Some q-supercongruences related to Swisher's (H.3) conjecture

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Abstract. We first give a q-analogue of a supercongruence of Sun, which is a generalization of Van Hamme's (H.2) supercongruence for any prime $p \equiv 3 \pmod 4$. We also give a further generalization of this q-supercongruence, which may also be considered as a generalization of a q-supercongruence recently conjectured by the second author and Zudilin. Then, by combining these two q-supercongruences, we obtain q-analogues of the following two results: for any integer $d > 1$ and prime p with $p \equiv -1 \pmod{2d}$,

$$\sum_{k=0}^{(p^2-1)/d} \frac{(\frac{1}{d})_k^3}{k!^3} \equiv p^2 \pmod{p^4},$$
$$\sum_{k=0}^{p^2-1} \frac{(\frac{1}{d})_k^3}{k!^3} \equiv p^2 \pmod{p^4},$$

which are generalizations of Swisher's (H.3) conjecture modulo p^4 for $r = 2$. The key ingredients in our proof are the 'creative microscoping' method, the q-Dixon sum, Watson's terminating ${}_8\phi_7$ transformation, and properties of the p-adic Gamma function.

Keywords: q-supercongruences; cyclotomic polynomial; q-Dixon sum; Watson's transformation; p-adic Gamma function.

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1. Introduction

In 1997, Van Hamme [25] listed 13 supercongruences related to truncated forms of Ramanujan's and Ramanujan-like formulas for $1/\pi$. Van Hamme himself proved three of them, including the following supercongruence [25, (H.2)]: for any prime $p \equiv 3 \pmod 4$,

$$\sum_{k=0}^{(p-1)/2} \frac{(\frac{1}{2})_k^3}{k!^3} \equiv 0 \pmod{p^2}, \tag{1.1}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol. Since $(\frac{1}{2})_k/k! \equiv 0 \pmod p$ for $(p+1)/2 \leqslant k \leqslant p-1$, we may compute the sum in (1.1) for k up to

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$p-1$. In recent years, all kinds of generalizations of (1.1) have been given by different authors [4, 9, 11, 12, 14–16, 19, 22, 23]. For example, Sun [23, Theorem 1.3] proved that, for any integer $d>1$ and prime p with $p\equiv -1 \pmod{2d}$,

$$\sum_{k=0}^{p-1}\frac{(\frac{1}{d})_k^3}{k!^3}\equiv 0 \pmod{p^2}. \tag{1.2}$$

In 2016, Swisher [24, (H.3) with $r=2$] conjectured that, for primes $p\equiv 3 \pmod 4$ and $p>3$,

$$\sum_{k=0}^{(p^2-1)/2}\frac{(\frac{1}{2})_k^3}{k!^3}\equiv p^2 \pmod{p^5}, \tag{1.3}$$

Motivated by Sun's supercongruence (1.2) and Swisher's conjecture (1.3), we shall prove the following results: for any integer $d>1$ and prime p with $p\equiv -1 \pmod{2d}$,

$$\sum_{k=0}^{(p^2-1)/d}\frac{(\frac{1}{d})_k^3}{k!^3}\equiv p^2 \pmod{p^4}, \tag{1.4}$$

$$\sum_{k=0}^{p^2-1}\frac{(\frac{1}{d})_k^3}{k!^3}\equiv p^2 \pmod{p^4}. \tag{1.5}$$

Note that the $d=3,4$ cases of the supercongruence (1.4) were already observed by He [13, Theorems 1.1 and 1.2]. However, He's proofs of them are incorrect due to errors in his derivations of (3.2) and (3.8) in [13].

It is known that many supercongruences have nice q-analogues, and mathematicians may have more ways to deal with q-congruences than to treat classical supercongruences. Recently, the second author and Zudilin [10] devised a method, called 'creative microscoping', to prove plenty of q-congruences. For other recent progress on q-congruences, the reader may consult [3, 4, 7–9, 11, 12, 16–18, 20, 27–29].

In this paper, we shall give a q-analogue of (1.2) in the following theorem. Note that the $d=2$ case was already obtained by the second author and Zudilin [12, Theorem 1.1].

