0 \leq a_0 \leq \cdots \leq a_r \geq \cdots \geq a_d \geq 0$. The following is an elementary but crucial property of reciprocal and unimodal polynomials (see, for example, [1] or [15, Proposition 1]).

**Lemma 2.2** If $A(q)$ and $B(q)$ are reciprocal and unimodal polynomials, then so is their product $A(q)B(q)$.

Similarly to the proof of [7, Theorem 3.1], we also need the following result. We refer the reader to [13, Proposition 10.1.(iii)] and [3, Proof of Theorem 2] for similar mathematical ideas.

**Lemma 2.3** [7, Lemma 5.1] Let $P(q)$ be a reciprocal and unimodal polynomial and $m$ and $n$ positive integers with $m \leq n$. Furthermore, assume that $\frac{1-q^n}{1-q^m} P(q)$ is a polynomial in $q$. Then $\frac{1-q^n}{1-q^m} P(q)$ has non-negative coefficients.

**Proof of Theorem 2.1.** It is well known that the $q$-binomial coefficients are reciprocal and unimodal polynomials (see, for example, [16, Ex. 7.75.d]). By Lemma 2.2, so is the product of three $q$-binomial coefficients. In view of Lemma 2.3, to prove Theorem 2.1, it suffices to show that the expression (2.1) is a polynomial in $q$. We shall accomplish this by considering a count of cyclotomic polynomials.

Recall that

$$q^n - 1 = \prod_{d|n} \Phi_d(q),$$

where $\Phi_d(q)$ denotes the $d$-th cyclotomic polynomial in $q$. Therefore,

$$\frac{1-q}{1-q^{m+n}} \binom{m+n-2}{m-1} \binom{n}{m} \binom{2n}{n} = \prod_{d=2}^{2n} \Phi_d(q)^{e_d},$$

with

$$e_d = -\chi(d \mid m+n) + \left\lfloor \frac{m+n-2}{d} \right\rfloor + \left\lfloor \frac{2n}{d} \right\rfloor$$

$$- \left\lfloor \frac{m-1}{d} \right\rfloor - \left\lfloor \frac{n-1}{d} \right\rfloor - \left\lfloor \frac{m}{d} \right\rfloor - \left\lfloor \frac{n}{d} \right\rfloor - \left\lfloor \frac{n-m}{d} \right\rfloor,$$

where $\chi(S) = 1$ if $S$ is true and $\chi(S) = 0$ otherwise. The number $e_d$ is obviously non-negative, unless $d \mid m+n$.

So, let us assume that $d \mid m+n$ and $d \geq 2$. We consider two cases: If $d \mid m$, then $d \mid n$, and so

$$\left\lfloor \frac{m+n-2}{d} \right\rfloor - \left\lfloor \frac{m-1}{d} \right\rfloor - \left\lfloor \frac{n-1}{d} \right\rfloor = \frac{m+n-d}{d} - \frac{m-d}{d} - \frac{n-d}{d} = 1.$$
Namely, \( e_d = 0 \) is non-negative; if \( d \nmid m \), then
\[
\left\lfloor \frac{2n}{d} \right\rfloor - \left\lfloor \frac{m}{d} \right\rfloor - \left\lfloor \frac{n}{d} \right\rfloor - \left\lfloor \frac{n-m}{d} \right\rfloor = \frac{m+n}{d} - \left\lfloor \frac{m}{d} \right\rfloor - \left\lfloor \frac{n}{d} \right\rfloor = 1.
\]
That is, \( e_d = 0 \) is still non-negative. This completes the proof of polynomiality of (2.1).

\[\square\]

3 Some preliminary results

**Lemma 3.1** Let \( n \) be a non-negative integer. Then
\[
\left( \frac{x}{n} \right)^2 = \sum_{k=0}^{n} \left( \frac{x}{n+k} \right) \left( \frac{n+k}{k} \right) \left( \frac{n}{k} \right).
\]

