Some q-supercongruences from Gasper's Karlsson–Minton type summation

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Abstract. We give a q-analogue of a supercongruence of Deines–Fuselier–Long–Swisher–Tu by using the 'creative microscoping' method and Gasper's Karlsson–Minton type summation. As a conclusion, we obtain a new supercongruence modulo p^2, where p is an odd prime. We also establish another two q-supercongruences along the same lines.

Keywords: supercongruence; p-adic Gamma function; cyclotomic polynomials; creative microscoping.

2020 Mathematics Subject Classifications: 33D15; 11A07; 11B65.

1. Introduction

Rodriguez-Villegas [18] investigated hypergeometric families of Calabi–Yau manifolds, and found (numerically) many possible supercongruences. His simplest supercongruence is as follows: for any odd prime p,

$$\sum_{k=0}^{p-1} \frac{(\frac{1}{2})_k^2}{k!^2} \equiv (-1)^{(p-1)/2} \pmod{p^2}, \tag{1.1}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ denotes the Pochhammer symbol. Mortenson [16] first proved this supercongruence. Guo and Zeng [10] gave a q-analogue of (1.1):

$$\sum_{k=0}^{p-1} \frac{(q;q^2)_k^2}{(q^2;q^2)_k^2} q^{2k} \equiv (-1)^{(p-1)/2} q^{(p^2-1)/4} \pmod{[p]^2} \quad \text{for any odd prime } p. \tag{1.2}$$

Here and in what follows, $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ is the *q-shifted factorial*, and $[n] = 1+q+\cdots+q^{n-1}$ is the *q-integer*. For simplicity, we also use the condensed notation $(a_1, a_2, \dots, a_m; q)_n = (a_1;q)_n (a_2;q)_n \cdots (a_m;q)_n$. The author [6] established the following extension of (1.1): Let $d \geqslant 2$ and $r \leqslant d-2$ be integers such that $\gcd(r,d) = 1$. Then, for all positive integers n with $n \equiv -r \pmod{d}$ and $n \geqslant d-r$, we have

$$\sum_{k=0}^{n-1} \frac{(q^r;q^d)_k^d q^{dk}}{(q^d;q^d)_k^d} \equiv 0 \pmod{\Phi_n(q)^2}. \tag{1.3}$$

Here $\Phi_n(q)$ is the n-th *cyclotomic polynomial* in q given by

$$\Phi_n(q) = \prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}} (q-\zeta^k),$$

where ζ is an n-th primitive root of unity. For more recent q-congruences, we refer the reader to [1, 7–9, 11–14, 17, 22–24].

On the other hand, Deines et al. [2] proved the following interesting generalization of (1.1): for any integer $d>1$ and prime $p\equiv 1 \pmod{d}$,

$$\sum_{k=0}^{p-1} \frac{(\frac{d-1}{d})_k^d}{k!^d} \equiv -\Gamma_p(\tfrac{1}{d})^d \pmod{p^2}. \tag{1.4}$$

Here $\Gamma_p(x)$ denotes the p-adic Gamma function (for any odd prime p), which may be defined as follows: for any positive integer n,

$$\Gamma_p(n) := (-1)^n \prod_{\substack{1\leqslant k\leqslant n-1\\ \gcd(k,p)=1}} k,$$

$\Gamma_p(0):=1$, and for any p-adic integer $x\neq 0$,

$$\Gamma_p(x) := \lim_{n\to x} \Gamma_p(n).$$

where n ranges over any sequence of positive integers p-adically approaching x.

In this paper, we shall give a q-analogue of (1.4), which is also a generalization of (1.2) and can be deemed a complement to (1.3).

Theorem 1.1. *Let $d,n>1$ be integers with $n\equiv 1 \pmod{d}$. Then, modulo $\Phi_n(q)^2$,*

$$\sum_{k=0}^{n-1} \frac{(q^{d-1};q^d)_k^d q^{dk}}{(q^d;q^d)_k^d} \equiv \frac{(q^d;q^d)_{(d-1)(n-1)/d}\, q^{(d-1)(n-1)(d+n-1)/(2d)}}{(q^d;q^d)_{(n-1)/d}^{d-1}(-1)^{(d-1)(n-1)/d}}. \tag{1.5}$$

