PROOF OF SOME SUPERCONGRUENCES THROUGH A q-MICROSCOPE

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Abstract. Recently, Z.-W. Sun proposed the following conjecture: for any odd prime p and positive odd integer m,

$$\frac{1}{m^2\binom{m-1}{(m-1)/2}^2}\left(\sum_{k=0}^{(mp-1)/2}\frac{\binom{2k}{k}^2}{16^k}-\left(\frac{-1}{p}\right)\sum_{k=0}^{(m-1)/2}\frac{\binom{2k}{k}^2}{16^k}\right)\equiv 0 \pmod{p^2}.$$

In this note, employing the "creative microscoping" method, introduced by the first author and Zudilin, we confirm the above conjecture of Sun, as well as another four similar supercongruences conjectured by Sun.

1. Introduction

Let p be an odd prime and let $(\frac{a}{b})$ denote the Kronecker symbol. Mortenson [16,17] proved the following four supercongruences:

$$\sum_{k=0}^{p-1}\frac{1}{16^k}\binom{2k}{k}^2\equiv\left(\frac{-1}{p}\right)\pmod{p^2}\quad\text{for } p>2, \tag{1.1}$$

$$\sum_{k=0}^{p-1}\frac{1}{27^k}\binom{3k}{2k}\binom{2k}{k}\equiv\left(\frac{-3}{p}\right)\pmod{p^2}\quad\text{for } p>3, \tag{1.2}$$

$$\sum_{k=0}^{p-1}\frac{1}{64^k}\binom{4k}{2k}\binom{2k}{k}\equiv\left(\frac{-2}{p}\right)\pmod{p^2}\quad\text{for } p>2, \tag{1.3}$$

$$\sum_{k=0}^{p-1}\frac{1}{432^k}\binom{6k}{3k}\binom{3k}{k}\equiv\left(\frac{-1}{p}\right)\pmod{p^2}\quad\text{for } p>3, \tag{1.4}$$

which were originally conjectured by Rodriguez-Villegas [22, (36)]. Note that the sum in (1.1) can be truncated at $(p-1)/2$, since $\binom{2k}{k}\equiv 0 \pmod{p}$ for $(p-1)/2<k\leqslant p-1$. For a simple proof of (1.1)–(1.4), we refer the reader to [23]. Some q-analogues of (1.1)–(1.4) are given in [2, 6, 9, 19]. More recent q-supercongruences can be found in [3–5, 7, 8, 10, 11, 13–15, 18, 20, 21, 27–31].

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Recently, Sun [25, Conjecture 11(i)] made the following conjecture: For any prime $p>3$ and positive odd integer m,

$$\frac{4^{m-1}}{m^2\binom{m-1}{(m-1)/2}^2}\left(\sum_{k=0}^{(mp-1)/2}\frac{\binom{2k}{k}^2}{16^k}-\left(\frac{-1}{p}\right)\sum_{k=0}^{(m-1)/2}\frac{\binom{2k}{k}^2}{16^k}\right)\equiv p^2E_{p-3}\pmod{p^3},\quad(1.5)$$

where E_n is the n-th Euler number. It is clear that (1.5) is a generalization of (1.1), and the $m=1$ case was already proved by Sun himself [24].

In this paper, we shall prove the supercongruence (1.5) modulo p^2 by establishing the following q-counterpart.

Theorem 1.1. *Let m and n be positive odd integers with $n>1$. Then*

$$\frac{1}{[m]_{q^n}^2{\binom{m-1}{(m-1)/2}}_{q^n}^2}\left(\sum_{k=0}^{(mn-1)/2}\frac{2(q;q^2)_k^2q^{2k}}{(q^2;q^2)_k^2(1+q^{2k})}-\left(\frac{-1}{n}\right)\sum_{k=0}^{(m-1)/2}\frac{2(q^n;q^{2n})_k^2q^{2nk}}{(q^{2n};q^{2n})_k^2(1+q^{2nk})}\right)$$
$$\equiv 0\pmod{\Phi_n(q)^2}.\quad(1.6)$$

Moreover, the denominator of (the reduced form of) the left-hand side of (1.6) is relatively prime to $\Phi_{n^j}(q)$ for any index $j\geqslant 2$.