*Proof.* Applying Chu-Vandermonde’s identity (see, for example, [9, p. 32])
\[
\left( \frac{x}{n} \right) = \sum_{k=0}^{n} \left( \frac{x-n}{k} \right) \left( \frac{n}{k} \right),
\]
and noticing that \( \left( \frac{x}{n} \right) \left( \frac{x-n}{k} \right) = \left( \frac{x}{n+k} \right) \left( \frac{n+k}{k} \right) \), we obtain (3.1). In fact, Eq. (3.1) is a special case of [14, p. 15, Eq. (9)]. \[\square\]

**Lemma 3.2** Let \( n \) be a positive integer and let \( 0 \leq k \leq n \). Then
\[
\sum_{m=k}^{n-1} (4m + 3) \binom{m}{k} = (4n-1) \binom{n}{k+1} - 4 \binom{n}{k+2},
\]
\[
\sum_{m=k}^{n-1} (8m^2 + 12m + 5) \binom{m}{k} = (8n^2 - 4n + 1) \binom{n}{k+1} - (16n - 12) \binom{n}{k+2} + 16 \binom{n}{k+3}.
\]

*Proof.* Proceed by induction on \( n \). \[\square\]

**Lemma 3.3** Let \( m \) and \( n \) be non-negative integers. Then
\[
\binom{m+n}{m} \binom{n+1}{m} \binom{2n}{n} \frac{3m^2 + n^2 + m + n}{(m+n)(n+1)} \equiv 0 \pmod{m+n+1}.
\]

*Proof.* Note that
\[
\frac{3m^2 + n^2 + m + n}{(m+n)(n+1)} = \frac{3m(m+n+1)}{(m+n)(n+1)} + \frac{n(m+n+1)}{(m+n)(n+1)} - \frac{2m(2n+1)}{(m+n)(n+1)}.
\]
Since \( \frac{1}{n+1} \binom{2n}{n+1} = \binom{2n}{n} - \binom{2n}{n-1} \) is an integer (the \( n \)-th Catalan number), we see that
\[
\binom{m+n}{m} \binom{2n}{n} \frac{m}{(m+n)(n+1)} = \binom{m+n-1}{m-1} \binom{2n}{n} \frac{1}{n+1},
\]
an\dand
\[
\binom{m+n}{m} \binom{2n}{n} \frac{n}{(m+n)(n+1)} = \binom{m+n-1}{m} \binom{2n}{n} \frac{1}{n+1},
\]
are both integers. It remains to show that
\[
\binom{m+n}{m} \binom{n+1}{m} \binom{2n}{n} \frac{2m(2n+1)}{(m+n)(n+1)} \equiv 0 \pmod{m+n+1},
\]
i.e.,
\[
\binom{m+n-1}{m-1} \binom{n+1}{m} \binom{2n+2}{n+1} \equiv 0 \pmod{m+n+1}.
\]
But this is just the \( n \to n+1 \) case of the congruence (1.5).

\[\square\]

**Lemma 3.4** Let
\[
S_n = f_{n-3}(-1) = \sum_{k=0}^{n-3} (-1)^k \binom{2k}{k} \binom{n-3}{k} \binom{k}{n-k-3}.
\]

Then there hold the following congruences:
\[
S_{3n} \equiv S_{3n+1} \equiv -S_{3n+2} \pmod{3}, \quad (3.2)
\]
\[
S_{4n+2} \equiv 0 \pmod{4}, \quad (3.3)
\]
\[
S_{n+2} + 12S_{n+1} + 16S_n \equiv 0 \pmod{n}. \quad (3.4)
\]

**Proof.** Zeilberger’s algorithm [9, 12] gives the following recurrence relation for \( S_n \):
\[
(5n^3 - 8n^2)S_{n+3} + (45n^3 - 117n^2 + 90n - 24)S_{n+2} + (200n^3 - 720n^2 + 824n - 288)S_{n+1} + (160n^3 - 736n^2 + 1024n - 384)S_n = 0. \quad (3.5)
\]
Replacing \( n \) by \( 3n - 1 \) in (3.5), we obtain
\[
-S_{3n+2} - S_{3n} \equiv 0 \pmod{3},
\]
while replacing \( n \) by \( 3n + 1 \) in (3.5), we get
\[
S_{3n+2} + S_{3n+1} \equiv 0 \pmod{3}.
\]
This proves (3.2). Similarly, replacing \( n \) by \( 4n - 1 \) in (3.5), we are led to (3.3).

