New q-analogues of a congruence of Sun and Tauraso

Victor J. W. Guo

School of Mathematics and Statistics, Huaiyin Normal University, Huai'an 223300, Jiangsu, People's Republic of China

jwguo@math.ecnu.edu.cn

Abstract. Recently, by making use of a formula of Carlitz, the author confirmed a q-congruence conjectured by Tauraso. In this note we use Carlitz's formula again to establish two new q-analogues of a congruence of Sun and Tauraso. We also give another generalization of Sun and Tauraso's congruence, along with its q-analogue.

Keywords: congruence; q-congruence; cyclotomic polynomial; Carlitz's formula.

AMS Subject Classifications: 33D15; 11A07; 11B65

1. Introduction

In 2010, Sun and Tauraso [13] proved that, for any odd prime p and positive integer r,

$$\sum_{k=0}^{p^r-1} \frac{1}{2^k}\binom{2k}{k} \equiv (-1)^{(p^r-1)/2} \pmod{p}. \tag{1.1}$$

In the same year, Sun [12] showed that the above congruence is also true modulo p^2, i.e.,

$$\sum_{k=0}^{p^r-1} \frac{1}{2^k}\binom{2k}{k} \equiv (-1)^{(p^r-1)/2} \pmod{p^2}. \tag{1.2}$$

In 2019, using a q-identity of Carlitz [1], the author [3] established the following q-analogue of (1.2): for any positive odd integer n,

$$\sum_{k=0}^{n-1} \frac{(q;q^2)_k}{(q;q)_k} q^k \equiv (-1)^{(n-1)/2} q^{(n^2-1)/4} \pmod{\Phi_n(q)^2}, \tag{1.3}$$

where $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ denotes the *q-shifted factorial* and $\Phi_n(q)$ stands for the n-th *cyclotomic polynomial* in q, which may be defined as

$$\Phi_n(q) = \prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}} (q-\zeta^k),$$

where ζ is an n-th primitive root of unity. The q-congruence (1.3) was observed by Tauraso [14] for n being an odd prime, and its weaker form modulo $\Phi_n(q)$ was due to the author and Zeng [6, Corollary 4.2]. Some q-congruences related to (1.3) were given in [2, 15, 16], and some other recent progress on q-congruences can be found in [4, 5, 7, 8, 10, 11, 17].

In this note we shall give the following two new q-analogues of (1.1).

Theorem 1. *Let n be a positive odd integer. Then*

$$\sum_{k=0}^{n-1}\frac{(q;q^2)_k(-1;q^2)_k}{(q^2;q^2)_k}q^{2k}\equiv(-1)^{(n-1)/2}\pmod{\Phi_n(q)},\tag{1.4}$$

$$\sum_{k=0}^{n-1}\frac{(q;q^2)_k(-q^2;q^2)_k}{(q^2;q^2)_k}q^{2k+1}\equiv(-1)^{(n-1)/2}\pmod{\Phi_n(q)}.\tag{1.5}$$

Let $(a)_m=a(a+1)\dots(a+m-1)$ be the *rising factorial*. For any positive integer n and rational number x such that the denominator of x is coprime with n, we let $\langle x\rangle_n$ denote the *least non-negative residue* of x modulo n. We shall also give the following congruence: for all positive integers d,r, arbitrary integer s, and any prime p coprime with d,

$$\sum_{k=0}^{p^r-1}\frac{(\frac{s}{d})_k 2^k}{k!}\equiv(-1)^{\langle -s/d\rangle_{p^r}}\pmod{p}.\tag{1.6}$$

It is clear that the $(d,s)=(2,1)$ case of (1.6) reduces to (1.1).

More generally, the following q-analogue of (1.6) is true.

Theorem 2. *Let d, n and r be positive integers such that* $\gcd(d,n)=1$*, and let s be an arbitrary integer. Then*

$$\sum_{k=0}^{n-1}\frac{(q^{2s};q^{2d})_k}{(q^d;q^d)_k}q^{-(k+1)((k-2)d/2+2s)}=(-1)^{\langle -s/d\rangle_n}q^{(s-d)\langle (s-d)/d\rangle_n}\pmod{\Phi_n(q)}.\tag{1.7}$$

It is easy to see that (1.6) follows from (1.7) by taking $n=p^r$ and $q\to 1$.

We shall prove Theorems 1 and 2 in Sections 2 and 3 by making use of the aforementioned q-identity due to Carlitz [1]. In Section 4, we propose two conjectures on generalizations of (1.4) and (1.5) modulo $\Phi_n(q)^2$.