Theorem 1.1. *Let d and n be positive integers with $d>1$. Then, modulo $\Phi_n(q)^2$,*

$$\sum_{k=0}^{n-1}\frac{(1+q^{2dk+1})(q^2;q^{2d})_k^3}{(1+q)(q^{2d};q^{2d})_k^3}q^{(2d-3)k}$$
$$\equiv\begin{cases}\dfrac{[(d-1)n]_{q^2}(q^3;q^{2d})_{(dn-n-1)/d}}{(q^{2d+1};q^{2d})_{(dn-n-1)/d}}q^{(1+n-dn)/d}, & \text{if } n\equiv -1 \pmod d,\\[2ex] \dfrac{[n]_{q^2}(q^3;q^{2d})_{(n-1)/d}}{(q^{2d+1};q^{2d})_{(n-1)/d}}q^{(1-n)/d}, & \text{if } n\equiv 1 \pmod d.\end{cases} \tag{1.6}$$

At the moment, we already need to be familiar with the standard basic hypergeometric notation. The *q-shifted factorial* is defined as $(a;q)_0 = 1$ and $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n \geqslant 1$ or $n=\infty$. The *q-integer* is given by $[n]=[n]_q = 1+q+\cdots+q^{n-1}$, and $\Phi_n(q)$ stands for the n-th *cyclotomic polynomial* in q, which may be defined by

$$\Phi_n(q) = \prod_{\substack{1\leqslant k\leqslant n\\ \gcd(n,k)=1}} (q-\zeta^k),$$

where ζ is an n-th primitive root of unity.

We now assume that $n \equiv -1 \pmod{2d}$. Then $(q^3;q^{2d})_{(dn-n-1)/d}$ contains the factor $1-q^{(2d-3)n}$. Since $1-q^n \equiv 0 \pmod{\Phi_n(q)}$, we always have $[(d-1)n]_{q^2} \equiv (q^3;q^{2d})_{(dn-n-1)/d} \equiv 0 \pmod{\Phi_n(q)}$, while $(q^{2d+1};q^{2d})_{(dn-n-1)/d}$ is relatively prime to $\Phi_n(q)$. By the first case of (1.6), we have

$$\sum_{k=0}^{n-1} \frac{(1+q^{2dk+1})(q^2;q^{2d})_k^3}{(1+q)(q^{2d};q^{2d})_k^3} q^{(2d-3)k} \equiv 0 \pmod{\Phi_n(q)^2}.$$

Thus, if we further let $n=p$ be a prime and take $q\to 1$, then we arrive at (1.2).

We shall also prove the following result, which is a generalization of [3, Theorem 1.1] (or equivalently, [11, Conjuecture 2]).

Theorem 1.2. *Let $d>1$ be an integer and let n be a positive integer with $n \equiv -1 \pmod{2d}$. Then, for any positive integer m,*

$$\sum_{k=0}^{mn-1} \frac{(1+q^{2dk+1})(q^2;q^{2d})_k^3}{(1+q)(q^{2d};q^{2d})_k^3} q^{(2d-3)k} \equiv 0 \pmod{\Phi_n(q)^2}, \tag{1.7}$$

$$\sum_{k=0}^{mn+(dn-n-1)/d} \frac{(1+q^{2dk+1})(q^2;q^{2d})_k^3}{(1+q)(q^{2d};q^{2d})_k^3} q^{(2d-3)k} \equiv 0 \pmod{\Phi_n(q)^2}. \tag{1.8}$$

With the help of Theorems 1.1 and 1.2, we shall prove (1.4) and (1.5) by establishing the following q-analogues of them.

Theorem 1.3. *Let $d>1$ be an integer and let n be a positive integer with $n \equiv -1 \pmod{2d}$. Then, modulo $\Phi_n(q)^2\Phi_{n^2}(q)^2$,*

$$\sum_{k=0}^{(n^2-1)/d} \frac{(1+q^{2dk+1})(q^2;q^{2d})_k^3}{(1+q)(q^{2d};q^{2d})_k^3} q^{(2d-3)k} \equiv \frac{[n^2]_{q^2}(q^3;q^{2d})_{(n^2-1)/d}}{(q^{2d+1};q^{2d})_{(n^2-1)/d}} q^{(1-n^2)/d}, \tag{1.9}$$

$$\sum_{k=0}^{n^2-1} \frac{(1+q^{2dk+1})(q^2;q^{2d})_k^3}{(1+q)(q^{2d};q^{2d})_k^3} q^{(2d-3)k} \equiv \frac{[n^2]_{q^2}(q^3;q^{2d})_{(n^2-1)/d}}{(q^{2d+1};q^{2d})_{(n^2-1)/d}} q^{(1-n^2)/d}. \tag{1.10}$$

The proofs of Theorems 1.1–1.3 will be given in Sections 2–4, respectively.

