A bijective proof of the Shor recurrence

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Abstract. In an approach to the Cayley formula for counting trees, Shor discovered a refined recurrence relation concerning the number of improper edges. Chen and the author gave a bijection for the Shor recurrence based on the combinatorial interpretations of Zeng, answering a question of Shor. In this paper, we present a new bijective proof of the Shor recurrence by applying Shor's formula for counting forests of rooted trees with roots $1, \ldots, r$ and with a given number of improper edges.

Keywords: Cayley's formula; bijection; tree; forest; improper edge.

MR Subject Classifications: 05A15, 05C05

1 Introduction

The famous Cayley's formula for counting trees states that the number of labeled trees on n nodes is n^{n-2}. Various proofs of Cayley's formula are known (see [1, 2, 5, 6, 8–10, 12–15, 17]). Shor [16] presented a new proof of Cayley's formula. His proof is based on a difficult combinatorial identity. Define a function $Q(i,j)$ as

$$Q(1,0)=1, \quad Q(i,-1)=0, \quad i \geqslant 1, \quad Q(1,j)=0, \quad j \geqslant 1,$$

and

$$Q(i,j)=(i-1)Q(i-1,j)+(i+j-2)Q(i-1,j-1), \quad \text{otherwise.}$$

Then

$$\sum_{j=0}^{i-1} Q(i,j)=i^{i-1}. \tag{1.1}$$

The above identity (1.1) is a special case ($k=0$) of the following identity due to Meir [11, p. 259]. Define a function $Q(i,j,k)$ as

$$Q(1,0,k)=1, \quad Q(i,-1,k)=0, \quad i \geqslant 1, \quad Q(1,j,k)=0, \quad j \geqslant 1,$$

and

$$Q(i,j,k)=(i+k-1)Q(i-1,j,k)+(i+j-2)Q(i-1,j-1,k), \quad \text{otherwise.} \tag{1.2}$$

Then

$$\sum_{j=0}^{i-1} Q(i,j,k) = (i+k)^{i-1}. \tag{1.3}$$

Introducing the concept of *improper edges* (defined in the next section), Shor [16] established the following results.

Theorem 1.1 (Shor) *The number of labeled rooted trees on n nodes with k improper edges is $Q(n,k)$.*

Theorem 1.2 (Shor) *The number of forests of r rooted trees on n labeled nodes with k improper edges and with roots $1,\dots,r$ is $rQ(n-r,k,r)$.*

Note that Dumont and Ramamonjisoa [7] used the grammatical method introduced by Chen in [3] to obtain the same combinatorial interpretation in Theorem 1.1. At the end of his paper, Shor [16] mentioned that besides the recurrence (1.2), the function $Q(i,j,k)$ also satisfies the following recurrence

$$Q(i,j,k) = (k-j+1)Q(i-1,j,k+1) + (i+j-2)Q(i-1,j-1,k+1), \tag{1.4}$$

from which one can easily deduce the identity (1.3), and asked for a combinatorial interpretation of this recurrence. A bijective proof of the recurrence (1.4) has been found by Chen and Guo [4] based on the interpretations of Zeng [18]. The aim of this paper is to give a new bijective proof of (1.4) with the help of Theorems 1.1 and 1.2.

2 The interpretation under Shor

We follow most notation in Zeng [18] and Chen and Guo [4]. The sets of rooted trees and rooted trees with root 1 on $[n] := \{1,\dots,n\}$ are denoted by $\mathscr{R}_n$ and $\mathscr{T}_n$, respectively. If $T \in \mathscr{R}_n$ and x is a node of T, then we denote by T_x the subtree rooted at x. We let $\beta(x)$, or $\beta_T(x)$ be the smallest node on T_x. We call a node z of T a descendant of x, if z is a node of T_x, and is denoted by $z \prec x$. If (x,y) is an edge of a tree T and y is a node of T_x, then we say that x is the father of y, and y is a child of x. Suppose that $e=(x,y)$ is an edge of a tree T, and y is a child of x, we say that e is a *proper* edge, if $x < \beta_T(y)$. Otherwise, e is called an *improper* edge. Denote by $\mathscr{R}_{n,k}$ and $\mathscr{T}_{n,k}$ the sets of rooted trees and rooted trees with root 1 on $[n]$ having k improper edges, respectively. Denote by $\mathscr{F}^r_{n,k}$ the set of forests of r rooted trees on $[n]$ having k improper edges with roots $1,\dots,r$. The degree of a node x in a rooted tree T is the number of children of x, and is denoted by $\deg(x)$, or $\deg_T(x)$. In addition, we may put some conditions on the set $\mathscr{F}^r_{n,k}$ (or $\mathscr{T}_{n,k}$) to denote the subset of forests (or trees) that satisfy these conditions. For instance, $\mathscr{F}^r_{n,k}[n \prec 1]$ represents the subset of $\mathscr{F}^r_{n,k}$ subject to the condition $n \prec 1$.

