PROOF OF A SUPERCONGRUENCE MODULO p^{2r}

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Abstract. Employing Watson's terminating ${}_8\phi_7$ transformation, we present a q-analogue of the following supercongruence: for any prime $p \equiv 1 \pmod 4$ and positive integer r,

$$\sum_{k=0}^{p^r-1} \frac{4k+1}{64^k} \binom{2k}{k}^3 \equiv 0 \pmod{p^{2r}},$$

which was conjectured by Z.-W. Sun in 2011, thus confirming Sun's conjecture. Further, applying a very-well-poised ${}_6\phi_5$ summation and the creative microscoping method introduced by the author and Zudilin, we extend this supercongruence to the modulo p^{2r+1} case. We also give some similar results for primes $p \equiv 3 \pmod 4$. Finally, we propose two conjectures on relevant supercongruences for further study.

1. Introduction

In 1997, Van Hamme [20, (B.2), (H.2)] proposed the following supercongruence: for any odd prime p,

$$\sum_{k=0}^{(p-1)/2} \frac{4k+1}{64^k} \binom{2k}{k}^3 \equiv (-1)^{(p-1)/2} p \pmod{p^3}, \tag{1.1}$$

$$\sum_{k=0}^{(p-1)/2} \frac{1}{64^k} \binom{2k}{k}^3 \equiv \begin{cases} -\Gamma_p(\frac{1}{4})^4 \pmod{p^2}, & \text{if } p \equiv 1 \pmod 4, \\ 0 \pmod{p^2}, & \text{if } p \equiv 3 \pmod 4, \end{cases} \tag{1.2}$$

where $\Gamma_p(x)$ is the p-adic Gamma function (see [13]). It is easy to see that (1.1) and (1.2) also hold when these two sums are over k up to $p-1$, since $\binom{2k}{k} \equiv 0 \pmod p$ for $(p-1)/2 < k \leqslant p-1$. The supercongruence (1.1) was first proved by Mortenson [12] using a ${}_6F_5$ transformation, and later received a Wilf–Zeilberger (WZ) proof by Zudilin [27] with the WZ pair borrowed from [2]. The supercongruence (1.2) was established by Van Hamme himself, and was extended to the modulus p^3 case by Long and Ramakrishna [11]. For some recent generalizations of (1.2), we refer to the reader to [10, 14, 21, 25, 26].

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In 2011, Z.-W. Sun [16, Conjecture 5.9] made the following conjecture: for any prime $p\equiv 1 \pmod 4$ and positive integer r,

$$\sum_{k=0}^{p^r-1}\frac{4k+1}{64^k}\binom{2k}{k}^3\equiv 0 \pmod{p^{2r}}. \tag{1.3}$$

In 2013, Z.-W. Sun's twin brother Z.-H. Sun [15, Theorem 3.5] proved that, for any odd prime p, modulo p^2,

$$\sum_{k=0}^{(p-1)/2}\frac{k}{64^k}\binom{2k}{k}^3\equiv\begin{cases}\frac{p}{2}-x^2, & \text{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ (2p-2+2^{p-1})\binom{(p-3)/2}{(p-3)/4}^2, & \text{if } p\equiv 3 \pmod 4.\end{cases} \tag{1.4}$$

Note that, for $p=x^2+4y^2\equiv 1 \pmod 4$,

$$-\Gamma_p(\tfrac14)^4\equiv\frac{1}{2^{p-1}}\binom{(p-1)/2}{(p-1)/4}^2\equiv 4x^2-2p \pmod{p^2}$$

(see [3, 19]). Combining (1.2) and (1.4) leads to the following congruence: for any prime $p\equiv 1 \pmod 4$,

$$\sum_{k=0}^{(p-1)/2}\frac{4k+1}{64^k}\binom{2k}{k}^3\equiv 0 \pmod{p^2}. \tag{1.5}$$

Therefore, Z.-H. Sun [15] has proved (1.3) for $r=1$, though he has not mentioned this explicitly in his paper. In 2017, He [9] reproved the $r=1$ case of (1.3) in a different way. For any p-adic integer x, let $\langle x\rangle_p$ stand for the least nonnegative residue of x modulo p. Recently, Wang and Z.-W. Sun [24, Corollary 1.1] proved the following general conjecture [17, Conjecture 19] of Z.-W. Sun which clearly implies (1.5): for any odd prime p and positive integer b with $p\equiv \pm 1 \pmod b$ and $\langle -1/b\rangle_p\equiv 0 \pmod 2$,

$$\sum_{k=0}^{p-1}(b^2k+b-1)\frac{(\frac12)_k(\frac1b)_k(1-\frac1b)_k}{k!^3}\equiv 0 \pmod{p^2},$$

where $(a)_n=a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol. However, Z.-W. Sun's original conjecture (1.3) still remains open so far.