Let $n = p$ be a prime and take $q \to 1$ in Theorem 1.3. Then $\Phi_p(1) = \Phi_{p^2}(1) = p$, and the left-hand sides of (1.9) and (1.10) reduce to those of (1.4) and (1.5), respectively. Moreover, the right-hand sides of (1.9) and (1.10) become

$$\lim_{q\to 1} \frac{[n^2]_{q^2}(q^3;q^{2d})_{(p^2-1)/d}}{(q^{2d+1};q^{2d})_{(p^2-1)/d}} = p^2 \frac{(\frac{3}{2d})_{(p^2-1)/d}}{(\frac{2d+1}{2d})_{(p^2-1)/d}}.$$

To illustrate that (1.9) and (1.10) reduce to (1.4) and (1.5), respectively, we need to prove the following result.

Theorem 1.4. *Let $d > 1$ be an integer and let p be a prime with $p \equiv -1 \pmod{2d}$. Then*

$$\frac{(\frac{3}{2d})_{(p^2-1)/d}}{(\frac{2d+1}{2d})_{(p^2-1)/d}} \equiv 1 \pmod{p^2}. \tag{1.11}$$

Our proof of Theorem 1.4 is similar to Wang and Pan's proof for the $d = 2$ case in [26]. For the reader's convenience, we will give a detailed proof in Section 5.

2. Proof of Theorem 1.1

We need to establish a parametric generalization of Theorem 1.1. The following is the $n \equiv -1 \pmod{d}$ case.

Theorem 2.1. *Let $d > 1$ be an integer and let n be a positive integer with $n \equiv -1 \pmod{d}$. Then, modulo $(1 - aq^{(2d-2)n})(a - q^{(2d-2)n})$,*

$$\sum_{k=0}^{n-1} \frac{(1+q^{2dk+1})(aq^2;q^{2d})_k(q^2/a;q^{2d})_k(q^2;q^{2d})_k}{(1+q)(aq^{2d};q^{2d})_k(q^{2d}/a;q^{2d})_k(q^{2d};q^{2d})_k} q^{(2d-3)k}$$
$$\equiv \frac{[(d-1)n]_{q^2}(q^3;q^{2d})_{(dn-n-1)/d}}{(q^{2d+1};q^{2d})_{(dn-n-1)/d}} q^{(1+n-dn)/d}. \tag{2.1}$$

Proof. Making the parameter substitutions $q \mapsto q^{2d}$, $a \mapsto q^2$, $b \mapsto bq^2$ and $c \mapsto cq^2$ in the q-Dixon sum [2, Appendix (II.13)], we have

$$\sum_{k=0}^{\infty} \frac{(1+q^{2dk+1})(q^2;q^{2d})_k(bq^2;q^{2d})_k(cq^2;q^{2d})_k}{(1+q)(q^{2d}/b;q^{2d})_k(q^{2d}/c;q^{2d})_k(q^{2d};q^{2d})_k}\left(\frac{q^{2d-3}}{bc}\right)^k$$
$$= \frac{(q^{2d+2};q^{2d})_\infty(q^{2d-1}/b;q^{2d})_\infty(q^{2d-1}/c;q^{2d})_\infty(q^{2d-2}/bc;q^{2d})_\infty}{(q^{2d}/b;q^{2d})_\infty(q^{2d}/c;q^{2d})_\infty(q^{2d+1};q^{2d})_\infty(q^{2d-3}/bc;q^{2d})_\infty}. \tag{2.2}$$