In order to prove (3.4), we need to consider four cases:
• If \( \gcd(n, 24) = 1 \), then (3.5) means that 
\[-24(S_{n+2} + 12S_{n+1} + 16S_n) \equiv 0 \pmod{n}, \]
i.e., the congruence (3.4) holds.

• If \( \gcd(n, 24) = 2, 4, 8 \), then (3.5) means that 
\[\pm 2nS_{n+2} - 24(S_{n+2} + 12S_{n+1} + 16S_n) \equiv 0 \pmod{8n}. \]
By (3.3), we have \( 2nS_{n+2} \equiv 0 \pmod{8n} \) in this case, and so the congruence (3.4) holds.

• If \( \gcd(n, 24) = 3 \), then (3.5) means that 
\[2nS_{n+1} + nS_n - 24(S_{n+2} + 12S_{n+1} + 16S_n) \equiv 0 \pmod{3n}. \]
By (3.2), we have \( 2nS_{n+1} + nS_n \equiv 0 \pmod{3n} \) in this case, and so the congruence (3.4) holds.

• If \( \gcd(n, 24) = 6, 12, 24 \), then (3.5) means that 
\[30nS_{n+2} + 8nS_{n+1} + 16nS_n - 24(S_{n+2} + 12S_{n+1} + 16S_n) \equiv 0 \pmod{24n}, \]
or
\[18nS_{n+2} + 8nS_{n+1} + 16nS_n - 24(S_{n+2} + 12S_{n+1} + 16S_n) \equiv 0 \pmod{24n}. \]
By (3.3), we have \( 30nS_{n+2} \equiv 18nS_{n+2} \equiv 0 \pmod{24n} \) and \( 8nS_{n+1} + 16nS_n \equiv 0 \pmod{24n} \) in this case, and so the congruence (3.4) still holds.

\[\square\]

### 4 Proof of Theorem 1.1

**First Proof of (1.1).** By Lemmas 3.1 and 3.2, we have

\[
\sum_{m=0}^{n-1} (4m + 3) g_m(x) \\
= \sum_{m=0}^{n-1} (4m + 3) \sum_{k=0}^{m} \binom{m}{k}^2 \binom{2k}{k} x^k \\
= \sum_{m=0}^{n-1} (4m + 3) \sum_{k=0}^{m} \binom{2k}{k} x^k \sum_{i=0}^{k} \binom{m}{k+i} \binom{k+i}{i} \binom{k}{i} \\
= \sum_{k=0}^{n-1} \binom{2k}{k} x^k \sum_{i=0}^{k} \binom{k+i}{i} \binom{k}{i} \sum_{m=k+i}^{n-1} (4m + 3) \binom{m}{k+i} \\
= \sum_{k=0}^{n-1} \binom{2k}{k} x^k \sum_{i=0}^{k} \binom{k+i}{i} \binom{k}{i} \binom{n}{k+i+1} - 4 \binom{n}{k+i+2} \binom{2k}{k+i} \binom{k}{i}. \quad (4.1)
\]
For any non-negative integer \( k \leq n - 1 \), to prove that the coefficient of \( x^k \) in the right-hand side of (4.1) is a multiple of \( n \), it suffices to show that
\[
\binom{2k}{k} \sum_{i=0}^{k} \left( \binom{n}{k+i+1} + 4 \binom{n}{k+i+2} \right) \binom{k+i}{i} \binom{k}{i} \equiv 0 \pmod{n}. \tag{4.2}
\]

We shall accomplish the proof of (4.2) by using a minor trick. Rewrite the left-hand side of (4.2) as
\[
\binom{2k}{k} \sum_{i=0}^{k+1} \binom{n}{k+i+1} \left( \binom{k+i}{i} \binom{k}{i} \right) + 4 \binom{k+i-1}{i-1} \binom{k}{i-1}
= \sum_{i=0}^{k+1} \binom{n}{k+i+1} \binom{k+i}{i} \binom{k+1}{i} \frac{2k^2 + 3i^2 + k + i}{(k+i)(k+1)}.
\]
Then, by Lemma 3.3, for each \( i \leq k+1 \), the expression
\[
\binom{k+i}{i} \binom{k+1}{i} \frac{2k^2 + 3i^2 + k + i}{(k+i)(k+1)}
\]
is a multiple of \( k+i+1 \). Finally, noticing that
\[
\binom{n}{k+i+1}(k+i+1) = n \binom{n-1}{k+i} \equiv 0 \pmod{n},
\]
we complete the proof.