Here and in what follows, the *q-integer* is defined by $[n]_q=1+q+\cdots+q^{n-1}$, the *q-shifted factorial* is defined by $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n=0,1,\dots,$ and the *q-binomial coefficient* is defined as

$${\begin{bmatrix}m\\n\end{bmatrix}}_q=\begin{cases}\dfrac{(q;q)_m}{(q;q)_n(q;q)_{m-n}}, & \text{if } 0\leqslant n\leqslant m,\\ 0, & \text{otherwise.}\end{cases}$$

Moreover, $\Phi_n(q)$ is the *n-th cyclotomic polynomial* in q given by

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}}(q-\zeta^k),$$

where ζ is an n-th primitive root of unity.

It is easy to see that $\Phi_n(1)=p$ if $n=p^r$ $(r\geqslant 1)$ is a prime power and $\Phi_n(1)=1$ otherwise. Meanwhile, the denominator of (1.6) is clearly a product of cyclotomic polynomials. This implies that the supercongruence (1.5) modulo p^2 immediately follows from (1.6) by putting $n=p$ and taking $q\to 1$, since $\lim_{q\to1}(q;q^2)_k/(q^2;q^2)_k=\binom{2k}{k}/4^k$.

Sun (see [26, Conjecture 5.4]) also proposed the following generalizations of (1.1)–(1.4): For any prime $p>3$ and positive integer m,

$$\frac{1}{m^2\binom{2m}{m}^2}\left(\sum_{k=0}^{mp-1}\frac{\binom{2k}{k}^2}{16^k}-\left(\frac{-1}{p}\right)\sum_{k=0}^{m-1}\frac{\binom{2k}{k}^2}{16^k}\right)\equiv 0\pmod{p^2},\quad(1.7)$$

$$\frac{1}{m^2\binom{3m}{m}\binom{2m}{m}}\left(\sum_{k=0}^{mp-1}\frac{\binom{3k}{k}\binom{2k}{k}}{27^k}-\left(\frac{-3}{p}\right)\sum_{k=0}^{m-1}\frac{\binom{3k}{k}\binom{2k}{k}}{27^k}\right)\equiv 0\pmod{p^2},\quad(1.8)$$

$$\frac{1}{m^2\binom{4m}{2m}\binom{2m}{m}}\left(\sum_{k=0}^{mp-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{64^k}-\left(\frac{-2}{p}\right)\sum_{k=0}^{m-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{64^k}\right)\equiv 0 \pmod{p^2}, \tag{1.9}$$

$$\frac{1}{m^2\binom{6m}{3m}\binom{3m}{m}}\left(\sum_{k=0}^{mp-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{432^k}-\left(\frac{-1}{p}\right)\sum_{k=0}^{m-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{432^k}\right)\equiv 0 \pmod{p^2}. \tag{1.10}$$

Note that the supercongruences (1.7)–(1.10), without the fractions before the brackets, have been obtained by Liu [12]. Moreover, in the case where $m=p^{r-1}$, the supercongruence (1.7) was proved by the first author and Zudilin [11], and the supercongruences (1.8)–(1.10) were confirmed by the second author [18].

Let $\langle x\rangle_n$ denote the least nonnegative residue of x modulo n. In this paper, we shall prove (1.7)–(1.10) by establishing the following q-supercongruence.

Theorem 1.2. *Let s,d,m be positive integers with $s<d$. Let $n>1$ be an odd integer with $n\equiv \pm 1 \pmod d$. Then, modulo $\Phi_n(q)^2$,*

$$\frac{(q^{dn};q^{dn})_m^2}{[m]_{q^n}^2(q^{sn};q^{dn})_m(q^{(d-s)n};q^{dn})_m}\left(\sum_{k=0}^{mn-1}\frac{2(q^s;q^d)_k(q^{d-s};q^d)_k q^{dk}}{(q^d;q^d)_k^2(1+q^{dk})}\right.$$
$$\left.-(-1)^{\langle -s/d\rangle_n}\sum_{k=0}^{m-1}\frac{2(q^{sn};q^{dn})_k(q^{(d-s)n};q^{dn})_k q^{dnk}}{(q^{dn};q^{dn})_k^2(1+q^{dnk})}\right)\equiv 0. \tag{1.11}$$

Moreover, the denominator of (the reduced form of) the left-hand side of (1.11) is relatively prime to $\Phi_{n^j}(q)$ for any index $j\geqslant 2$.