Since $n\equiv -1 \pmod d$, putting $b=q^{-(2d-2)n}$ and $c=q^{(2d-2)n}$ in (2.2) we conclude that the left-hand side terminates and is equal to

$$\sum_{k=0}^{(dn-n-1)/d}\frac{(1+q^{2dk+1})(q^{2-(2d-2)n};q^{2d})_k(q^{2+(2d-2)n};q^{2d})_k(q^2;q^{2d})_k}{(1+q)(q^{2d-(2d-2)n};q^{2d})_k(q^{2d+(2d-2)n};q^{2d})_k(q^{2d};q^{2d})_k}q^{(2d-3)k}$$

$$=\sum_{k=0}^{n-1}\frac{(1+q^{2dk+1})(q^{2-(2d-2)n};q^{2d})_k(q^{2+(2d-2)n};q^{2d})_k(q^2;q^{2d})_k}{(1+q)(q^{2d-(2d-2)n};q^{2d})_k(q^{2d+(2d-2)n};q^{2d})_k(q^{2d};q^{2d})_k}q^{(2d-3)k},$$

while the right-hand side becomes

$$\frac{(q^{2d-1-(2d-2)n};q^{2d})_{(dn-n-1)/d}(q^{2d+2};q^{2d})_{(dn-n-1)/d}}{(q^{2d-(2d-2)n};q^{2d})_{(dn-n-1)/d}(q^{2d+1};q^{2d})_{(dn-n-1)/d}}$$

$$=\frac{(1-q^{(2d-2)n})(q^3;q^{2d})_{(dn-n-1)/d}}{(1-q^2)(q^{2d+1};q^{2d})_{(dn-n-1)/d}}q^{(1+n-dn)/d}.$$

This implies that the q-congruence (2.1) is true modulo $1-aq^{(2d-2)n}$ and $a-q^{(2d-2)n}$. □

We now give the $n\equiv 1 \pmod d$ case.

Theorem 2.2. *Let $d>1$ be an integer and let n be a positive integer with $n\equiv 1 \pmod d$. Then, modulo $(1-aq^{2n})(a-q^{2n})$,*

$$\sum_{k=0}^{n-1}\frac{(1+q^{2dk+1})(aq^2;q^{2d})_k(q^2/a;q^{2d})_k(q^2;q^{2d})_k}{(1+q)(aq^{2d};q^{2d})_k(q^{2d}/a;q^{2d})_k(q^{2d};q^{2d})_k}q^{(2d-3)k}\equiv\frac{[n]_{q^2}(q^3;q^{2d})_{(n-1)/d}}{(q^{2d+1};q^{2d})_{(n-1)/d}}q^{(1-n)/d}. \tag{2.3}$$

Proof. Since $n\equiv 1 \pmod d$, letting $b=q^{-2n}$ and $c=q^{2n}$ in (2.2) we see that the left-hand side terminates and is equal to

$$\sum_{k=0}^{(n-1)/d}\frac{(1+q^{2dk+1})(q^{2-2n};q^{2d})_k(q^{2+2n};q^{2d})_k(q^2;q^{2d})_k}{(1+q)(q^{2d-2n};q^{2d})_k(q^{2d+2n};q^{2d})_k(q^{2d};q^{2d})_k}q^{(2d-3)k}$$

$$=\sum_{k=0}^{n-1}\frac{(1+q^{2dk+1})(q^{2-2n};q^{2d})_k(q^{2+2n};q^{2d})_k(q^2;q^{2d})_k}{(1+q)(q^{2d-2n};q^{2d})_k(q^{2d+2n};q^{2d})_k(q^{2d};q^{2d})_k}q^{(2d-3)k},$$

while the right-hand side reduces to

$$\frac{(q^{2d-1-2n};q^{2d})_{(n-1)/d}(q^{2d+2};q^{2d})_{(n-1)/d}}{(q^{2d-2n};q^{2d})_{(n-1)/d}(q^{2d+1};q^{2d})_{(n-1)/d}}=\frac{(1-q^{2n})(q^3;q^{2d})_{(n-1)/d}}{(1-q^2)(q^{2d+1};q^{2d})_{(n-1)/d}}q^{(1-n)/d}.$$

This proves the q-congruence (2.3). □

Proof of Theorem 1.1. Let $a=1$ in (2.1). Then the left-hand side of (2.1) reduces to the left-hand side of (1.6), and the denominators of the left-hand side are relatively prime to $\Phi_n(q)$ since $\gcd(d,n)=1$. Moreover, the modulus $(1-aq^{(2d-2)n})(a-q^{(2d-2)n})$ becomes $(1-q^{(2d-2)n})^2$, which has the factor $\Phi_n(q)^2$. This proves the first case of (1.6). Similarly, letting $a=1$ in (2.3), we are led to the second case of (1.6). □

