NEW QUADRATIC IDENTITIES FOR BASIC HYPERGEOMETRIC SERIES AND q-CONGRUENCES

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Abstract. We derive new quadratic summations and transformations for basic hypergeometric series. These are applied to prove several biparametric q-congruences for truncated basic hypergeometric series modulo the square of a cyclotomic polynomial. We also prove a couple of q-congruences of a different type for truncated sums involving q-binomial coefficients.

1. Introduction

2. New congruences modulo the square of a cyclotomic polynomial

Theorem 1. *Let $d \geqslant 2$ be an integer, r an arbitrary integer coprime to d, and s and n positive integers satisfying $n > r + d$ and $n \equiv r + d \pmod{2d}$. Then*

$$\sum_{k=0}^{M} \frac{(q^r;q^d)_k^2}{(q^d;q^d)_k^2} \frac{(q^{-d};q^{2d})_k}{(q^{2r+3d};q^{2d})_k} q^{3dk} \equiv 0 \pmod{\Phi_n(q)^2}, \tag{2.1}$$

where $(s-1)n + (n-r)/d \leqslant M \leqslant sn - 1$.

3. New basic hypergeometric identities

4. Proofs of the new q-congruences

5. Conclusion

Conjecture 1. Let n be a positive integer and r an arbitrary integer. Then

$$\sum_{k=0}^{n-1} q^{r(n-k)^2+(r-1)k} \begin{bmatrix} n+k \\ k \end{bmatrix}^{2r} \begin{bmatrix} n-1 \\ k \end{bmatrix}^{2r}$$
$$\equiv q^{(r-1)n+1}[n] - \frac{r(2r-1)(n-1)^2\, q(1-q)^2}{4}[n]^3 \pmod{[n]\Phi_n(q)^3}. \tag{5.1}$$

Date: April 9, 2021.

1991 *Mathematics Subject Classification.* 11A07, 11B65, 11F33, 11Y60, 33D15.

Key words and phrases. basic hypergeometric series; quadratic identities; congruences; supercongruences; q-analogue; cyclotomic polynomials.

The first author was partially supported by the National Natural Science Foundation of China (grant 11771175). The second author was partially supported by Austrian Science Fund grant P32305.

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