It is easy to see that, for any odd prime p,

$$(-1)^{\langle -1/2\rangle_p}=\left(\frac{-1}{p}\right),\quad (-1)^{\langle -1/4\rangle_p}=\left(\frac{-2}{p}\right),$$

and for any prime $p>3$,

$$(-1)^{\langle -1/3\rangle_p}=\left(\frac{-3}{p}\right),\quad (-1)^{\langle -1/6\rangle_p}=\left(\frac{-1}{p}\right).$$

For $d=2,3,4,6$, and any prime $p>3$, we always have $p\equiv\pm1 \pmod d$. Thus, for $d=2,3,4,6$, $s=1$, letting n be a prime and then taking $q\to1$ in Theorem 1.2, we obtain the supercongruences (1.7)–(1.10).

Moreover, for general d and s, letting $n=p$ and $q\to 1$ in Theorem 1.2, we are led to the following result, which confirms the first part of [25, Conjecture 10].

Corollary 1.3. *Let s,d,m be positive integers with $s<d$. Let p be an odd prime with $p\equiv\pm1 \pmod d$. Then*

$$\frac{\sum_{k=0}^{pm-1}\binom{-s/d}{k}\binom{(s-d)/d}{k}-(-1)^{\langle -s/d\rangle_p}\sum_{k=0}^{m-1}\binom{-s/d}{k}\binom{(s-d)/d}{k}}{p^2m^2\binom{-s/d}{m}\binom{-(d-s)/d}{m}}$$

is a p-adic integer.

We shall prove Theorem 1.1 in the next section by using the 'creative microscoping' method devised by the first author and Zudilin [10]. More precisely, we shall first give a generalization of Theorem 1.1 with an additional parameter a, and then we deduce Theorem 1.1 from this generalization by choosing $a = 1$. Finally, we shall prove Theorem 1.2 in Section 3 using the same method.

2. Proof of Theorem 1.1

We need the following result, which was proved by the first author [2, Corollary 1.4].

Lemma 2.1. *Let d, n, and s be positive integers with $\gcd(d,n) = 1$ and n odd. Then, modulo $(1 - aq^{s+d\langle -s/d\rangle_n})(a - q^{d-s+d\langle (s-d)/d\rangle_n})$,*

$$\sum_{k=0}^{n-1} \frac{2(aq^s;q^d)_k(q^{d-s}/a;q^d)_k q^{dk}}{(q^d;q^d)_k^2(1+q^{dk})} \equiv (-1)^{\langle -s/d\rangle_n}. \tag{2.1}$$

We have the following parametric generalization of Theorem 1.1.

Theorem 2.2. *Let m and n be positive odd integers with $n > 1$. Then, modulo*

$$\prod_{j=0}^{(m-1)/2} (1 - aq^{(2j+1)n})(a - q^{(2j+1)n}), \tag{2.2}$$

we have

$$\sum_{k=0}^{(mn-1)/2} \frac{2(aq;q^2)_k(q/a;q^2)_k q^{2k}}{(q^2;q^2)_k^2(1+q^{2k})} \equiv \left(\frac{-1}{n}\right) \sum_{k=0}^{(m-1)/2} \frac{2(aq^n;q^{2n})_k(q^n/a;q^{2n})_k q^{2nk}}{(q^{2n};q^{2n})_k^2(1+q^{2nk})}. \tag{2.3}$$

Proof. It suffices to prove that both sides of (2.3) are identical for $a = q^{-(2j+1)n}$ and $a = q^{(2j+1)n}$ with $j = 0, 1, \dots, (m-1)/2$, i.e.,

$$\sum_{k=0}^{(mn-1)/2} \frac{2(q^{1-(2j+1)n};q^2)_k(q^{1+(2j+1)n};q^2)_k}{(q^2;q^2)_k^2(1+q^{2k})} q^{2k}$$
$$= \left(\frac{-1}{n}\right) \sum_{k=0}^{(m-1)/2} \frac{2(q^{-2jn};q^{2n})_k(q^{(2j+2)n};q^{2n})_k}{(q^{2n};q^{2n})_k^2(1+q^{2nk})} q^{2nk}. \tag{2.4}$$

Note that the congruence (2.1) with $d = 2$ and $s = 1$ reduces to

$$\sum_{k=0}^{n-1} \frac{2(aq;q^2)_k(q/a;q^2)_k q^{2k}}{(q^2;q^2)_k^2(1+q^{2k})} \equiv \left(\frac{-1}{n}\right) \pmod{(1 - aq^n)(a - q^n)}. \tag{2.5}$$