If we write $(k-j+1)Q(i-1,j,k+1)$ as

$$(i+k)Q(i-1,j,k+1) - (i+j-1)Q(i-1,j,k+1),$$

then from (1.2) we see that (1.4) is equivalent to

$$Q(i,j,k) = Q(i,j,k+1) - (i+j-1)Q(i-1,j,k+1).$$

Replacing (i,j,k) by $(n-r,k,r-1)$, we get

$$Q(n-r,k,r-1) = Q(n-r,k,r) - (n+k-r)Q(n-1-r,k,r). \tag{2.1}$$

In order to give a combinatorial interpretation of the recurrence (2.1), we need the following lemmas. From now on, we assume that $r \leqslant n-1$.

Lemma 2.1 *The number of forests of r rooted trees on $[n]$ having k improper edges with roots $1,\ldots,r$ for which n is a descendant of the node 1 is $Q(n-r,k,r)$, i.e., we have*

$$|\mathscr{F}_{n,k}^{r}[n \prec 1]| = Q(n-r,k,r) = \frac{1}{r}|\mathscr{F}_{n,k}^{r}|. \tag{2.2}$$

Proof. The case $r=1$ reduces to Theorem 1.2. For any $2 \leqslant p \leqslant r$, exchanging the labels of nodes 1 and p, we establish a bijection between $\mathscr{F}_{n,k}^{r}[n \prec 1]$ and $\mathscr{F}_{n,k}^{r}[n \prec p]$. The proof then follows from Theorem 1.2. □

Moreover, set $\mathscr{F}_{n,k}^{0}[n \prec 1] = \mathscr{R}_{n,k}$. Then by Theorem 1.1, the first equality of (2.2) also holds for $r=0$.

Lemma 2.2 *For $r \geqslant 1$, we have*

$$|\mathscr{F}_{n,k+1}^{r}[n \prec 1, \deg(n) > 0]| = (n+k-r)Q(n-1-r,k,r).$$

Proof. It follows from the proof of Theorem 1.2 that

$$|\mathscr{F}_{n,k+1}^{r}[\deg(n) > 0]| = (n+k-r)|\mathscr{F}_{n-1,k}^{r}|.$$

Exactly in the same way as for Lemma 2.1, we complete the proof. □

Lemma 2.3 *For $r \geqslant 1$, we have*

$$|\mathscr{F}_{n-1,k}^{r-1}[n-1 \prec 1]| = |\mathscr{F}_{n,k}^{r}[n \prec 1, \deg(r+1) = 0]|.$$

Proof. Suppose $r \geqslant 2$, $F \in \mathscr{F}_{n-1,k}^{r-1}[n-1 \prec 1]$. First, relabel a node i by $i+1$, for any $r \leqslant i \leqslant n-1$. Second, introduce a node r as a new root of F and move all the subtrees of $r+1$ and make them as subtrees of r. Third, if n is not a descendant of 1 in the new forest, then n must be a descendant of r, and in this case we exchange the labels 1 and r. Note that the third step happens if and only if $r+1$ is a descendant of r in the new forest. It is clear that the construction is reversible. Thus we establish the desired bijection (see Figure 1). For $r=1$, the proof is analogous, and is left to the interested reader. □

It follows from Lemmas 2.1–2.3 that the recurrence (2.1) is equivalent to the following identity.

Theorem 2.4 *For $r \geqslant 1$, we have*

$$|\mathscr{F}_{n,k}^{r}[n \prec 1, \deg(r+1) > 0]| = |\mathscr{F}_{n,k+1}^{r}[n \prec 1, \deg(n) > 0]|. \tag{2.3}$$

We will give a bijective proof of (2.3) in the next section.