2. Proof of Theorem 1

Proof of (1.4). Letting $q\mapsto q^{-1}$, we see that the congruence (1.4) can be written as

$$\sum_{k=0}^{n-1}\frac{(q;q^2)_k(-1;q^2)_k}{(q^2;q^2)_k}q^{-k^2}\equiv(-1)^{(n-1)/2}\pmod{\Phi_n(q)}.\tag{2.1}$$

Moreover, taking $q\mapsto q^2$, $a=q$, $b=-1$, and $n\mapsto n-1$ in Carlitz's formula (see [1]):

$$\sum_{k=0}^{n}\frac{(a;q)_k(b;q)_k}{(q;q)_k}(-ab)^{n-k}q^{(n-k)(n+k-1)/2}=\sum_{k=0}^{n}\frac{(a;q)_{n+1}(-b)^k q^{\binom{k}{2}}}{(q;q)_k(q;q)_{n-k}(1-aq^{n-k})},\tag{2.2}$$

we obtain

$$\sum_{k=0}^{n-1} \frac{(q;q^2)_k(-1;q^2)_k}{(q^2;q^2)_k} q^{(n-1)^2-k^2} = \sum_{k=0}^{n-1} \frac{(q;q^2)_n q^{k^2-k}}{(q^2;q^2)_k(q^2;q^2)_{n-k-1}(1-q^{2n-2k-1})}. \tag{2.3}$$

It is clear that the q-shifted factorial $(q;q^2)_n$ contains the factor $1-q^n$ and is therefore divisible by $\Phi_n(q)$. Moreover, the product $(q^2;q^2)_k(q^2;q^2)_{n-k-1}$ is coprime with $\Phi_n(q)$ for k in the range $0 \leqslant k \leqslant n-1$. Thus, each summand on the right-hand side of (2.3) is congruent to 0 modulo $\Phi_n(q)$ except for $k=(n-1)/2$. Namely, modulo $\Phi_n(q)$, the identity (2.3) reduces to

$$\begin{aligned}\sum_{k=0}^{n-1} \frac{(q;q^2)_k(-1;q^2)_k}{(q^2;q^2)_k} q^{1-k^2} &\equiv \frac{(q;q^2)_n q^{(n-1)(n-3)/4}}{(q^2;q^2)^2_{(n-1)/2}(1-q^n)} \\ &= \begin{bmatrix} n-1 \\ \frac{n-1}{2} \end{bmatrix}_{q^2} \begin{bmatrix} 2n-1 \\ n-1 \end{bmatrix}_q \frac{q^{(n-1)(n-3)/4}}{(-q;q)^2_{n-1}},\end{aligned} \tag{2.4}$$

where the *q-binomial coefficients* ${n \brack k}_q$ are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} & \text{if } 0 \leqslant k \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

Now, a special case of a q-analogue of Morley's congruence [9, (1.5)] gives

$$\begin{bmatrix} n-1 \\ \frac{n-1}{2} \end{bmatrix}_{q^2} \equiv (-1)^{(n-1)/2} q^{(1-n^2)/4} (-q;q)^2_{n-1} \pmod{\Phi_n(q)^2} \tag{2.5}$$

(we only need the modulus $\Phi_n(q)$ case here), and, in view of $q^n \equiv 1 \pmod{\Phi_n(q)}$,

$$\begin{bmatrix} 2n-1 \\ n-1 \end{bmatrix}_q = \prod_{k=1}^{n-1} \frac{1-q^{2n-k}}{1-q^k} \equiv \prod_{k=1}^{n-1} \frac{1-q^{n-k}}{1-q^k} = 1 \pmod{\Phi_n(q)}. \tag{2.6}$$

Substituting the above two q-congruences into (2.4), we are led to (2.1). □

Proof of (1.5). The proof is similar to that of (1.4). Replacing q by q^{-1}, we can write the congruence (1.5) as

$$\sum_{k=0}^{n-1} \frac{(q;q^2)_k(-q^2;q^2)_k}{(q^2;q^2)_k} q^{-(k+1)^2} \equiv (-1)^{(n-1)/2} \pmod{\Phi_n(q)}. \tag{2.7}$$

Then we substitute $q \mapsto q^2$, $a=q$, $b=-q^2$, and $n \mapsto n-1$ in (2.2) to get

$$\sum_{k=0}^{n-1} \frac{(q;q^2)_k(-q^2;q^2)_k}{(q^2;q^2)_k} q^{n^2-(k+1)^2} = \sum_{k=0}^{n-1} \frac{(q;q^2)_n q^{k^2+k}}{(q^2;q^2)_k(q^2;q^2)_{n-k-1}(1-q^{2n-2k-1})}. \tag{2.8}$$

As before, the q-shifted factorial $(q;q^2)_n$ has the factor $1-q^n$, and $1-q^{2n-2k+1}\not\equiv 0 \pmod{\Phi_n(q)}$ for $0\leqslant k\leqslant n-1$ and $k\neq (n-1)/2$. Thus, modulo $\Phi_n(q)$, the identity (2.8) reduces to

$$\sum_{k=0}^{n-1}\frac{(q;q^2)_k(-q^2;q^2)_k}{(q^2;q^2)_k}q^{-(k+1)^2}\equiv\frac{(q;q^2)_n q^{(n^2-1)/4}}{(q^2;q^2)^2_{(n-1)/2}(1-q^n)}$$
$$=\begin{bmatrix}n-1\\ \frac{n-1}{2}\end{bmatrix}_{q^2}\begin{bmatrix}2n-1\\ n-1\end{bmatrix}_q\frac{q^{(n^2-1)/4}}{(-q;q)^2_{n-1}}.$$