For n prime, letting $q\to 1$ in Theorem 1.1, we get the following supercongruence: For any integer $d>1$ and prime $p\equiv 1 \pmod{d}$,

$$\sum_{k=0}^{p-1} \frac{(\frac{d-1}{d})_k^d}{k!^d} \equiv (-1)^{(d-1)(p-1)/d} \frac{\left(\frac{(d-1)(p-1)}{d}\right)!}{(\frac{p-1}{d})!^{d-1}} \pmod{p^2}. \tag{1.6}$$

Since $p-1$ is even, we have $(-1)^{(p-1)/d} = (-1)^{(d-1)(p-1)/d}$. Moreover, by [2, (18) with $y=-1$], there holds

$$\frac{(p-1)!}{(1-p)_{(p-1)/d}(\frac{p-1}{d})!^{d-1}} \equiv -\Gamma_p(\tfrac{1}{d})^d \pmod{p^2}. \tag{1.7}$$

It is clear that

$$\frac{(p-1)!}{(1-p)_{(p-1)/d}} = (-1)^{(p-1)/d}\big(\tfrac{(d-1)(p-1)}{d}\big)!. \tag{1.8}$$

Thus, the supercongruence (1.6) is equivalent to (1.4).

We shall also give the following supercongruence similar to (1.4).

Theorem 1.2. *Let $d>1$ be an integer and let $p\equiv 1 \pmod d$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{k(\frac{d-1}{d})_k^d}{k!^d} \equiv \frac{(d-1)\Gamma_p(\frac{1}{d})^d}{2d} \pmod{p^2}. \tag{1.9}$$

Since $\Gamma_p(\frac{1}{2})^2 = (-1)^{(p+1)/2}$, for $d=2$, the supercongruence (1.9) reduces to

$$\sum_{k=0}^{p-1} \frac{k(\frac{1}{2})_k^2}{k!^2} \equiv \frac{(-1)^{(p+1)/2}}{4} \pmod{p^2}, \tag{1.10}$$

which was generalized to the modulus p^3 case for $p>3$ by Sun [19, Theorem 1.2, Equations (1.8) and (1.10)].

One of the referees found that the following supercongruence related to (1.10) seems to be true.

Conjecture 1.3. *Let $p\equiv 1 \pmod 4$ be a prime and $r\geqslant 1$. Then*

$$\sum_{k=0}^{p^r-1} \left(k-\frac{p^{2r}-1}{4}\right)\frac{(\frac{1}{2})_k^2}{k!^2} \equiv 0 \pmod{p^{2r+1}}. \tag{1.11}$$

Note that the $r=1$ case of (1.11) follows directly from [19, Theorem 1.2, Equations (1.7)–(1.10)].

The rest of the paper is arranged as follows. We shall prove Theorems 1.1 and 1.2 in Sections 2 and 3, respectively. Two similar q-supercongruence will be given in Section 4. Finally, we put forward three related conjectures in Section 5.

2. Proof of Theorem 1.1

We will utilize a Karlsson–Minton type summation due to Gasper (see [5, (1.9.9)]; and see also [4, (5.13)] for a more general form): for nonnegative integers $n_1,\dots,n_m$,

$$\sum_{k=0}^{N} \frac{(q^{-N}, b_1q^{n_1},\dots,b_mq^{n_m};q)_k}{(q,b_1,\dots,b_m;q)_k} q^k = (-1)^N \frac{(q;q)_N b_1^{n_1}\cdots b_m^{n_m}}{(b_1;q)_{n_1}\cdots(b_m;q)_{n_m}} q^{\binom{n_1}{2}+\cdots+\binom{n_m}{2}}, \tag{2.1}$$

where $N=n_1+\cdots+n_m$.

We first establish the following parametric generalization of Theorem 1.1 by using the method of 'creative microscoping' introduced in [11].

Theorem 2.1. *Let $d,n>1$ be integers with $n\equiv 1 \pmod d$. Then, modulo $(1-aq^n)(a-q^n)$,*

$$\sum_{k=0}^{n-1}\frac{(a^{d-1}q^{d-1},a^{d-3}q^{d-1},\dots,a^{3-d}q^{d-1},a^{1-d}q^{d-1};q^d)_k q^{dk}}{(a^{d-2}q^d,a^{d-4}q^d,\dots,a^{4-d}q^d,a^{2-d}q^d;q^d)_k(q^d;q^d)_k}$$
$$\equiv\frac{(-1)^{(d-1)(n-1)/d}(q^d;q^d)_{(d-1)(n-1)/d}\,q^{(d-1)(n-1)(d+n-1)/(2d)}}{(a^{d-2}q^d,a^{d-4}q^d,\dots,a^{4-d}q^d,a^{2-d}q^d;q^d)_{(n-1)/d}}. \tag{2.2}$$