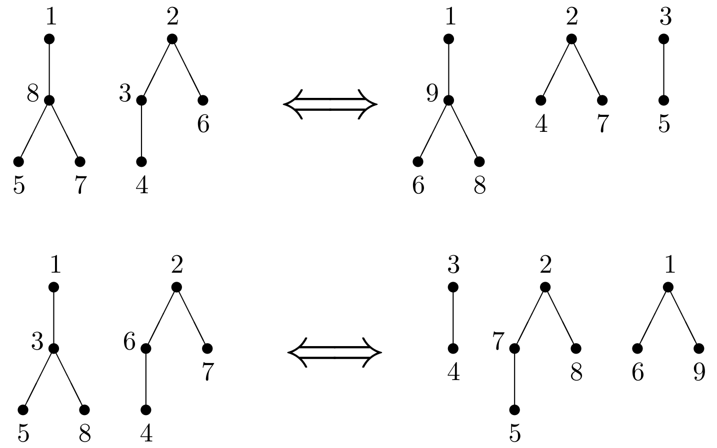

Figure 1: Examples of Lemma 2.3 for $n = 9$ and $r = 3$.

3 The Bijections

The case $r = 1$ of (2.3) reduces to the following statement.

Theorem 3.1 *For $0 \leqslant k \leqslant n-3$, we have the following bijection:*

$$\mathscr{T}_{n,k}[\deg(2) > 0] \longleftrightarrow \mathscr{T}_{n,k+1}[\deg(n) > 0]. \tag{3.1}$$

Note that a proof of a refined version of (3.1) has been given in [4, Theorem 3.8]. Here we shall give a new proof of (3.1).

Lemma 3.2 *For $0 \leqslant k \leqslant n-3$, we have the following bijection:*

$$\mathscr{T}_{n,k}[\deg(2) > 0] \longleftrightarrow \mathscr{T}_{n,k}[\deg(1) > 1]. \tag{3.2}$$

Proof. Suppose that $T \in \mathscr{T}_{n,k}[\deg(2) > 0]$. The set of subtrees of 1 that do not contain the node 2 is denoted by R, and the set of subtrees of 2 is denoted by S. Exchanging R and S, we obtain a tree $T' \in \mathscr{T}_{n,k}[\deg(1) > 1]$. This sets up the desired bijection. ☐

It is clear that (3.1) may be deduced from (3.2) and the following statement.

Theorem 3.3 *For $0 \leqslant k \leqslant n-3$, we have the following bijection:*

$$\mathscr{T}_{n,k}[\deg(1) > 1] \longleftrightarrow \mathscr{T}_{n,k+1}[\deg(n) > 0]. \tag{3.3}$$

Let $\mathscr{T}_{n,k}^{(i)}$ be the set of trees in $\mathscr{T}_{n,k}$ such that there are i proper edges on the path from 1 to n. We need to consider two cases in the construction of a bijection for (3.3).

Theorem 3.4 *For $i \geqslant 2$, we have the following bijection:*

$$\mathscr{T}_{n,k}^{(i)}[\deg(1) > 1] \longleftrightarrow \mathscr{T}_{n,k+1}^{(i-1)}[\deg(1) > 1, \deg(n) > 0]. \tag{3.4}$$

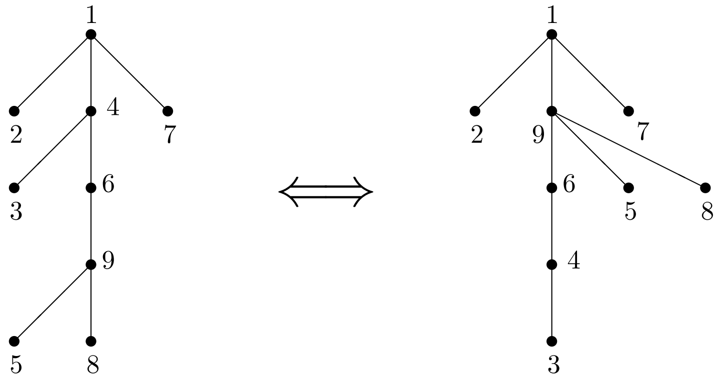

Figure 2: An example of Theorem 3.4 for $n = 9, i = 2, k = 4$.