3. Proof of Theorem 1.2

Recall that Watson's terminating ${}_8\phi_7$ transformation formula (see [2, Section 2] and [2, Appendix (III.18)]) can be stated as follows:

$$
\begin{aligned}
&{}_8\phi_7\left[\begin{matrix} a, & qa^{\frac12}, & -qa^{\frac12}, & b, & c, & d, & e, & q^{-n} \\ & a^{\frac12}, & -a^{\frac12}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1}\end{matrix}; q, \frac{a^2q^{n+2}}{bcde}\right] \\
&\quad = \frac{(aq;q)_n(aq/de;q)_n}{(aq/d;q)_n(aq/e;q)_n}\, {}_4\phi_3\left[\begin{matrix} aq/bc,\ d,\ e,\ q^{-n} \\ aq/b,\ aq/c,\ deq^{-n}/a\end{matrix}; q, q\right],
\end{aligned} \tag{3.1}
$$

where the basic hypergeometric ${}_{r+1}\phi_r$ series is defined by

$$
{}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, \dots, b_r\end{matrix}; q, z\right] := \sum_{k=0}^{\infty} \frac{(a_1;q)_k(a_2;q)_k\dots(a_{r+1};q)_k}{(q;q)_k(b_1;q)_k\cdots(b_r;q)_k} z^k.
$$

We write the left-hand side of (1.8) with $m \geqslant 0$ as a terminating basic hypergeometric series:

$$
{}_8\phi_7\left[\begin{matrix} q^2, & q^{2d+1}, & -q^{2d+1}, & q^2, & q, & q^2, & q^{2d+(2dm+2d-2)n}, & q^{2-(2dm+2d-2)n} \\ & q, & -q, & q^{2d}, & q^{2d+1}, & q^{2d}, & q^{2-(2dm+2d-2)n}, & q^{2d+(2dm+2d-2)n}\end{matrix}; q^{2d}, q^{2d-3}\right]. \tag{3.2}
$$

By Watson's transformation (3.1) with $q \mapsto q^{2d}$, $a = b = d = q^2$, $c = q$, $e = q^{2d+(2dm+2d-2)n}$, and $n \mapsto mn + (dn - n - 1)/d$, the series (3.2) is equal to

$$
\begin{aligned}
&\frac{(q^{2d+2};q^{2d})_{mn+(dn-n-1)/d}(q^{-(2dm+2d-2)n};q^{2d})_{mn+(dn-n-1)/d}}{(q^{2d};q^{2d})_{mn+(dn-n-1)/d}(q^{2-(2dm+2d-2)n};q^{2d})_{mn+(dn-n-1)/d}} \\
&\quad\times {}_4\phi_3\left[\begin{matrix} q^{2d-1}, & q^2, & q^{2d+(2dm+2d-2)n}, & q^{2-(2dm+2d-2)n} \\ & q^{2d}, & q^{2d+1}, & q^{2d+2}\end{matrix}; q^{2d}, q^{2d}\right].
\end{aligned} \tag{3.3}
$$

It is easy to see that there are exactly $m+1$ factors of the form $1 - q^{an}$ (a is an integer) among the $mn + (dn - n - 1)/d$ factors of $(q^{2d+2};q^{2d})_{mn+(dn-n-1)/d}$. So are the q-shifted factorial $(q^{-(2dm+2d-2)n};q^{2d})_{mn+(dn-n-1)/d}$. But both $(q^{2d};q^{2d})_{mn+(dn-n-1)/d}$ and $(q^{2-(2dm+2d-2)n};q^{2d})_{mn+(dn-n-1)/d}$ have merely m factors of the form $1 - q^{an}$. Since $\Phi_n(q)$ is a factor of $1 - q^N$ if and only if N is a multiple of n, we deduce that the fraction before the ${}_4\phi_3$ series is congruent to 0 modulo $\Phi_n(q)^2$.