Clearly, $(mn-1)/2 \geqslant ((2j+1)n-1)/2$ for $0 \leqslant j \leqslant (m-1)/2$, and $(q^{1-(2j+1)n};q^2)_k = 0$ for $k > ((2j+1)n-1)/2$. In view of (2.5), the left-hand side of (2.4) is equal to $\left(\frac{-1}{(2j+1)n}\right)$. Similarly, the right-hand side of (2.4) is equal to

$$\left(\frac{-1}{n}\right)\left(\frac{-1}{2j+1}\right) = \left(\frac{-1}{(2j+1)n}\right),$$

where $\left(\frac{-1}{1}\right)$ is understood to be 1. This establishes the identity (2.4), and so the q-congruence (2.3) holds. $\square$

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. It is well known that

$$q^N - 1 = \prod_{d|N} \Phi_d(q).$$

Let $\lfloor x \rfloor$ stand for the integral part of a real number x. For any positive integer j, there exist $\lfloor m/n^{j-1} \rfloor - \lfloor (m-1)/(2n^{j-1}) \rfloor$ multiples of n^{j-1} in the arithmetic progression $1, 3, \ldots, m$. Thus, the $a = 1$ case of (2.2) has the factor

$$\prod_{j=1}^{\infty} \Phi_{n^j}(q)^{2\lfloor m/n^{j-1} \rfloor - 2\lfloor (m-1)/(2n^{j-1}) \rfloor}.$$

On the other hand, the least common denominator of the left-hand side of (2.3) is a multiple of that of the right-hand side of (2.3). The former is at most equal to $(q^2; q^2)^2_{(mn-1)/2}(-q^2; q^2)_{(mn-1)/2}$ and its factor related to $\Phi_n(q), \Phi_{n^2}(q), \ldots$ is just

$$\prod_{j=1}^{\infty} \Phi_{n^j}(q)^{2\lfloor (mn-1)/(2n^j) \rfloor},$$

since $(-q^2; q^2)_{(mn-1)/2}$ is relatively prime to $\Phi_{n^j}(q)$ for any $j \geqslant 1$.

Furthermore, writing $[m]_q = (q; q)_m/((1-q)(q; q)_{m-1})$, the central q-binomial coefficient $\left[{m-1 \atop (m-1)/2}\right]$ as a product of different cyclotomic polynomials (see [1]), and then utilizing the fact $\Phi_{n^j}(q^n) = \Phi_{n^{j+1}}(q)$, we see that the polynomial $[m]_{q^n}^2 \left[{m-1 \atop (m-1)/2}\right]_{q^n}^2$ only has the following factor

$$\prod_{j=2}^{\infty} \Phi_{n^j}(q)^{2\lfloor m/n^{j-1} \rfloor - 4\lfloor (m-1)/(2n^{j-1}) \rfloor}$$

related to $\Phi_n(q), \Phi_{n^2}(q), \ldots$.

It is clear that

$$2\lfloor m/n^{j-1} \rfloor - 2\lfloor (m-1)/(2n^{j-1}) \rfloor - 2\lfloor (mn-1)/(2n^j) \rfloor = 2 \quad \text{for } j = 1,$$

and

$$\lfloor (mn-1)/(2n^j) \rfloor = \lfloor (m-1)/(2n^{j-1}) \rfloor \quad \text{for } j > 1.$$

Therefore, letting $a = 1$ in (2.3), we conclude that the q-congruence (1.6) holds, and the denominator on the left-hand side of (1.6) is not divisible by $\Phi_{n^j}(q)$ for any index $j \geqslant 2$. This completes the proof. $\square$

3. Proof of Theorem 1.2

We first establish the following parametric generalization of Theorem 1.2 for $n \equiv 1 \pmod{d}$.