Proof. Since $\gcd(d,n)=1$, none of the numbers $d,2d,\dots(n-1)d$ are multiples of n. This implies that the denominators of the left-hand side of (2.2) do not contain the factor $1-aq^n$ nor $1-a^{-1}q^n$. Hence, for $a=q^{-n}$ or $a=q^n$, the left-hand side of (2.2) can be written as

$$\sum_{k=0}^{(d-1)(n-1)/d}\frac{(q^{-(d-1)(n-1)},q^{d-1-(d-3)n},\dots,q^{d-1+(d-3)n},q^{(d-1)(n+1)};q^d)_k q^{dk}}{(q^{d-(d-2)n},q^{d-(d-4)n},\dots,q^{d+(d-4)n},q^{d+(d-2)n};q^d)_k(q^d;q^d)_k}, \tag{2.3}$$

where we have used the fact that $(q^{-(d-1)(n-1)};q^d)_k=0$ for $k>(d-1)(n-1)/d$. Letting $q\mapsto q^d$, $N=(d-1)(n-1)/d$, $m=d-1$, $b_j=q^{d-(d-2j)n}$ and $n_j=(n-1)/d$ $(1\leqslant j\leqslant d-1)$ in (2.1), we see that (2.3) is equal to

$$\frac{(-1)^{(d-1)(n-1)/d}(q^d;q^d)_{(d-1)(n-1)/d}q^{(d-1)(n-1)+(d-1)\binom{(n-1)/d}{2}}}{(q^{d-(d-2)n},q^{d-(d-4)n},\dots,q^{d+(d-4)n},q^{d+(d-2)n};q^d)_{(n-1)/d}},$$

which is just the $a=q^{-n}$ or $a=q^n$ case of (2.2). Namely, the congruence (2.2) holds. □

Proof of Theorem 1.1. It is well known that $\Phi_n(q)$ is a factor of $1-q^m$ if and only if n divides m. Therefore, when $a=1$, the denominators of (2.2) are relatively prime to $\Phi_n(q)$. On the other hand, when $a=1$, the polynomial $(1-aq^n)(a-q^n)=(1-q^n)^2$ contains the factor $\Phi_n(q)^2$. Thus, the congruence (1.5) immediately follows from the $a=1$ case of (2.2). □

3. Proof of Theorem 1.2

Let $n>1$ be an integer with $n\equiv 1 \pmod d$. Making the substitution $q\mapsto q^{-1}$ in (1.5), we obtain its dual form: Modulo $\Phi_n(q)^2$,

$$\sum_{k=0}^{n-1}\frac{(q^{d-1};q^d)_k^d}{(q^d;q^d)_k^d}\equiv\frac{(q^d;q^d)_{(d-1)(n-1)/d}\,q^{-(d-1)(n-1)(dn-n+1)/(2d)}}{(q^d;q^d)_{(n-1)/d}^{d-1}(-1)^{(d-1)(n-1)/d}}. \tag{3.1}$$

Subtracting (1.5) from (3.1) and dividing both sides by $1-q$, we arrive at

$$\sum_{k=0}^{n-1}\frac{(q^{d-1};q^d)_k^d(1-q^{dk})}{(q^d;q^d)_k^d(1-q)}$$

$$\equiv \frac{(q^d;q^d)_{(d-1)(n-1)/d}\, q^{-(d-1)(n-1)(dn-n+1)/(2d)}(1-q^{(d-1)(n^2-1)/2})}{(q^d;q^d)_{(n-1)/d}^{d-1}(-1)^{(d-1)(n-1)/d}(1-q)} \pmod{\Phi_n(q)^2}.$$

Letting $n=p$ be a prime and taking the limit as $q\to 1$ in the above q-supercongruence, we are led to the following result: for any integer $d>1$ and prime $p\equiv 1 \pmod{d}$,

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{(\frac{d-1}{d})_k^d}{k!^d}dk &\equiv \frac{\big(\frac{(d-1)(p-1)}{d}\big)!(d-1)(p^2-1)}{2(-1)^{(d-1)(p-1)/d}(\frac{p-1}{d})!^{d-1}}\\ &\equiv \frac{-\big(\frac{(d-1)(p-1)}{d}\big)!(d-1)}{2(-1)^{(p-1)/d}(\frac{p-1}{d})!^{d-1}} \pmod{p^2}.\end{aligned}$$

The proof of (1.9) then follows from (1.7) and (1.8).