For any integer x, let $f_{d,n}(x)$ be the least non-negative integer k such that $(q^x;q^{2d})_k \equiv 0$ modulo $\Phi_n(q)$. Since $n \equiv -1 \pmod{2d}$, we have $f_{d,n}(2) = (dn-n+d-1)/d$, $f_{d,n}(2d-1) = (n+1)/(2d)$, $f_{d,n}(2d) = n$, $f_{d,n}(2d+1) = (2dn-n-1)/(2d)$, $f_{d,n}(2d+2) = (dn-n-1)/d$, and so

$$
f_{d,n}(2d-1) < f_{d,n}(2d+2) < f_{d,n}(2) \leqslant f_{d,n}(2d+1) < f_{d,n}(2d).
$$

This means that the denominator of the reduced form of the k-th term

$$
\frac{(q^{2d-1};q^{2d})_k(q^2;q^{2d})_k(q^{2d+(2dm+2d-2)n};q^{2d})_k(q^{2-(2dm+2d-2)n};q^{2d})_k}{(q^{2d};q^{2d})_k^2(q^{2d+1};q^{2d})_k(q^{2d+2};q^{2d})_k} q^{2dk}
$$

in the ${}_4\phi_3$ series is always relatively prime to $\Phi_n(q)$ for $k \geqslant 0$. Therefore, the expression (3.3) (i.e. (3.2)) is congruent to 0 modulo $\Phi_n(q)^2$, thus establishing (1.8) for $m \geqslant 0$.

Finally, we observe that $(q^2;q^{2d})_k/(q^{2d};q^{2d})_k$ is congruent to 0 modulo $\Phi_n(q)$ for $mn+(dn-n-1)/d < k \leqslant (m+1)n-1$. The proof of (1.7) with $m \mapsto m+1$ then follows from (1.8) immediately.

4. Proof of Theorem 1.3

Since $n \equiv -1 \pmod{2d}$, we have $n^2 \equiv 1 \pmod{d}$. By the second case of (1.6), the q-congruence (1.10) is true modulo $\Phi_{n^2}(q)^2$. It is easy to see that $(q^2;q^{2d})_k \equiv 0 \pmod{\Phi_{n^2}(q)}$ for $(n^2-1)/d < k \leqslant n-1$, we conclude that (1.9) is also true modulo $\Phi_{n^2}(q)^2$.

It is easily seen that, for $n \equiv -1 \pmod{2d}$,

$$\frac{[n^2]_{q^2}(q^3;q^{2d})_{(n^2-1)/d}}{(q^{2d+1};q^{2d})_{(n^2-1)/d}}q^{(1-n^2)/d} \equiv 0 \pmod{\Phi_n(q)^2}$$

because $[n^2]_{q^2} = (1-q^{2n^2})/(1-q^2)$ is divisible by $\Phi_n(q)$, and $(q^3;q^{2d})_{(n^2-1)/d}$ has $(n+1)/d$ factors of the form $1-q^{an}$ (a is an integer), while $(q^{2d+1};q^{2d})_{(n^2-1)/d}$ only contains $(n-d+1)/d$ such factors. Moreover, in view of Theorem 1.2, the left-hand sides of (1.9) and (1.10) are both congruent to 0 modulo $\Phi_n(q)^2$ since $(n^2-1)/d = (n-d+1)n/d+(dn-n-1)/d$. It follows that the q-congruences (1.9) and (1.10) are true modulo $\Phi_n(q)^2$. Since $\Phi_n(q)$ and $\Phi_{n^2}(q)$ are relatively prime polynomials, we complete the proof of the theorem.

5. Proof of Theorem 1.4

We first recall some basic properties of Morita's p-adic Gamma function [1,21]. Let p be an odd prime. For any integer $n \geq 1$, the p-adic Gamma function is defined by

$$\Gamma_p(n) := (-1)^n \prod_{\substack{0<k<n \\ p\nmid k}} k.$$

In particular, set $\Gamma_p(0) = 1$. Let $\mathbb{Z}_p$ denote the ring of all p-adic integers. Extend Γ_p to all $x \in \mathbb{Z}_p$ by defining

$$\Gamma_p(x) = \lim_{x_n \to x} \Gamma_p(x_n),$$

where x_n is any sequence of positive integers that p-adically approaches x. By the definition of p-adic Gamma function, we have

$$\frac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} -x, & p \nmid x, \\ -1, & p \mid x. \end{cases} \tag{5.1}$$

It is also known that, for any $x \in \mathbb{Z}_p$,

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{a_0(x)}, \tag{5.2}$$

where $a_0(x) \in \{1, 2, \ldots, p\}$ satisfies $a_0(x) \equiv x \pmod{p}$.

To prove Theorem 1.4, we also require the following result (see [19, Theorem 14]).