Theorem 3.1. *Let s, d, m be positive integers with $s < d$. Let $n > 1$ be an odd integer with $n \equiv 1 \pmod{d}$. Then, modulo*

$$\prod_{j=0}^{m-1}(1-aq^{n(dj+s)})(a-q^{n(dj+d-s)}), \tag{3.1}$$

we have

$$\sum_{k=0}^{mn-1}\frac{2(aq^s;q^d)_k(q^{d-s}/a;q^d)_k q^{dk}}{(q^d;q^d)_k^2(1+q^{dk})} \equiv (-1)^{\langle -s/d\rangle_n}\sum_{k=0}^{m-1}\frac{2(aq^{sn};q^{dn})_k(q^{(d-s)n}/a;q^{dn})_k q^{dnk}}{(q^{dn};q^{dn})_k^2(1+q^{dnk})}. \tag{3.2}$$

Proof. We need to show that both sides of (3.2) are equal for $a = q^{-(dj+s)n}$ and $a = q^{n(dj+d-s)}$ $(j = 0, 1, 2, \dots, m-1)$. Namely,

$$\sum_{k=0}^{mn-1}\frac{2(q^{s-n(dj+s)};q^d)_k(q^{d-s+n(dj+s)};q^d)_k q^{dk}}{(q^d;q^d)_k^2(1+q^{dk})} = (-1)^{\langle -s/d\rangle_n}\sum_{k=0}^{m-1}\frac{2(q^{sn-n(dj+s)};q^{dn})_k(q^{(d-s)n+n(dj+s)};q^{dn})_k q^{dnk}}{(q^{dn};q^{dn})_k^2(1+q^{dnk})}, \tag{3.3}$$

and

$$\sum_{k=0}^{mn-1}\frac{2(q^{s+n(dj+d-s)};q^d)_k(q^{d-s-n(dj+d-s)};q^d)_k q^{dk}}{(q^d;q^d)_k^2(1+q^{dk})} = (-1)^{\langle -s/d\rangle_n}\sum_{k=0}^{m-1}\frac{2(q^{sn+n(dj+d-s)};q^{dn})_k(q^{(d-s)n-n(dj+d-s)};q^{dn})_k q^{dnk}}{(q^{dn};q^{dn})_k^2(1+q^{dnk})}. \tag{3.4}$$

It is clear that $mn-1 \geqslant nj + s(n-1)/d$ and $mn-1 \geqslant nj + (d-s)(n-1)/d$ for $j = 0, 1, 2, \dots, m-1$. Since $n \equiv 1 \pmod{d}$, we get $\langle -s/d\rangle_n = s(n-1)/d$ and $\langle (s-d)/d\rangle_n = (d-s)(n-1)/d$. By Lemma 2.1, the left-hand side of (3.3) is equal to

$$(-1)^{nj+s(n-1)/d}.$$

Similarly, the right-hand side of (3.3) is equal to

$$(-1)^{(sn-s)/d}(-1)^j = (-1)^{j+s(n-1)/d},$$

thus establishing (3.3). In the same way, we can also prove that both sides of (3.4) are equal to $(-1)^{j+(d-s)(n-1)/d}$. This proves the q-congruence (3.2). $\square$

We now give a parametric generalization of Theorem 1.2 for $n \equiv -1 \pmod{d}$.

Theorem 3.2. *Let s, d, m be positive integers with $s < d$. Let $n > 1$ be an odd integer with $n \equiv -1 \pmod{d}$. Then, modulo* (3.1),

$$\sum_{k=0}^{mn-1} \frac{2(q^s/a;q^d)_k(aq^{d-s};q^d)_k q^{dk}}{(q^d;q^d)_k^2(1+q^{dk})} \equiv (-1)^{\langle -s/d\rangle_n} \sum_{k=0}^{m-1} \frac{2(aq^{sn};q^{dn})_k(q^{(d-s)n}/a;q^{dn})_k q^{dnk}}{(q^{dn};q^{dn})_k^2(1+q^{dnk})}. \tag{3.5}$$

Proof. For $a = q^{-n(dj+s)}$ with $0 \leqslant j \leqslant m-1$, by Lemma 2.1, the left-hand side of (3.5) is equal to

$$\sum_{k=0}^{mn-1} \frac{2(q^{s+n(dj+s)};q^d)_k(q^{d-s-n(dj+s)};q^d)_k q^{dk}}{(q^d;q^d)_k^2(1+q^{dk})} = (-1)^{j-1+s(n+1)/d},$$

Similarly, the right-hand side of (3.5) is equal to

$$(-1)^{\langle -s/d\rangle_n}(-1)^j = (-1)^{n-(n+1)s/d+j} = (-1)^{j-1+s(n+1)/d},$$

where we have used the fact that $\langle -s/d\rangle_n = n - s(n+1)/d$ since $n \equiv -1 \pmod{d}$. Hence, the q-congruence (3.5) is true modulo $\prod_{j=0}^{m-1}(1-aq^{n(dj+s)})$.