The proof of (2.7) then follows from (2.5) and (2.6). □

3. Proof of Theorem 2

Taking $q\mapsto q^d$, $a=q^s$, $b=-q^s$, and $n\mapsto n-1$ in (2.2), we have

$$\sum_{k=0}^{n-1}\frac{(q^{2s};q^{2d})_k}{(q^d;q^d)_k}q^{(n-k-1)((n+k-2)d/2+2s)}=\sum_{k=0}^{n-1}\frac{(q^s;q^d)_n q^{sk+d\binom{k}{2}}}{(q^d;q^d)_k(q^d;q^d)_{n-k-1}(1-q^{dn-dk-d+s})}. \tag{3.1}$$

Since $\gcd(d,n)=1$, the q-shifted factorial $(q^s;q^d)_n$ contains a factor of the form $1-q^{an}$ (a is an integer) and is thus divisible by $\Phi_n(q)$. Meanwhile, the product $(q^d;q^d)_k(q^d;q^d)_{n-k-1}$ is coprime with $\Phi_n(q)$ for $0\leqslant k\leqslant n-1$. Hence, each summand on the right-hand side of (3.1) is congruent to 0 modulo $\Phi_n(q)$ except for $k=\langle (s-d)/d\rangle_n$. This means that, modulo $\Phi_n(q)$, the identity (3.1) reduces to

$$\sum_{k=0}^{n-1}\frac{(q^s;q^d)_k(-q^s;q^d)_k}{(q^d;q^d)_k}q^{-(k+1)(k-2)(d/2+2s)}$$
$$\equiv\frac{(q^s;q^d)_n q^{s\langle (s-d)/d\rangle_n+d\binom{\langle (s-d)/d\rangle_n}{2}}}{(q^d;q^d)_{\langle (s-d)/d\rangle_n}(q^d;q^d)_{n-\langle (s-d)/d\rangle_n-1}(1-q^{dn-d\langle (s-d)/d\rangle_n-d+s})}$$
$$=\begin{bmatrix}n-1\\ \langle (s-d)/d\rangle_n\end{bmatrix}_{q^d}q^{s\langle (s-d)/d\rangle_n+d\binom{\langle (s-d)/d\rangle_n}{2}}, \tag{3.2}$$

where we have used the fact that

$$\frac{(q^s;q^d)_n}{1-q^{dn-d\langle (s-d)/d\rangle_n-d+s}}\equiv (q;q)_{n-1}\equiv (q^d;q^d)_{n-1}\pmod{\Phi_n(q)}.$$

Note that $\langle -s/d\rangle_n\equiv\langle (s-d)/d\rangle_n \pmod 2$. The proof then follows from (3.2) and

$$\begin{bmatrix}n-1\\ k\end{bmatrix}_{q^d}=\prod_{j=1}^{k}\frac{1-q^{d(n-j)}}{1-q^{dj}}\equiv\prod_{j=1}^{k}\frac{1-q^{-dj}}{1-q^{dj}}=(-1)^k q^{-d\binom{k+1}{2}}\pmod{\Phi_n(q)}$$

with $k=\langle (s-d)/d\rangle_n$.

4. Open problems

Numerical calculation implies that the following generalizations of the q-congruences in Theorem 1 should be true.

Conjecture 1. *Let n be a positive odd integer. Then, modulo $\Phi_n(q)^2$,*

$$\sum_{k=0}^{n-1}\frac{(q;q^2)_k(-1;q^2)_k}{(q^2;q^2)_k}q^{2k}\equiv\begin{cases}(-1)^{(n-1)/2}q^{\binom{n}{2}}, & \text{if } n\equiv 1 \pmod 4,\\ (-1)^{(n-1)/2}q^{\binom{n+1}{2}}, & \text{if } n\equiv 3 \pmod 4.\end{cases} \tag{4.1}$$

Conjecture 2. *Let n be a positive odd integer. Then, modulo $\Phi_n(q)^2$,*

$$\sum_{k=0}^{n-1}\frac{(q;q^2)_k(-q^2;q^2)_k}{(q^2;q^2)_k}q^{2k+1}\equiv\begin{cases}(-1)^{(n-1)/2}q^{\binom{n+1}{2}}, & \text{if } n\equiv 1 \pmod 4,\\ (-1)^{(n-1)/2}q^{\binom{n}{2}}, & \text{if } n\equiv 3 \pmod 4.\end{cases} \tag{4.2}$$

Note that these two conjectures also give new q-analogues of (1.1). Although the q-congruences (4.1) and (4.2) are very similar to (1.3), the method of proving (1.3) given in [3] does not work here. Therefore, to prove (4.1) and (4.2) we need a new technique, which is left to the interested reader.

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