PROOF OF A SUPERCONGRUENCE CONJECTURED BY SUN THROUGH A q-MICROSCOPE

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Abstract. Recently, Z.-W. Sun made the following conjecture: for any odd prime p and odd integer m,

$$\frac{1}{m^2\binom{m-1}{(m-1)/2}}\left(\sum_{k=0}^{(mp-1)/2}\frac{\binom{2k}{k}}{8^k}-\left(\frac{2}{p}\right)\sum_{k=0}^{(m-1)/2}\frac{\binom{2k}{k}}{8^k}\right)\equiv 0 \pmod{p^2}.$$

In this note, applying the "creative microscoping" method, introduced by the author and Zudilin, we confirm the above conjecture of Sun.

1. Introduction

During the past decade, congruences and supercongruences have been studied by quite a few authors. In 2011, Z.-W. Sun [14, (1.6)] proved that, for any odd prime p,

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{8^k}\equiv\left(\frac{2}{p}\right)+\left(\frac{-2}{p}\right)\frac{p^2}{4}E_{p-3} \pmod{p^3},$$

where $\left(\frac{\cdot}{\cdot}\right)$ denotes the Jacobi symbol and E_n is the n-th Euler number. Later, he [15, (1.7)] further proved that

$$\sum_{k=0}^{(p^r-1)/2}\frac{\binom{2k}{k}}{8^k}\equiv\left(\frac{2}{p^r}\right) \pmod{p^2}. \tag{1.1}$$

Recently, Sun [16, Conjecture 4(ii)] also proposed the following conjecture: for any odd prime p and odd integer m,

$$\frac{1}{m^2\binom{m-1}{(m-1)/2}}\left(\sum_{k=0}^{(mp-1)/2}\frac{\binom{2k}{k}}{8^k}-\left(\frac{2}{p}\right)\sum_{k=0}^{(m-1)/2}\frac{\binom{2k}{k}}{8^k}\right)\equiv 0 \pmod{p^2}, \tag{1.2}$$

which is clearly a generalization of (1.1).

In the past few years, q-analogues of congruences and supercongruences have caught the interests of a lot of people (see [3–13, 17]). In particular, the author and

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Liu [8] gave the following q-analogue of (1.1): for odd $n>1$,

$$\sum_{k=0}^{(n-1)/2} q^{k^2}\frac{(q;q^2)_k}{(q^4;q^4)_k} \equiv (-q)^{(1-n^2)/8} \pmod{\Phi_n(q)^2}. \tag{1.3}$$

Here and in what follows, $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$, $n=0,1,\ldots$, or $n=\infty$, is the *q-shifted factorial*, and $\Phi_n(q)$ is the *n-th cyclotomic polynomial* in q given by

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(n,k)=1}}(q-\zeta^k),$$

where ζ is an n-th primitive root of unity. Moreover, Gu and the author [4] gave some different q-analogues of (1.1), such as

$$\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k q^{2k}}{(q^2;q^2)_k(-q;q^2)_k} \equiv \left(\frac{2}{n}\right) q^{2\lfloor (n+1)/4\rfloor^2} \pmod{\Phi_n(q)^2}, \tag{1.4}$$

where $\lfloor x\rfloor$ denotes the largest integer not exceeding x. In order to prove Sun's conjecture (1.2), we need the following new q-analogue of (1.1).

Theorem 1.1. *Let $n>1$ be an odd integer. Then*

$$\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k(-1;q^4)_k}{(-q;q^2)_k(q^4;q^4)_k}q^{2k} \equiv \left(\frac{2}{n}\right) \pmod{\Phi_n(q)^2}. \tag{1.5}$$

Recall that the *q-integer* is defined by $[n]_q=1+q+\cdots+q^{n-1}$ and the *q-binomial coefficient* ${m \brack n}_q$ is defined as

$${m \brack n}_q=\begin{cases}\dfrac{(q;q)_m}{(q;q)_n(q;q)_{m-n}}, & \text{if } 0\leqslant n\leqslant m,\\ 0, & \text{otherwise.}\end{cases}$$

On the basis of (1.5), we are able to give the following q-analogue of (1.2).

Theorem 1.2. *Let m and n be positive odd integers with $n>1$. Then*

$$\begin{aligned}\frac{1}{[m]_{q^n}^2{m-1 \brack (m-1)/2}_{q^n}}\Bigg(&\sum_{k=0}^{(mn-1)/2}\frac{(q;q^2)_k(-1;q^4)_k}{(-q;q^2)_k(q^4;q^4)_k}q^{2k}\\ &-\left(\frac{2}{n}\right)\sum_{k=0}^{(m-1)/2}\frac{(q^n;q^{2n})_k(-1;q^{4n})_k}{(-q^n;q^{2n})_k(q^{4n};q^{4n})_k}q^{2nk}\Bigg)\\ \equiv 0 \pmod{\Phi_n(q)^2}.&\end{aligned} \tag{1.6}$$

Moreover, the denominator of (the reduced form of) the left-hand side of (1.6) is relatively prime to $\Phi_{n^j}(q)$ for any index $j\geqslant 2$.