Proof. Let T be a tree in $\mathscr{T}_{n,k}^{(i)}[\deg(1) > 0]$. We assume that $(n = v_1, v_2, \ldots, v_s = 1)$ is the path from n to 1, and v_j is the first node on this path such that (v_{j-1}, v_j) is a proper edge of T. Let $X(T_{v_j})$ be the rooted tree obtained from T_{v_j} by just taking n as the new root. We then obtain a rooted tree T' by replacing the subtree T_{v_j} by $X(T_{v_j})$. It is easy to see that T' has one more improper edge than T because of the edge (v_j, v_{j-1}). Moreover, we have $\deg_{T'}(n) = \deg_T(n) + 1$ (see Figure 2).

Conversely, for any tree T' in $\mathscr{T}_{n,k+1}^{(i-1)}[\deg(1) > 1, \deg(n) > 0]$, we may recover the tree T as follows. Assume that $(n = u_1, u_2, \ldots, u_t = \beta(n))$ is the path from n to $\beta(n)$. Let $u_h \neq n$ be the first node on this path such that every node in $T'_n - T'_{u_h}$ is greater than u_h. It is not difficult to see that such a node u_h can always be found because the node $\beta(n)$ is a candidate satisfying the above condition. Let $Y(T'_n)$ be the rooted tree obtained from T'_n by taking u_h as the new root. Replacing the subtree T'_n by $Y(T'_n)$ in T', we obtain the tree T. □

Theorem 3.5 *For $n \geqslant 3$ and $m \geqslant 2$, we have the following bijection:*

$$\mathscr{T}_{n,k}^{(1)}[\deg(1) = m] \longleftrightarrow \mathscr{T}_{n,k+1}^{(m-1)}[\deg(1) = 1, \deg(n) > 0]. \tag{3.5}$$

Note that Theorems 3.4 and 3.5 together lead to a refined version of Theorem 3.3. We now focus on the proof of (3.5). The proof of (3.4) actually implies the following assertion:

Lemma 3.6 *For $i \geqslant 2$, we have the following bijection:*

$$\mathscr{T}_{n,k}^{(i)}[\deg(1) = 1, \deg(n) = m] \longleftrightarrow \mathscr{T}_{n,k+1}^{(i-1)}[\deg(1) = 1, \deg(n) = m + 1]. \tag{3.6}$$

For any $m \geqslant 2$, by repeatedly using (3.6), we obtain

$$\mathscr{T}_{n,k+1}^{(m-1)}[\deg(1) = 1, \deg(n) \geqslant 1] \longleftrightarrow \mathscr{T}_{n,k+m-1}^{(1)}[\deg(1) = 1, \deg(n) \geqslant m - 1]. \tag{3.7}$$

By (3.7), we see that Theorem 3.5 is equivalent to the following result.

Theorem 3.7 *For $n \geqslant 3$ and $m \geqslant 2$, we have the following bijection:*

$$\mathscr{T}^{(1)}_{n,k}[\deg(1) = m] \longleftrightarrow \mathscr{T}^{(1)}_{n,k+m-1}[\deg(1) = 1, \deg(n) \geqslant m-1].$$

Proof. Suppose that $T \in \mathscr{T}^{(1)}_{n,k}[\deg(1) = m]$. Let the children of 1 be $a_1, a_2, \ldots, a_m$, and let n be a descendant of a_m. Denote by β_0 the minimum element of $\beta(a_1), \beta(a_2), \ldots, \beta(a_{m-1})$.

If $\deg_T(n) > 0$, assume that the children of n are $b_1, b_2, \ldots, b_s$, and denote by α the maximum element of $\beta(b_1), \beta(b_2), \ldots, \beta(b_s)$. We need to consider two cases (see Figure 3):

- $\beta_0 > \alpha$. Remove the subtrees $T_{a_i} (1 \leqslant i \leqslant m-1)$ and attach them to the node n as subtrees.

- $\beta_0 < \alpha$. Exchange the node n and the subtree T_α. Thus the degree of n becomes zero. All edges but the first one on the path from 1 to α are improper, while the first edge on the path from α to n is proper. Then remove the subtrees $T_{a_i} (1 \leqslant i \leqslant m-1)$ and attach them to the node n as subtrees.

If $\deg_T(n) = 0$, then n must be a child of 1, and deal with the tree T as the first case. Thus, we obtain a tree $T' \in \mathscr{T}^{(1)}_{n,k+m-1}[\deg(1) = 1, \deg(n) \geqslant m-1]$.