**Second Proof of (1.1).** This proof is motivated by [20, Lemma 3.4 and its proof]. It is clear that (1.1) is equivalent to the following congruence:
\[
\binom{2j}{j} \sum_{k=j}^{n-1} (4k+3) \binom{k}{j}^2 \equiv 0 \pmod{n}. \tag{4.3}
\]
Denote the left-hand side of (4.3) by \( u_j \). Then by Zeilberger’s algorithm [12], we have
\[
u_{j+1} - u_j = -\binom{2j}{j} \binom{n-1}{j}^2 \frac{(9j+6)(j+1)n^2 + (12j^2 - 8jn - 4n + 14j + 4)n^3}{(j+1)^3(j+2)}
- \binom{2j}{j} \binom{n+1}{j} \frac{9(j+6)n}{(j+1)(n+1)}
- \binom{2j}{j} \binom{n}{j+1} \frac{12j^2 - 8jn - 4n + 14j + 4}{(j+1)(n+1)}.
\tag{4.4}
\]
Noticing that \( \frac{1}{j+1} \binom{2j}{j} \) is an integer and \( n+1 \) is relatively prime to \( n \), from (4.4) we immediately get
\[
u_{j+1} - u_j \equiv 0 \pmod{n}.
\]
Since \( u_0 = 2n^2 + n \equiv 0 \pmod{n} \), we conclude that \( u_j \equiv 0 \pmod{n} \) for all \( j \). This proves (4.3).

**Proof of (1.2).** By Lemmas 3.1 and 3.2, similarly to (4.1), we have

\[
\sum_{m=0}^{n-1} (8m^2 + 12m + 5) g_m(-1) \\
= \sum_{k=0}^{n-1} \binom{2k}{k} (-1)^k \sum_{i=0}^k \left( (8n^2 - 4n + 1) \binom{n}{k+i+1} - (16n - 12) \binom{n}{k+i+2} + 16 \binom{n}{k+i+3} \right) \left( \frac{k+i}{i} \right) \left( \frac{k}{i} \right).
\]

(4.5)

In view of (4.2), it follows from (4.5) that

\[
\sum_{m=0}^{n-1} (8m^2 + 12m + 5) g_m(-1) \\
= \sum_{k=0}^{n-1} \binom{2k}{k} (-1)^k \sum_{i=0}^k \left( \binom{n}{k+i+1} + 12 \binom{n}{k+i+2} + 16 \binom{n}{k+i+3} \right) \times \left( \frac{k+i}{i} \right) \left( \frac{k}{i} \right) \\
= \sum_{m=0}^{n-1} \left( \binom{n}{m+1} + 12 \binom{n}{m+2} + 16 \binom{n}{m+3} \right) \\
\times \sum_{k=0}^{m} (-1)^k \binom{2k}{k} \binom{m}{k} \binom{k}{m-k} \pmod{2n^2}.
\]

(4.6)

Note that the right-hand side of (4.6) may be written as

\[
\sum_{m=0}^{n-1} \left( \binom{n}{m+1} + 12 \binom{n}{m+2} + 16 \binom{n}{m+3} \right) S_{m+3} \\
= \sum_{m=1}^{n} \binom{n}{m} (S_{m+2} + 12S_{m+1} + 16S_m),
\]

(4.7)

which is clearly congruent to 0 modulo \( n \) by (3.4) and the fact that \( m \binom{n}{m} = \binom{n-1}{m-1} \). \( \square \)

**5 A \( q \)-analogue of (1.1)**

Define the \( q \)-analogue of Sun polynomials as follows:

\[
g_n(x; q) = \sum_{k=0}^{n} \left[ \binom{n}{k} q \frac{[2k]}{q} \right] x^k.
\]
We have the following congruences related to $g_n(x; q)$.