It is well known that $\Phi_n(1)=p$ if n is a prime power p^r $(r\geqslant 1)$ and $\Phi_n(1)=1$ otherwise. Moreover, the denominator of (1.6) is no doubt a product of cyclotomic polynomials. This means that (1.2) immediately follows from (1.6) by letting $n=p$ and taking the limits as $q\to 1$.

We shall prove Theorem 1.1 in the next section. The proof of Theorem 1.2 will be given in Section 3 by using the method of "creative microscoping" recently introduced by the author and Zudilin [10]. More precisely, we shall first give a generalization of Theorem 1.2 with an extra parameter a, and Theorem 1.2 then follows from this generalization by taking $a\to 1$. Finally, in Section 4, we give some hints on solving another similar conjecture of Sun.

2. Proof of Theorem 1.1

It is easy to check that

$$(1-q^{n-2j+1})(1-q^{n+2j-1})+(1-q^{2j-1})^2q^{n-2j+1}=(1-q^n)^2,$$

and so

$$(1-q^{n-2j+1})(1-q^{n+2j-1})\equiv -(1-q^{2j-1})^2q^{n-2j+1}\pmod{\Phi_n(q)^2}$$

since $1-q^n\equiv 0\pmod{\Phi_n(q)}$. Thus, we have

$$\begin{aligned}(q^{1-n};q^2)_k(q^{1+n};q^2)_k&=(-1)^kq^{k^2-nk}\prod_{j=1}^{k}(1-q^{n-2j+1})(1-q^{n+2j-1})\\&\equiv q^{k^2-nk}\prod_{j=1}^{k}(1-q^{2j-1})^2q^{n-2j+1}\\&=(q;q^2)_k^2\pmod{\Phi_n(q)^2}.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k(-1;q^4)_k}{(-q;q^2)_k(q^4;q^4)_k}q^{2k}&\equiv\sum_{k=0}^{(n-1)/2}\frac{(q^{1-n};q^2)_k(q^{1+n};q^2)_k(-1;q^4)_k}{(q;q^2)_k(-q;q^2)_k(q^4;q^4)_k}q^{2k}\\&=\left(\frac{2}{n}\right)\pmod{\Phi_n(q)^2}.\end{aligned}\tag{2.1}$$

Here the last step in (2.1) follows from a terminating q-analogue of Whipple's ${}_3F_2$ sum [2, Appendix (II.19)]:

$$\begin{aligned}&\sum_{k=0}^{n}\frac{(q^{-n};q)_k(q^{n+1};q)_k(c;q)_k(-c;q)_k}{(e;q)_k(c^2q/e;q)_k(q;q)_k(-q;q)_k}q^k\\&\quad=\frac{(eq^{-n};q^2)_\infty(eq^{n+1};q^2)_\infty(c^2q^{1-n}/e;q^2)_\infty(c^2q^{n+2}/e;q^2)_\infty}{(e;q)_\infty(c^2q/e;q)_\infty}q^{n(n+1)/2}\end{aligned}$$

with $n\mapsto (n-1)/2$, $q\mapsto q^2$, $c^2=-1$ and $e=q$.

3. Proof of Theorem 1.2

We first establish the following parametric generalization of Theorem 1.2.

Theorem 3.1. *Let m and n be positive odd integers with $n>1$. Then, modulo*

$$\prod_{j=0}^{(m-1)/2}(1-aq^{(2j+1)n})(a-q^{(2j+1)n}), \tag{3.1}$$

we have

$$\sum_{k=0}^{(mn-1)/2}\frac{(aq;q^2)_k(q/a;q^2)_k(-1;q^4)_k}{(q;q^2)_k(-q;q^2)_k(q^4;q^4)_k}q^{2k}$$
$$\equiv\left(\frac{2}{n}\right)\sum_{k=0}^{(m-1)/2}\frac{(aq^n;q^{2n})_k(q^n/a;q^{2n})_k(-1;q^{4n})_k}{(q^n;q^{2n})_k(-q^n;q^{2n})_k(q^{4n};q^{4n})_k}q^{2nk}. \tag{3.2}$$

Proof. It suffices to prove that both sides of (3.2) are identical for $a=q^{-(2j+1)n}$ and $a=q^{(2j+1)n}$ with $j=0,1,\dots,(m-1)/2$, i.e.,

$$\sum_{k=0}^{(mn-1)/2}\frac{(q^{1-(2j+1)n};q^2)_k(q^{1+(2j+1)n};q^2)_k(-1;q^4)_k}{(q;q^2)_k(-q;q^2)_k(q^4;q^4)_k}q^{2k}$$
$$=\left(\frac{2}{n}\right)\sum_{k=0}^{(m-1)/2}\frac{(q^{-2jn};q^{2n})_k(q^{2-2jn};q^{2n})_k(-1;q^{4n})_k}{(q^n;q^{2n})_k(-q^n;q^{2n})_k(q^{4n};q^{4n})_k}q^{2nk}. \tag{3.3}$$