4. Two similar q-supercongruences

In this section, we give two more q-supercongruences using Gasper's Karlsson–Minton type summation (2.1) and the 'creative microscoping' method.

Theorem 4.1. *Let $d\geqslant 3$ be an odd integer. Let n be a positive integer with $n\equiv -1 \pmod{d}$ and $n\geqslant 2d-1$. Then, modulo $\Phi_n(q)^2$,*

$$\begin{aligned}&\sum_{k=0}^{n-1}\frac{(q^{d+1};q^d)_k^{d-1}(q^{1-d};q^d)_k q^{dk}}{(q^d;q^d)_k^d}\\ &\quad\equiv -\frac{(1-q)(1-q^{d-1})(q^d;q^d)_{n-1-(n+1)/d}}{(q^d;q^d)_{(n+1)/d}^{d-1}}q^{(d(d+n)(n+1)-(n+1)^2-2d)/(2d)}. \qquad (4.1)\end{aligned}$$

Proof. We first prove the following parametric congruence: modulo $(1-aq^n)(a-q^n)$,

$$\begin{aligned}&\sum_{k=0}^{n-1}\frac{(a^{d-1}q^{d+1},a^{d-3}q^{d+1},\dots,a^2q^{d+1},q^{1-d},a^{-2}q^{d+1},\dots,a^{3-d}q^{d+1},a^{1-d}q^{d+1};q^d)_k q^{dk}}{(a^{d-2}q^d,a^{d-4}q^d,\dots,a^{4-d}q^d,a^{2-d}q^d;q^d)_k(q^d;q^d)_k}\\ &\equiv -\frac{(1-q)(1-q^{d-1})(q^d;q^d)_{n-1-(n+1)/d}}{(a^{d-2}q^d,a^{d-4}q^d,\dots,a^{4-d}q^d,a^{2-d}q^d;q^d)_{(n+1)/d}}q^{(d(d+n)(n+1)-(n+1)^2-2d)/(2d)}. \qquad (4.2)\end{aligned}$$

In fact, for $a=q^{-n}$ or $a=q^n$, the left-hand side of (4.2) can be written as

$$\sum_{k=0}^{n-1}\frac{(q^{d+1-(d-1)n},q^{d+1-(d-3)n},\dots,q^{d+1-2n},q^{1-d},q^{d+1+2n},\dots,q^{d+1+(d-3)n},q^{d+1+(d-1)n};q^d)_k}{(q^{d-(d-2)n},q^{d-(d-4)n},\dots,q^{d+(d-4)n},q^{d+(d-2)n};q^d)_k(q^d;q^d)_k q^{-dk}}. \qquad (4.3)$$

Note that $(q^{d+1-(d-1)n};q^d)_k=0$ for $k>n-1-(n+1)/d$. Letting $q\mapsto q^d$, $N=n-1-(n+1)/d$, $m=d-1$, $b_j=q^{d-(d-2j)n}$ $(1\leqslant j\leqslant d-1)$, $n_{(d-1)/2}=(n+1)/d-2$, and $n_j=(n+1)/d$ ($1\leqslant j\leqslant d-1$ and $j\neq (d-1)/2$) in (2.1), we see that (4.3) is equal to

$$\frac{(q^d;q^d)_{n-1-(n+1)/d}\, q^{(d-1)(n+1)-2(d-n)+d(d-2)\binom{(n+1)/d}{2}+d\binom{(n+1)/d-2}{2}}}{(q^{d-(d-2)n},q^{d-(d-4)n},\dots,q^{d-3n},q^{d+n},q^{d+3n},\dots,q^{d+(d-4)n},q^{d+(d-2)n};q^d)_{(n+1)/d}}$$

$$\times \frac{1}{(q^{d-n};q^d)_{(n+1)/d-2}}$$
$$= \frac{-(1-q)(1-q^{d-1})(q^d;q^d)_{n-1-(n+1)/d}\, q^{(d(d+n)(n+1)-(n+1)^2-2d)/(2d)}}{(q^{d-(d-2)n}, q^{d-(d-4)n}, \dots, q^{d-3n}, q^{d-n}, q^{d+n}, q^{d+3n}, \dots, q^{d+(d-4)n}, q^{d+(d-2)n}; q^d)_{(n+1)/d}},$$

which is just the $a=q^{-n}$ or $a=q^n$ case of (4.2), thus establishing the expected congruence (4.2).