Conversely, for the above tree T', suppose that the children of n are $c_1, c_2, \ldots, c_p$ and $\beta(c_1) > \beta(c_2) > \cdots > \beta(c_p)$. We also have two cases:

- The child of 1 is n, or $\deg_{T'}(n) \geqslant m$. Remove the subtrees $T'_{c_i} (1 \leqslant i \leqslant m-1)$ and attach them back to the node 1 as subtrees.

- The child of 1 is not n, and $\deg_{T'}(n) = m-1$. Remove the subtrees $T'_{c_i} (1 \leqslant i \leqslant m-1)$ and attach them back to the node 1 as subtrees. Suppose that we obtain a tree T'' and that the path from 1 to n in T'' is $P : (1 = y_1, y_2, \ldots, y_s = n)$. Let (y_i, y_{i+1}) be the second proper edge on the path P. Assume that $R_1, R_2 \ldots, R_s$ are all of the subtrees of y_i such that $\beta(R_j) > y_i$, and $n \notin R_j$ for $j = 1, 2, \ldots, s$. Attach these subtrees to the node n, and exchange the labels of nodes y_i and n. □

Thus we have proved Theorem 3.1. The essence of Theorem 3.1 is the duality between the second minimum element and the maximum element in a tree with root 1. It is easy to understand that the labels of a rooted tree need not to be consecutive integers in order that the bijection holds. By applying Theorem 3.1, we can construct the main bijection of this paper, which leads to a combinatorial proof of Theorem 2.4.

Theorem 3.8 *For $1 \leqslant r \leqslant n-2$ and $0 \leqslant k \leqslant n-r-2$, we have the following bijection:*

$$\mathscr{F}^r_{n,k}[n \prec 1, \deg(r+1) > 0] \longleftrightarrow \mathscr{F}^r_{n,k+1}[n \prec 1, \deg(n) > 0].$$

Proof. The case $r = 1$ reduces to Theorem 3.1. We now assume that $r \geqslant 2$. Suppose that $F \in \mathscr{F}^r_{n,k}[n \prec 1, \deg(r+1) > 0]$, the tree in F with root i is denoted by T_i. Assume that $r+1$ is a node of T_x. Note that n is a node of T_1. We now proceed to construct a forest $F' \in \mathscr{F}^r_{n,k+1}[n \prec 1, \deg(n) > 0]$.

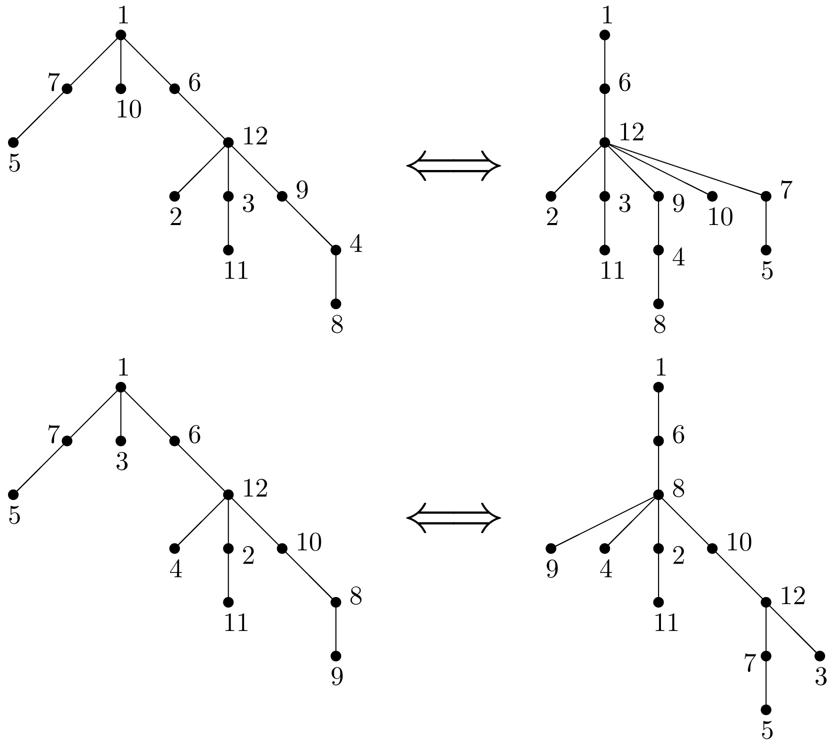

Figure 3: Examples of Theorem 3.7 for $n = 12$ and $m = 3$.

