A new proof of the q-Dixon identity

Victor J. W. Guo

School of Mathematical Sciences, Huaiyin Normal University, Huai'an, Jiangsu 223300, People's Republic of China

jwguo@hytc.edu.cn

Abstract. We give a new and elementary proof of Jackson's terminating q-analogue of Dixon's identity by using recurrences and induction.

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1. Introduction

Jackson's terminating q-analogue of Dixon's identity [2, 8]:

$$\sum_{k=-a}^{a}(-1)^k q^{\frac{3k^2+k}{2}} \begin{bmatrix} a+b \\ a+k \end{bmatrix} \begin{bmatrix} b+c \\ b+k \end{bmatrix} \begin{bmatrix} c+a \\ c+k \end{bmatrix} = \begin{bmatrix} a+b+c \\ a+b \end{bmatrix} \begin{bmatrix} a+b \\ a \end{bmatrix}, \tag{1.1}$$

where the q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)(1-q^2)\cdots(1-q^k)(1-q)(1-q^2)\cdots(1-q^{n-k})}, & \text{if } 0 \leqslant k \leqslant n, \\ 0, & \text{otherwise,} \end{cases}$$

is an important identity in combinatorics and number theory. Note that Dixon's identity (see [8], [12, p. 43, (IV)], or [9, p. 11, (2.6)]) is the $q = 1$ case of (1.1). Several short proofs of the Dixon or q-Dixon identity can be found in [4–7]. The q-Dixon identity can also be deduced from the q-Pfaff-Saalschütz identity (see [7, 13]).

Recently, Mickic [10, 11] gave an elementary proof of Dixon's identity and some other binomial coefficient identities by using recurrences and induction. The aim of this note is to give a new proof of (1.1) by generalizing the argument of [10, 11].

2. Proof of (1.1)

For any integer n, let $[n]=\frac{1-q^n}{1-q}$. Denote the left-hand side of (1.1) by $S(a,b,c)$. We introduce two auxiliary sums as follows:

$$P(a,b,c):=\sum_{k=-a}^{a}(-1)^k q^{\frac{3k^2+k}{2}}[a-k][a+k]\begin{bmatrix}a+b\\a+k\end{bmatrix}\begin{bmatrix}b+c\\b+k\end{bmatrix}\begin{bmatrix}c+a\\c+k\end{bmatrix}, \tag{2.1}$$

$$Q(a,b,c):=\sum_{k=-a}^{a}(-1)^k q^{\frac{3k^2+k}{2}}[b-k][b+k]\begin{bmatrix}a+b\\a+k\end{bmatrix}\begin{bmatrix}b+c\\b+k\end{bmatrix}\begin{bmatrix}c+a\\c+k\end{bmatrix}. \tag{2.2}$$

It is easy to see that $[k]\begin{bmatrix}n\\k\end{bmatrix}=[n]\begin{bmatrix}n-1\\k-1\end{bmatrix}$, and so, for $a,b,c\geqslant 1$,

$$\begin{aligned}P(a,b,c)&=[a+b][a+c]\sum_{k=-a+1}^{a-1}(-1)^k q^{\frac{3k^2+k}{2}}\begin{bmatrix}a-1+b\\a-1+k\end{bmatrix}\begin{bmatrix}b+c\\b+k\end{bmatrix}\begin{bmatrix}c+a-1\\c+k\end{bmatrix},\\&=[a+b][a+c]S(a-1,b,c).\end{aligned} \tag{2.3}$$

Similarly, we have

$$Q(a,b,c)=[a+b][b+c]S(a,b-1,c). \tag{2.4}$$

It follows from (2.1) and (2.2) that

$$P(a,b,c)-Q(a,b,c)q^{a-b}=[a+b][a-b]S(a,b,c). \tag{2.5}$$

If $a\neq b$, then from (2.3)–(2.5) we deduce that

$$S(a,b,c)=\frac{1}{[a-b]}\left([a+c]S(a-1,b,c)-[b+c]S(a,b-1,c)q^{a-b}\right). \tag{2.6}$$

We need to consider the case when $a=b=c$ separately. Noticing the well known relations (see, for example [1, (3.3.3) and (3.3.4)])

$$\begin{bmatrix}n\\k\end{bmatrix}=\begin{bmatrix}n-1\\k\end{bmatrix}q^k+\begin{bmatrix}n-1\\k-1\end{bmatrix}=\begin{bmatrix}n-1\\k\end{bmatrix}+\begin{bmatrix}n-1\\k-1\end{bmatrix}q^{n-k},$$