The proof of (4.1) then follows by letting $a=1$ in (4.2). □

We have the following conclusion.

Corollary 4.2. *For any odd integer $d \geqslant 3$ and prime $p \equiv -1 \pmod d$ with $p \geqslant 2d-1$,*

$$\sum_{k=0}^{p-1} \frac{(\frac{d+1}{d})_k^{d-1}(\frac{1-d}{d})_k}{k!^d} \equiv \frac{d-1}{d^2}\Gamma_p(-\tfrac{1}{d})^d \pmod{p^2}.$$

Proof. For $n=p$, letting $q \to 1$ in Theorem 4.1, we obtain

$$\sum_{k=0}^{p-1} \frac{(\frac{d+1}{d})_k^{d-1}(\frac{1-d}{d})_k}{k!^d} \equiv -\frac{(d-1)\big(p-1-\frac{p+1}{d}\big)!}{d^2(\frac{p+1}{d})!^{d-1}} \pmod{p^2}.$$

It remains to show that

$$\frac{\big(p-1-\frac{p+1}{d}\big)!}{(\frac{p+1}{d})!^{d-1}} = -\Gamma_p(-\tfrac{1}{d})^d.$$

Let $\mathbb{Z}_p$ be the ring of all p-adic integers. For $x \in \mathbb{Z}_p$, the function $\Gamma_p(x)$ has the following properties (see [15, Theorem 14]):

(i) $\Gamma_p(x+1)/\Gamma_p(x) = -x$ unless $x \in p\mathbb{Z}_p$ in which case the quotient equals -1;

(ii) $\Gamma_p(x)\Gamma_p(1-x) = (-1)^{a_0(x)}$, where $a_0(x)$ denotes the least positive residue of x;

(iii) For any prime $p \geqslant 5$, there exists $G_1(x) \in \mathbb{Z}_p$ such that for any $y \in \mathbb{Z}_p$,

$$\Gamma_p(x+yp) \equiv \Gamma_p(x)(1+G_1(x)yp) \pmod{p^2}.$$

Set $m = \frac{p+1}{d}$. Then m is even. By (i), we have $\Gamma_p(-m) = 1/m!$ and $\Gamma_p(-p) = -1/(p-1)!$. Moreover, by (i) and (ii),

$$\frac{1}{(1-p)_m} = \frac{(-1)^m\Gamma_p(1-p)}{\Gamma_p(1-p+\frac{p+1}{d})} = \frac{(-1)^{m-1}\Gamma_p(-p)}{\Gamma_p(1+\frac{1}{d}+\frac{p}{d}-p)} = \Gamma_p(-\tfrac{1}{d}+(1-\tfrac{1}{d})p)\Gamma_p(-p).$$

It follows that

$$\frac{\big(p-1-\frac{p+1}{d}\big)!}{(\frac{p+1}{d})!^{d-1}} = \frac{(p-1)!}{(1-p)_m m!^{d-1}} = -\Gamma_p(-\tfrac{p+1}{d})^{d-1}\Gamma_p(-\tfrac{1}{d}+(1-\tfrac{1}{d})p)$$

$$\equiv -\Gamma_p(-\tfrac{1}{d})^d(1-G_1(-\tfrac{1}{d})\tfrac{1}{d}p)^{d-1}(1+G_1(-\tfrac{1}{d})(1-\tfrac{1}{d})p)$$
$$\equiv -\Gamma_p(-\tfrac{1}{d})^d \pmod{p^2}, \tag{4.4}$$

as desired. □

Theorem 4.3. *Let $d \geqslant 4$ be an even integer and let n be a positive integer with $n \equiv -1 \pmod{d}$. Then, modulo $\Phi_n(q)^2$,*

$$\sum_{k=0}^{n-1} \frac{(q^{d+1};q^d)_k^{d-2}(q;q^d)_k^2 q^{dk}}{(q^d;q^d)_k^d}$$
$$\equiv (-1)^{(n+1)/d} \frac{(1-q)^2(q^d;q^d)_{n-1-(n+1)/d}}{(q^d;q^d)_{(n+1)/d}^{d-1}} q^{(d(d+n)(n+1)-(n+1)^2-4d)/(2d)}. \tag{4.5}$$