(i) $x = 1$. Then in the tree T_1, the second minimum element is $r+1$ and the maximum element n. Applying Theorem 3.1 on T_1, we are led to a tree T_1'. Replacing T_1 by T_1' in F, we obtain a forest $F' \in \mathscr{F}^r_{n,k+1}[n \prec 1, \deg(n) > 0]$.

(ii) $x \neq 1$ and $\deg_{T_1}(n) > 0$. Then applying Theorem 3.1 on T_x, we obtain a forest $F' \in \mathscr{F}^r_{n,k+1}[n \prec 1, \deg(n) > 0]$.

(iii) $x \neq 1$ and $\deg_{T_1}(n) = 0$. Let us relabel the subtrees T_x and T_1. Suppose T_x has nodes x and $r+1 = u_1 < u_2 < \cdots < u_i$ and T_1 has nodes $1 = v_1 < v_2 < \cdots < v_j$ and n. Let R be the tree obtained from T_x relabeled by $1 < u_2 < \cdots < u_i$ and n, and let S be the tree obtained from T_1 relabeled by x and $r+1 < v_2 < \cdots < v_j$. Applying Theorem 3.1 on R, we obtain a rooted tree R' with $\deg_{R'}(n) > 0$. Then replacing T_x by R' and T_1 by S, we are led to a forest F', which is clearly in $\mathscr{F}^r_{n,k+1}[n \prec 1, \deg(n) > 0]$.

It is easy to see that all the above steps are reversible. What is left is to classify the cases for a forest $F' \in \mathscr{F}^r_{n,k+1}[n \prec 1, \deg(n) > 0]$. The tree in F' with root i is denoted by T_i'.

(A) If $r+1$ is in the tree T_1', then we resort to the reverse of the case (i) to recover the forest $F \in \mathscr{F}^r_{n,k}[n \prec 1, \deg(r+1) > 0]$.

(B) Suppose that $x \neq 1$ and T_x' contains $r+1$. If the degree of the maximum element in T_x' is nonzero, then we may resort to the reverse of the construction in the case

(ii) to recover F. Otherwise, the degree of the maximum element in T'_x is equal to zero, and we may proceed by the reverse of the case (iii) to recover the desired F. □

4 Concluding remarks

At the end of the paper [4], Chen and Guo proposed two open problems and one conjecture for further study. Although we have given a new proof of the Shor recurrence (1.2), finding a short proof of (1.2) under Zeng's interpretations [4, Problem 4.1] still remains open. As far as we know, no one has made any progress in the other two questions [4, Conjecture 4.2 and Problem 4.4] by now. The reader is encouraged to continue to work on these questions.

Shor's Theorem 1.2 gives

$$|\mathscr{F}^r_{n,k}| = (n-1)|\mathscr{F}^r_{n-1,k}| + (n+k-r-2)|\mathscr{F}^r_{n-1,k-1}|. \tag{4.1}$$

The following problem is concerned with a refined version of (4.1). Suppose that F is a forest, n is the maximum node of F, $\deg(n) > 0$, and n is contained in the tree T. Let $\lambda(F) = u_h \neq n$ be the first node on the path from n to $\beta(n)$ such that every node in $T_n - T_{u_h}$ is greater than u_h (this node plays an important role in the proof of Theorem 3.4). Motivated by [4, Conjecture 4.2], we have the following conjecture.

Conjecture 4.1 *For $n \geqslant r+3$ and $r+1 \leqslant i \leqslant n-2$, we have the recurrence relation*

$$|\mathscr{F}^r_{n,k}[\lambda = i]| = (n-2)|\mathscr{F}^r_{n-1,k}[\lambda = i]| + (n+k-r-3)|\mathscr{F}^r_{n-1,k-1}[\lambda = i]|. \tag{4.2}$$

It is clear that

$$|\mathscr{F}^r_{n,k}[\lambda = n-1]| = |\mathscr{F}^r_{n-1,k-1}|, \quad 1 \leqslant k \leqslant n-r-1. \tag{4.3}$$

If Conjecture 4.1 is true, then we can use induction to derive the recurrence (4.1) from (4.2), (4.3), and the obvious relation

$$|\mathscr{F}^r_{n,k}[\deg(n) = 0]| = (n-1)|\mathscr{F}^r_{n-1,k}|.$$

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