Clearly, $(mn-1)/2\geqslant((2j+1)n-1)/2$ for $0\leqslant j\leqslant(m-1)/2$, and $(q^{1-(2j+1)n};q^2)_k=0$ for $k>((2j+1)n-1)/2$. By the identity in (2.1), the left-hand side of (3.3) is equal to $\left(\frac{2}{(2j+1)n}\right)$. Likewise, the right-hand side of (3.3) is equal to

$$\left(\frac{2}{n}\right)\left(\frac{2}{2j+1}\right)=\left(\frac{2}{(2j+1)n}\right),$$

where $\left(\frac{2}{1}\right)$ is understood to be 1. This establishes the identity (3.3), and so the q-congruence (3.2) holds. $\square$

We now can prove Theorem 1.2.

Proof of Theorem 1.2. It is easy to see that

$$q^N-1=\prod_{d\mid N}\Phi_d(q),$$

and there are $\lfloor m/n^{j-1}\rfloor-\lfloor(m-1)/(2n^{j-1})\rfloor$ multiples of n^{j-1} in the arithmetic progression $1,3,\dots,m$ for any positive integer j. Thus, the limit of (3.1) as $a\to1$ has the factor

$$\prod_{j=1}^{\infty}\Phi_{n^j}(q)^{2\lfloor m/n^{j-1}\rfloor-2\lfloor(m-1)/(2n^{j-1})\rfloor}.$$

On the other hand, the denominator of the left-hand side of (3.2) is divisible by that of the right-hand side of (3.2). The former is equal to $(q^2;q^2)_{mn-1}$ and its factor related to $\Phi_n(q), \Phi_{n^2}(q), \dots$ is just

$$\prod_{j=1}^{\infty} \Phi_{n^j}(q)^{\lfloor (mn-1)/n^j \rfloor}.$$

Moreover, writing $[m]_q = (q;q)_m/((1-q)(q;q)_{m-1})$, the q-binomial coefficient ${m-1 \brack (m-1)/2}$ as a product of cyclotomic polynomials (see, for example, [1]), and then using the fact $\Phi_{n^j}(q^n) = \Phi_{n^{j+1}}(q)$, we know that the polynomial $[m]_{q^n}^2 {m-1 \brack (m-1)/2}_{q^n}$ only has the following factor

$$\prod_{j=2}^{\infty} \Phi_{n^j}(q)^{2\lfloor m/n^{j-1} \rfloor - \lfloor (m-1)/n^{j-1} \rfloor - 2\lfloor (m-1)/(2n^{j-1}) \rfloor}$$

related to $\Phi_n(q), \Phi_{n^2}(q), \dots$.

It is clear that

$$2\lfloor m/n^{j-1} \rfloor - 2\lfloor (m-1)/(2n^{j-1}) \rfloor - \lfloor (mn-1)/n^j \rfloor = 2 \quad \text{for } j=1,$$

and

$$\lfloor (mn-1)/n^j \rfloor = \lfloor (m-1)/n^{j-1} \rfloor \quad \text{for } j \geqslant 1.$$

Therefore, letting $a \to 1$ in (3.2), we see that the q-congruence (1.6) holds, and the denominator of the left-hand side of (1.6) is relatively prime to $\Phi_{n^j}(q)$ for $j \geqslant 2$, as desired. $\square$

4. Concluding remarks

Sun [16, Conjecture 4(ii)] also made the following conjecture: for any odd prime p and odd integer m,

$$\frac{1}{m^2 \binom{m-1}{(m-1)/2}} \left(\sum_{k=0}^{(mp-1)/2} \frac{\binom{2k}{k}}{16^k} - \left(\frac{3}{p}\right) \sum_{k=0}^{(m-1)/2} \frac{\binom{2k}{k}}{16^k} \right) \equiv 0 \pmod{p^2}, \tag{4.1}$$

of which the $m=1$ case was already proved by Sun [15] himself. Although Gu and the author [4] gave the following q-analogue of (4.1) for $m=1$: for odd $n>1$,

$$\sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k q^{2k}}{(q^4;q^4)_k (-q;q^2)_k} \equiv \left(\frac{3}{n}\right) q^{(n^2-1)/12} \pmod{\Phi_n(q)^2}, \tag{4.2}$$

we cannot utilize (4.2) to give a q-analogue of (4.1) similar to Theorem 1.2 because $(n^2-1)/12$ is not a linear function of n. Anyway, we believe that such a q-analogue of (4.1) should exist, which is left to the interested reader.

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