**Theorem 5.1** Let $n$ be a positive integer. Then

\[
(1 + q)^2 \sum_{k=0}^{n-1} q^{2k} [k + 1]_q^2 g_k(x; q^2) \equiv \sum_{k=0}^{n-1} q^{2k} g_k(x; q^2) \pmod{\prod_{d|n} \Phi_d(q))}, \quad (5.1)
\]

\[
\sum_{k=0}^{n-1} q^k g_k(x; q) \equiv 0 \pmod{\prod_{d|n} \Phi_d(q))}, \quad (5.2)
\]

\[
\sum_{j=0}^{n-1} x^j [j + 1]_q^2 \sum_{k=j}^{n-1} q^k \left[ \begin{array}{c} k \\ j \end{array} \right]_q \left[ \begin{array}{c} k + 1 \\ j + 1 \end{array} \right]_q \equiv 0 \pmod{\prod_{d|n} \Phi_d(q)), \quad (5.3)
\]

where $[n]_q = \frac{1-q^n}{1-q}$ denotes a $q$-integer.

**Proof.** It is clear that

\[
\sum_{k=0}^{n-1} q^{2k} [k + 1]_q^2 g_k(x; q^2) = \sum_{k=0}^{n-1} q^{2k} [k + 1]_q^2 \sum_{j=0}^{k} \left[ \begin{array}{c} k \\ j \end{array} \right]_q^2 \left[ \begin{array}{c} 2j \\ j \end{array} \right]_q^2 x^j
\]

\[
= \sum_{j=0}^{n-1} [j + 1]_q^2 \left[ \begin{array}{c} 2j \\ j \end{array} \right]_q^2 x^j \sum_{k=j}^{n-1} q^k \left[ \begin{array}{c} k + 1 \\ j + 1 \end{array} \right]_q^2 \left[ \begin{array}{c} k \\ j \end{array} \right]_q^2.
\]

Suppose that $d | n$ and $d$ is odd. It is easy to see that $\Phi_d(q)$ divides $\Phi_d(q^2)$. Write $j = \gamma d + \delta$, where $0 \leq \delta \leq d - 1$. If $d \leq 2\delta$, then by the $q$-Lucas theorem (see Olive [11], Désarménien [4, Proposition 2.2] or Guo and Zeng [8, Proposition 2.1]),

\[
\left[ \begin{array}{c} 2j \\ j \end{array} \right]_q^2 \equiv \left( \frac{2\gamma + 1}{\gamma} \right) \left[ \begin{array}{c} 2\delta - d \\ \delta \end{array} \right]_q^2 = 0 \pmod{\Phi_d(q))}.
\]

Now assume that $\delta \leq \frac{d-1}{2}$. Then applying the $q$-Lucas theorem, we have

\[
\sum_{k=j}^{n-1} q^k \left[ \begin{array}{c} k + 1 \\ j + 1 \end{array} \right]_q^2 \left[ \begin{array}{c} k \\ j \end{array} \right]_q^2 = \sum_{\alpha=0}^{n-1} \sum_{\beta=0}^{d-1} q^{2(\alpha d + \beta)} \left[ \begin{array}{c} \alpha d + \beta + 1 \\ \gamma d + \delta + 1 \end{array} \right]_q \left[ \begin{array}{c} \alpha d + \beta \\ \gamma d + \delta \end{array} \right]_q^2
\]

\[
\equiv \sum_{\alpha=0}^{n-1} \left( \frac{\alpha}{\gamma} \right) \sum_{\beta=0}^{d-1} q^{2\beta} \left[ \begin{array}{c} \beta + 1 \\ \delta + 1 \end{array} \right]_q \left[ \begin{array}{c} \beta \\ \delta \end{array} \right]_q^2 \pmod{\Phi_d(q))}.
\]