Lemma 5.1. *For any odd prime p and $a, m \in \mathbb{Z}_p$, we have*

$$\Gamma_p(a + mp) \equiv \Gamma_p(a) + \Gamma_p'(a)mp \pmod{p^2}. \tag{5.3}$$

Proof of Theorem 1.4. Let $\Gamma(x)$ be the classical Gamma function. By (5.1), we have

$$\begin{aligned}
\frac{(\frac{3}{2d})_{(p^2-1)/d}}{(\frac{2d+1}{2d})_{(p^2-1)/d}} &= \frac{\Gamma(\frac{2p^2+1}{2d})\Gamma(\frac{2d+1}{2d})}{\Gamma(\frac{3}{2d})\Gamma(\frac{2p^2+2d-1}{2d})} \\
&= \frac{\frac{(2d-3)p}{2d}\cdot\frac{(4d-3)p}{2d}\cdot\cdots\cdot\frac{(2p-1)p}{2d}}{\frac{(2d-1)p}{2d}\cdot\frac{(4d-1)p}{2d}\cdot\cdots\cdot\frac{(2p-2d+1)p}{2d}}\cdot\frac{\Gamma_p(\frac{2p^2+1}{2d})\Gamma_p(\frac{2d+1}{2d})}{\Gamma_p(\frac{3}{2d})\Gamma_p(\frac{2p^2+2d-1}{2d})} \\
&\equiv \frac{p(\frac{2d-3}{2d})_{(p+1)/d}}{(\frac{2d-1}{2d})_{(p-d+1)/d}}\cdot\frac{\Gamma_p(\frac{1}{2d})\Gamma_p(\frac{2d+1}{2d})}{\Gamma_p(\frac{3}{2d})\Gamma_p(\frac{2d-1}{2d})} \pmod{p^2},
\end{aligned}$$

and

$$\begin{aligned}
\frac{p(\frac{2d-3}{2d})_{(p+1)/d}}{(\frac{2d-1}{2d})_{(p-d+1)/d}} &= \frac{p(-1)^{(p+1)/d}\Gamma_p(\frac{2p+2d-1}{2d})\Gamma_p(\frac{2d-1}{2d})}{\frac{p}{2d}(-1)^{(p-d+1)/d}\Gamma_p(\frac{2d-3}{2d})\Gamma_p(\frac{2p+1}{2d})} \\
&= -\frac{2d\Gamma_p(\frac{2p+2d-1}{2d})\Gamma_p(\frac{2d-1}{2d})}{\Gamma_p(\frac{2d-3}{2d})\Gamma_p(\frac{2p+1}{2d})}.
\end{aligned}$$

It follows from (5.1)–(5.3) that

$$\begin{aligned}
\frac{(\frac{3}{2d})_{(p^2-1)/d}}{(\frac{2d+1}{2d})_{(p^2-1)/d}} &\equiv \frac{\Gamma_p(\frac{2p+2d-1}{2d})\Gamma_p(\frac{1}{2d})^2}{\Gamma_p(\frac{2d-3}{2d})\Gamma_p(\frac{2p+1}{2d})\Gamma_p(\frac{3}{2d})} \\
&\equiv \frac{\Gamma_p(\frac{2p+2d-1}{2d})\Gamma_p(\frac{2d-1-2p}{2d})(-1)^{\frac{p+1}{2d}}\Gamma_p(\frac{1}{2d})^2}{(-1)^{\frac{(2d-3)(p+1)}{2d}}} \\
&\equiv \Gamma_p\left(\frac{2d-1}{2d}\right)^2\Gamma_p\left(\frac{1}{2d}\right)^2 = 1 \pmod{p^2},
\end{aligned}$$

as desired. □

6. An open problem

Motivated by Swisher's conjectural supercongruence (1.3), it is natural to propose the following conjecture, which is also a generalization of [13, Conjecture 1.3].

Conjecture 6.1. *For any integer $d > 1$ and prime p with $p \equiv -1 \pmod{2d}$, we have*

$$\sum_{k=0}^{(p^2-1)/d} \frac{(\frac{1}{d})_k^3}{k!^3} \equiv p^2 \pmod{p^5},$$

$$\sum_{k=0}^{p^2-1} \frac{(\frac{1}{d})_k^3}{k!^3} \equiv p^2 \pmod{p^5}.$$

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