For $a = q^{n(dj+d-s)}$ with $0 \leqslant j \leqslant m-1$, the left-hand side of (3.5) is equal to

$$\sum_{k=0}^{mn-1} \frac{2(q^{s-n(dj+d-s)};q^d)_k(q^{(d-s)+n(dj+d-s)};q^d)_k q^{dk}}{(q^d;q^d)_k^2(1+q^{dk})} = (-1)^{j-1+s(n+1)/d},$$

which is the same as the right-hand side of (3.5). This proves (3.5) modulo $\prod_{j=0}^{m-1}(a-q^{n(dj+d-s)})$. □

Proof of Theorem 1.2. We first consider the $n \equiv 1 \pmod{d}$ case. For any positive integer j, there exist exactly

$$\left\lfloor \frac{(m-1)-s(n^{j-1}-1)/d}{n^{j-1}} \right\rfloor + 1$$

multiples of n^{j-1} in the set $\{dj+s : j = 0, \dots, m-1\}$, and there exist exactly

$$\left\lfloor \frac{(m-1)-(d-s)(n^{j-1}-1)/d}{n^{j-1}} \right\rfloor + 1$$

multiples of n^{j-1} in the set $\{dj+d-s : j = 0, \dots, m-1\}$. Thus, the $a = 1$ case of (3.1) has the factor

$$\prod_{j\geqslant 1} \Phi_{n^j}(q)^{\left\lfloor \frac{(m-1)-s(n^{j-1}-1)/d}{n^{j-1}} \right\rfloor + \left\lfloor \frac{(m-1)-(d-s)(n^{j-1}-1)/d}{n^{j-1}} \right\rfloor + 2}.$$

Similarly, using the fact $\Phi_{n^j}(q^n) = \Phi_{n^{j+1}}(q)$, we conclude that the denominator of the reduced form of the fraction

$$\frac{(q^{dn};q^{dn})_m^2}{[m]_{q^n}^2(q^{sn};q^{dn})_m(q^{(d-s)n};q^{dn})_m}$$

only has the following factor

$$\prod_{j\geqslant 2}\Phi_{n^j}(q)^{\left\lfloor\frac{(m-1)-s(n^{j-1}-1)/d}{n^{j-1}}\right\rfloor+\left\lfloor\frac{(m-1)-(d-s)(n^{j-1}-1)/d}{n^{j-1}}\right\rfloor+2-2\left\lfloor\frac{m-1}{n^{j-1}}\right\rfloor}$$

related to $\Phi_n(q), \Phi_{n^2}(q), \dots$.

On the other hand, the least common denominator of the left-hand side of (3.2) is divisible by that of the right-hand side of (3.2). The former is at most equal to $(q^d;q^d)_{mn-1}^2\prod_{k=1}^{mn-1}(1+q^{dk})$ and its factor related to $\Phi_n(q), \Phi_{n^2}(q), \dots$ is just

$$\prod_{j=1}^{\infty}\Phi_{n^j}(q)^{2\lfloor(mn-1)/n^j\rfloor}.$$

It is clear that

$$2\left\lfloor(m-1)/n^{j-1}\right\rfloor+2-2\left\lfloor(mn-1)/n^{j}\right\rfloor=2\quad\text{for } j=1, \tag{3.6}$$

and

$$\left\lfloor(m-1)/n^{j-1}\right\rfloor=\left\lfloor(mn-1)/n^{j}\right\rfloor\quad\text{for } j>1. \tag{3.7}$$

Thus, taking $a=1$ in (3.2), we see that the $n\equiv 1 \pmod{d}$ case of (1.11) is true modulo $\Phi_n(q)^2$, and the denominator of (the reduced form of) the left-hand side of (1.11) is relatively prime to $\Phi_{n^j}(q)$ for any index $j\geqslant 2$.

We now consider the $n\equiv -1 \pmod{d}$ case. The proof is similar to that of the $n\equiv 1 \pmod{d}$ case (firstly let $a=1$ in (3.5) and finally employ (3.6) and (3.7)). However, there is no need to give the factor of the $a=1$ case of (3.1) related to $\Phi_n(q), \Phi_{n^2}(q), \dots$ (it is not so easy as before). This is because the $a=1$ case of (3.1) is just $(q^{sn};q^{dn})_m(q^{(d-s)n};q^{dn})_m$, which appears in the denominator of the fraction before the summation in (1.11). □

Data availability. All data generated during this study are included in the published article.

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