(5.4)
It is easy to see that
\[
\sum_{\beta=0}^{d-1} q^{2\beta} \beta \binom{\beta+1}{\delta+1} \binom{\beta}{\delta} = \sum_{r=0}^{d-1-\delta} q^{2r+2\delta} \binom{r+\delta+1}{\delta+1} \binom{r+\delta}{\delta} \mod \Phi_d(q),
\]
where we have used the \(q\)-Chu-Vandemonde identity (see \([2, (3.3.10)]\)) in the last step. Furthermore, we have
\[
\sum_{j=0}^{n-1} q^{2j} \binom{k+1}{j+1} \binom{k}{j} \mod \Phi_d(q),
\]
for \(\delta \leq \frac{d-3}{2}\). This proves that
\[
\sum_{j=0}^{n-1} q^{2j} \binom{k+1}{j+1} \binom{k}{j} \equiv \begin{cases} \sum_{\alpha=0}^{\frac{d-1}{2}} \left(\frac{\alpha}{\gamma}\right)^2 q^{\frac{d-1}{2}} \binom{d-2}{\frac{d-1}{2}} q^2, & \text{if } j \equiv \frac{d-1}{2} \pmod{d}, \\ 0, & \text{otherwise.} \end{cases}
\]
Hence, writing \(j = \gamma d + \frac{d-1}{2}\) and applying the \(q\)-Lucas theorem, we obtain
\[
\sum_{j=0}^{n-1} q^{2j} \binom{k+1}{j+1} \binom{k}{j} \equiv \sum_{\gamma=0}^{\frac{d-1}{2}} \sum_{\alpha=0}^{\frac{d-1}{2}} \left(\frac{\alpha}{\gamma}\right)^2 x^{\gamma d + \frac{d-1}{2}} q^{\frac{d-5}{2}} \binom{d+1}{\frac{d-1}{2}} q^2 \binom{d-1}{\frac{d-1}{2}} q^{\frac{d-2}{2}},
\]
where we have used the congruence
\[
\binom{d-1}{k} q^k \equiv (-1)^k q^{-k(k+1)} \mod \Phi_d(q) \text{ for } 0 \leq k \leq d-1.
\]
On the other hand, we have

\[
\sum_{k=0}^{n-1} q^{2k} g_k(x; q^2) = \sum_{j=0}^{2j} x^{j \sum_{k=j}^{n-1} q^{2k} \left[ \frac{k}{j} \right]^2}.
\]

Similarly as before, if \( j = \gamma d + \delta \) and \( \delta \leq \frac{d-1}{2} \), then

\[
\sum_{k=j}^{n-1} q^{2k} \left[ \frac{k}{j} \right]^2 \equiv \sum_{\alpha=0}^{\frac{n}{2}-1} \sum_{\beta=0}^{\frac{n}{2}-1} \left( \frac{\alpha}{\gamma} \right)^2 q^{2\beta+4r+4r\delta+2r^2} \left[ -\delta - 1 \right]^2 \left( \frac{d}{d-1} \right)^2 q^2 \equiv \sum_{\alpha=0}^{\frac{n}{2}-1} \left( \frac{\alpha}{\gamma} \right)^2 q^{-2d-2\delta^2-2} \left[ 2d - 2\delta - 2 \right]^2 \left( \frac{d-1}{d-1-\delta} \right)^2 \left( \mod \Phi_d(q) \right).
\]

It is obvious that the right-hand side of (5.6) divisible by \( \Phi_d(q) \) for \( \delta \leq \frac{d-3}{2} \), which means that

\[
\sum_{k=0}^{n-1} q^{2k} g_k(x; q^2) \equiv \sum_{\alpha=0}^{\frac{n}{2}-1} \sum_{\gamma=0}^{\frac{n}{2}-1} \left( 2\gamma \right) \left( \frac{\alpha}{\gamma} \right)^2 x^{\gamma d + \frac{d-1}{2} \frac{d-3}{2}} \left( \frac{d-1}{d-1-\delta} \right)^2 q^2 \equiv \sum_{\alpha=0}^{\frac{n}{2}-1} \sum_{\gamma=0}^{\frac{n}{2}-1} \left( 2\gamma \right) \left( \frac{\alpha}{\gamma} \right)^2 x^{\gamma d + \frac{d-1}{2}} \left( \mod \Phi_d(q) \right).
\]

This proves (5.1).