Proof. This time we need to prove the following parametric generalization of (4.5): modulo $(1-aq^n)(a-q^n)$,

$$\sum_{k=0}^{n-1} \frac{(a^{d-1}q^{d+1}, a^{d-3}q^{d+1}, \dots, a^3q^{d+1}, aq, a^{-1}q, a^{-3}q^{d+1}, \dots, a^{3-d}q^{d+1}, a^{1-d}q^{d+1}; q^d)_k q^{dk}}{(a^{d-2}q^d, a^{d-4}q^d, \dots, a^{4-d}q^d, a^{2-d}q^d; q^d)_k (q^d;q^d)_k}$$
$$\equiv (-1)^{(n+1)/d} \frac{(1-aq)(1-q/a)(q^d;q^d)_{n-1-(n+1)/d}\, q^{(d(d+n)(n+1)-(n+1)^2-4d)/(2d)}}{(a^{d-2}q^d, a^{d-4}q^d, \dots, a^{4-d}q^d, a^{2-d}q^d; q^d)_{(n+1)/d}(q^d;q^d)_{(n+1)/d}}. \tag{4.6}$$

It is equivalent to say that both sides are equal for $a = q^{-n}$ and $a = q^n$. But this follows from the $q \mapsto q^d$, $N = n-1-(n+1)/d$, $m = d-1$, $b_j = q^{d-(d-2j)n}$ $(1 \leqslant j \leqslant d-1)$, $n_{(d-2)/2} = n_{d/2} = (n+1)/d-1$, and $n_j = (n+1)/d$ $(1 \leqslant j \leqslant d-1$ and $j \neq (d-2)/2, d/2)$ case of (2.1) and the relations

$$(q^{d-2n};q^d)_{(n+1)/d-1} = (q^{d-2n};q^d)_{(n+1)/d}/(1-q^{1-n}),$$
$$(q^d;q^d)_{(n+1)/d-1} = (q^d;q^d)_{(n+1)/d}/(1-q^{1+n}).$$

Like before, letting $a=1$ in (4.6), we are led to (4.5). □

Similarly, we have the following supercongruences.

Corollary 4.4. *For any even integer $d \geqslant 4$ and prime $p \equiv -1 \pmod{d}$,*

$$\sum_{k=0}^{p-1} \frac{(\frac{d+1}{d})_k^{d-2}(\frac{1}{d})_k^2}{k!^d} \equiv -\frac{\Gamma_p(-\frac{1}{d})^d}{d^2} \pmod{p^2}. \tag{4.7}$$

Proof. For $n = p$, taking $q \to 1$ in Theorem 4.3, we get

$$\sum_{k=0}^{p-1} \frac{(\frac{d+1}{d})_k^{d-2}(\frac{1}{d})_k^2}{k!^d} \equiv (-1)^{(p+1)/d} \frac{(p-1-\frac{p+1}{d})!}{d^2(\frac{p+1}{d})!^{d-1}} \pmod{p^2}.$$

This time ($(p+1)/d$ is not necessarily even), we have

$$\frac{\big(p-1-\frac{p+1}{d}\big)!}{\big(\frac{p+1}{d}\big)!^{d-1}} \equiv -(-1)^{(p+1)/d}\Gamma_p(-\tfrac{1}{d})^d \pmod{p^2} \tag{4.8}$$

for $p \geqslant 5$, and (4.7) holds in this case. On the other hand, for $d=4$ and $p=3$, we can readily check (4.7). □

Note that (4.7) also holds for $d=2$, which reduces to (1.1).

5. Concluding remarks and open problems

Wang and Pan [21] proved that, for $d \geqslant 3$, the supercongruence (1.4) is also true modulo p^3, which was originally conjectured by Deines–Fuselier–Long–Swisher–Tu [2]. However, the supercongruence (1.9) does not hold modulo p^3 for $d \geqslant 3$ in general. We do not know whether there is a complete q-analogue of Wang and Pan's result.

Motivated by Dwork's work [3] and Swisher's conjecture [20, (H.3)], we would like to propose the following Dwork-type supercongruence conjecture.