we have

$$\begin{aligned}&S(a,a,a)\\&=\sum_{k=-a}^{a}(-1)^k q^{\frac{3k^2+k}{2}}\left(\begin{bmatrix}2a-1\\a+k\end{bmatrix}q^{a+k}+\begin{bmatrix}2a-1\\a+k-1\end{bmatrix}\right)\left(\begin{bmatrix}2a-1\\a+k\end{bmatrix}+\begin{bmatrix}2a-1\\a+k-1\end{bmatrix}q^{a-k}\right)^2\\&=\sum_{k=-a}^{a}(-1)^k q^{\frac{3k^2+k}{2}}\Bigg(\begin{bmatrix}2a-1\\a+k\end{bmatrix}^3q^{a+k}+\begin{bmatrix}2a-1\\a+k-1\end{bmatrix}^3q^{2a-2k}\\&\qquad+\begin{bmatrix}2a\\a+k\end{bmatrix}\begin{bmatrix}2a-1\\a+k\end{bmatrix}\begin{bmatrix}2a-1\\a+k-1\end{bmatrix}(1+q^{a-k}+q^{2a})\Bigg).\end{aligned} \tag{2.7}$$

By the symmetry of q-binomial coefficients, it is clear that

$$\sum_{k=-a}^{a}(-1)^k q^{\frac{3k^2+k}{2}} \begin{bmatrix} 2a-1 \\ a+k \end{bmatrix}^3 q^k = \sum_{k=-a}^{a-1}(-1)^k q^{\frac{3k^2+3k}{2}} \begin{bmatrix} 2a-1 \\ a+k \end{bmatrix}^3 = 0,$$

$$\sum_{k=-a}^{a}(-1)^k q^{\frac{3k^2+k}{2}} \begin{bmatrix} 2a-1 \\ a+k-1 \end{bmatrix}^3 q^{-2k} = \sum_{k=-a+1}^{a}(-1)^k q^{\frac{3k^2-3k}{2}} \begin{bmatrix} 2a-1 \\ a+k-1 \end{bmatrix}^3 = 0,$$

and

$$\begin{aligned}
&\sum_{k=-a}^{a}(-1)^k q^{\frac{3k^2+k}{2}} \begin{bmatrix} 2a \\ a+k \end{bmatrix} \begin{bmatrix} 2a-1 \\ a+k \end{bmatrix} \begin{bmatrix} 2a-1 \\ a+k-1 \end{bmatrix} q^{a-k} \\
&= \sum_{k=1-a}^{a-1}(-1)^k q^{\frac{3k^2-k}{2}} \begin{bmatrix} 2a \\ a+k \end{bmatrix} \begin{bmatrix} 2a-1 \\ a+k \end{bmatrix} \begin{bmatrix} 2a-1 \\ a+k-1 \end{bmatrix} q^{a} \\
&= q^a S(a,a,a-1).
\end{aligned}$$

Therefore, (2.7) implies that

$$S(a,a,a) = (1+q^a+q^{2a})S(a,a,a-1). \tag{2.8}$$

We now give a proof of (1.1) by induction on $a+b+c$. It is clear that (1.1) is true for $a=b=c=1$. Assume that (1.1) holds for all non-negative integers a, b and c with $a+b+c=n$. Let a, b and c be non-negative integers satisfying $a+b+c=n+1$. We consider three cases:

- If at least one of the numbers a, b and c is equal to 0, then (1.1) is obviously true.
- If $a=b=c$, then by the induction hypothesis, we have

  $$S(a,a,a-1) = \begin{bmatrix} 3a-1 \\ 2a \end{bmatrix} \begin{bmatrix} 2a \\ a \end{bmatrix}.$$

  Therefore, by (2.8), we obtain

  $$S(a,a,a) = (1+q^a+q^{2a}) \begin{bmatrix} 3a-1 \\ 2a \end{bmatrix} \begin{bmatrix} 2a \\ a \end{bmatrix} = \begin{bmatrix} 3a \\ 2a \end{bmatrix} \begin{bmatrix} 2a \\ a \end{bmatrix}.$$

- If $a \neq b$, then by (2.6) and the induction hypothesis, we get

  $$\begin{aligned}
  S(a,b,c) &= \frac{[a+c]}{[a-b]} \begin{bmatrix} a+b+c-1 \\ a+b-1 \end{bmatrix} \begin{bmatrix} a+b-1 \\ a-1 \end{bmatrix} \\
  &\qquad - \frac{[b+c]}{[a-b]} \begin{bmatrix} a+b+c-1 \\ a+b-1 \end{bmatrix} \begin{bmatrix} a+b-1 \\ a \end{bmatrix} q^{a-b} \\
  &= \begin{bmatrix} a+b+c \\ a+b \end{bmatrix} \begin{bmatrix} a+b \\ a \end{bmatrix},
  \end{aligned}$$

as desired. If $a = b$, then $a \neq c$, and we can do similarly as before by noticing the symmetry of a, b and c in $S(a, b, c)$.

Hence, (1.1) holds for $a + b + c = n + 1$, and by induction, it holds for all non-negative integers a, b and c.

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