Now assume that \( d \) is an even divisor of \( n \). Similarly to (5.6), we have

\[
\sum_{k=j}^{n-1} q^k \left[ \frac{k}{j} \right]^2 q \equiv 0 \left( \mod \Phi_d(q) \right) \text{ for } j = \gamma d + \delta \text{ and } 0 \leq \delta \leq \frac{d}{2} - 1.
\]

On the other hand, if \( j = \gamma d + \delta \) with \( \frac{d}{2} \leq \delta \leq d-1 \), then by the \( q \)-Lucas theorem, we obtain

\[
\left[ \frac{2j}{j} \right]_q = \left[ 2\gamma d + 2\delta \right] \prod_j \left( \mod \Phi_d(q) \right).
\]

This proves (5.2).

Suppose that \( d > 2 \) is even and \( d \mid n \). Similarly to (5.4) and (5.5), we get

\[
\sum_{k=j}^{n-1} q^k \left[ \frac{k}{j} \right] \left[ \frac{k+1}{j+1} \right] q \equiv \sum_{\alpha=0}^{\frac{n}{2}-1} \left( \frac{\alpha}{\gamma} \right)^2 q^{-2\delta-\delta^2-2} \left[ 2d - 2\delta - 3 \right] \left( \frac{d-1}{d-1-\delta} \right)^2 q \equiv 0 \left( \mod \Phi_d(q) \right).
\]

This proves (5.2).
for \( j = \gamma d + \delta \) and \( 0 \leq \delta \leq \frac{d}{2} - 2 \). On the other hand, if \( j = \gamma d + \delta \) with \( \frac{d}{2} \leq \delta \leq d - 1 \), then
\[
\left[ \frac{2j}{j} \right]_q \equiv 0 \pmod{\Phi_d(q)}.
\]
while if \( j = \gamma d + \frac{d}{2} - 1 \), then
\[
[j + 1]_q^2 \equiv \frac{1 - q^d}{1 - q} \equiv 0 \pmod{\Phi_d(q)}.
\]
This proves (5.3).

Recall that for \( d > 1 \), we have
\[
\Phi_d(1) = \begin{cases} 
p, & \text{if } d = p^\alpha \text{ is a prime power,} \\
1 & \text{otherwise.}
\end{cases}
\]
Write \( n = 2^r n_1 \), where \( n_1 \) is an odd integer. Then
\[
\prod_{d \mid n \text{ and } d > 1 \text{ is odd}} \Phi_d(1) = n_1, \quad \text{and} \quad \prod_{d \mid n \text{ and } d \text{ is even}} \Phi_d(1) = 2^r.
\]

Letting \( q = 1 \) in (5.1)–(5.3), we immediately get
\[
\sum_{k=0}^{n-1} (4k + 3)g_k(x) \equiv 0 \pmod{n_1}, \quad (5.7)
\]
and
\[
2 \sum_{k=0}^{n-1} kg_k(x) \equiv \sum_{k=0}^{n-1} g_k(x) \equiv 0 \pmod{2^r}. \quad (5.8)
\]
It is clear that (1.1) follows from (5.7) and (5.8). Therefore, Theorem 5.1 may be deemed a \( q \)-analogue of (1.1).

### 6 An open problem

Numerical calculation suggests the following conjecture on congruences involving \( S_n \).

**Conjecture 6.1** Let \( n \) be a positive integer and \( p \) a prime. Then
\[
\sum_{k=1}^{n} (-1)^k \frac{S_{k+2} + 12S_{k+1} + 16S_k}{k} \equiv 0 \pmod{n}, \quad (6.1)
\]
\[
\sum_{k=1}^{p} (-1)^k \frac{S_{k+2} + 12S_{k+1} + 16S_k}{k} \equiv 2p(-1)^{\frac{p+1}{2}} \pmod{p^2}.
\]
By (4.6)–(4.7), it is easy to see that if the $n = p$ case of (6.1) is true, then we have
\[ \sum_{k=0}^{p-1} (8k^2 + 12k + 5)g_k(-1) \equiv 0 \pmod{p^2}. \]

which is a special case of (1.3) and (1.4) conjectured by Z.-W. Sun.

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References


[20] Z.-W. Sun, Congruences involving \( g_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} x^k \), Ramanujan J., doi:10.1007/s11139-015-9727-3.