Conjecture 5.1. *Let $d \geqslant 3$ be an integer and let $p \equiv 1 \pmod{d}$ be a prime. Then, for $r \geqslant 1$,*

$$\sum_{k=0}^{p^r-1} \frac{(\frac{d-1}{d})_k^d}{k!^d} \equiv -\Gamma_p(\tfrac{1}{d})^d \sum_{k=0}^{p^{r-1}-1} \frac{(\frac{d-1}{d})_k^d}{k!^d} \pmod{p^{3r}}, \tag{5.1}$$

$$\sum_{k=0}^{(d-1)(p^r-1)/d} \frac{(\frac{d-1}{d})_k^d}{k!^d} \equiv -\Gamma_p(\tfrac{1}{d})^d \sum_{k=0}^{(d-1)(p^{r-1}-1)/d} \frac{(\frac{d-1}{d})_k^d}{k!^d} \pmod{p^{3r}}. \tag{5.2}$$

It should be pointed out that, for $d=2$, the supercongruences (5.1) and (5.2) only hold modulo p^{2r} in general, and this was recently confirmed by the author and Zudilin [12, Theorem 3.12] through establishing q-analogues of them. Moreover, letting $n=p^r$ and $q \to 1$ in Theorem 1.1, we see that the left-hand sides of (5.1) and (5.2) are congruent to

$$(-1)^{(p^r-1)/d}\frac{\big(\frac{(d-1)(p^r-1)}{d}\big)!}{\big(\frac{p^r-1}{d}\big)!^{d-1}} \pmod{p^2},$$

from which we can verify that (5.1) and (5.2) are true modulo p^2.

Numerical calculations suggest that the following two conjectures should be true.

Conjecture 5.2. *Let $d \geqslant 4$ be an even integer. Let n be a positive integer with $n \equiv -1 \pmod{d}$ and $n \geqslant 2d-1$. Then, modulo $\Phi_n(q)^2$,*

$$\sum_{k=0}^{n-1} \frac{(q^{d+1};q^d)_k^{d-1}(q^{1-d};q^d)_k q^{dk}}{(q^d;q^d)_k^d}$$

$$\equiv (-1)^{n-(n+1)/d}\frac{(1-q)(1-q^{d-1})(q^d;q^d)_{n-1-(n+1)/d}}{(q^d;q^d)_{(n+1)/d}^{d-1}}q^{(d(d+n)(n+1)-(n+1)^2-2d)/(2d)}. \quad (5.3)$$

In particular, for any prime $p\equiv -1 \pmod{d}$ *with* $p\geqslant 2d-1$,

$$\sum_{k=0}^{p-1}\frac{(\frac{d+1}{d})_k^{d-1}(\frac{1-d}{d})_k}{k!^d}\equiv \frac{d-1}{d^2}\Gamma_p(-\tfrac{1}{d})^d \pmod{p^2}. \quad (5.4)$$

Conjecture 5.3. *The* q*-supercongruence* (4.5) *also holds for any odd integer* $d\geqslant 3$. *In particular, for any odd prime* $p\equiv -1 \pmod{d}$,

$$\sum_{k=0}^{p-1}\frac{(\frac{d+1}{d})_k^{d-2}(\frac{1}{d})_k^2}{k!^d}\equiv -\frac{\Gamma_p(-\frac{1}{d})^d}{d^2} \pmod{p^2}. \quad (5.5)$$

For $d=2$, Conjectures 5.2 is true. In fact, noticing that $(1-q^{1-n})(1-q^{1+n})\equiv(1-q)^2 \pmod{\Phi_n(q)^2}$ (see [8, Section 4]), we see that (5.3) reduces to

$$\sum_{k=0}^{n-1}\frac{(q^3;q^2)_k(q^{-1},q^2)_k}{(q^2;q^2)_k^2}q^{2k}\equiv(-1)^{(n+1)/2}q^{(n^2+3)/4} \pmod{\Phi_n(q)^2},$$

which is a special case of [17, Theorem 1.1 (see Remark 1.1 with $x=q^{-d}$)]. Note that the parametric generalizations of Theorems 4.1 and 4.3 are symmetric in a and a^{-1}. However, it seems difficult to find such parametric generalizations of the q-supercongruences in Conjectures 5.2 and 5.3. We hope that an interested reader can make progress on these two conjectures, at least for the n prime and $q\to 1$ cases (5.4) and (5.5).

Data Availability Statements. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Acknowledgments. The author thanks the anonymous referees for helpful comments on this paper.

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