In this paper, we first prove the following results.

Theorem 1.1. *Let $p\equiv 1 \pmod 4$ be a prime and let $r\geqslant 1$. Then*

$$\sum_{k=0}^{(p^r-1)/2}\frac{4k+1}{64^k}\binom{2k}{k}^3\equiv (-1)^r p^{2r}\Gamma_p(\tfrac34)^{4r} \pmod{p^{2r+1}}, \tag{1.6}$$

$$\sum_{k=0}^{p^r-1}\frac{4k+1}{64^k}\binom{2k}{k}^3\equiv (-1)^r p^{2r}\Gamma_p(\tfrac34)^{4r} \pmod{p^{2r+1}}. \tag{1.7}$$

It is clear that the supercongruences (1.6) and (1.7) modulo p^{2r} reduce to (1.3) and its companion: for any prime $p\equiv 1 \pmod 4$ and positive integer r,

$$\sum_{k=0}^{(p^r-1)/2}\frac{4k+1}{64^k}\binom{2k}{k}^3\equiv 0 \pmod{p^{2r}}. \tag{1.8}$$

We shall also prove the following similar supercongruences.

Theorem 1.2. *Let $p\equiv 3 \pmod 4$ be a prime and let $r\geqslant 1$. Then*

$$\sum_{k=0}^{(p^{2r}-1)/d}\frac{4k+1}{64^k}\binom{2k}{k}^3\equiv p^{2r} \pmod{p^{2r+1}}, \tag{1.9}$$

$$\sum_{k=0}^{(p^{2r-1}-1)/d}\frac{4k+1}{64^k}\binom{2k}{k}^3\equiv -16p^{2r-2}\Gamma_p(\tfrac{3}{4})^4 \pmod{p^{2r}}, \tag{1.10}$$

where $d=1,2$.

The paper is arranged as follows. In the next section, we shall give q-analogues of (1.3) and (1.8) by using Watson's terminating ${}_8\phi_7$ transformation. In Section 3, we shall give q-analogues of (1.6) and (1.7) by employing a very-well-poised ${}_6\phi_5$ summation and the creative microscoping method devised by the author and Zudilin [7]. Then we give a proof of Theorem 1.1 from its q-analogue and properties of the p-adic Gamma function in Section 4, and give a proof of Theorem 1.2 in Section 5. Finally, in Section 6, we put forward some open problems on related supercongruences for further study.

2. q-Analogues of (1.3) and (1.8)

Throughout the paper, the *q-shifted factorial* is defined by $(a;q)_0=1$ and $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n=1,2,\dots$, and the n-th *cyclotomic polynomial* $\Phi_n(q)$ is given by

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(n,k)=1}}(q-\zeta^k),$$

where ζ is an n-th primitive root of unity. Furthermore, the *q-integer* is defined as $[n]=[n]_q=1+q+\cdots+q^{n-1}$.

In order to present q-analogues of (1.3) and (1.8), we first give the following q-congruences.

Theorem 2.1. *Let m and n be positive integers with $n\equiv 1 \pmod 4$ and $n>1$. Then*

$$\sum_{k=0}^{mn-1}[4k+1]_{q^2}\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3}q^{-k}\equiv 0 \pmod{\Phi_n(q)^2}, \tag{2.1}$$

$$\sum_{k=0}^{mn+(n-1)/2}[4k+1]_{q^2}\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3}q^{-k}\equiv 0 \pmod{\Phi_n(q)^2}. \tag{2.2}$$

Proof. Recall that Watson's terminating ${}_8\phi_7$ transformation (see, for example, [4, Appendix (III.18)]) can be stated as follows:

$$
\begin{aligned}
&{}_8\phi_7\left[\begin{matrix} a, & qa^{\frac12}, & -qa^{\frac12}, & b, & c, & d, & e, & q^{-n} \\ & a^{\frac12}, & -a^{\frac12}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1}\end{matrix}; q, \frac{a^2q^{n+2}}{bcde}\right] \\
&\quad = \frac{(aq;q)_n(aq/de;q)_n}{(aq/d;q)_n(aq/e;q)_n}\, {}_4\phi_3\left[\begin{matrix} aq/bc, \ d,\ e,\ q^{-n} \\ aq/b,\ aq/c,\ deq^{-n}/a\end{matrix}; q,\ q\right],
\end{aligned}
\tag{2.3}
$$

where the basic hypergeometric ${}_{r+1}\phi_r$ series is defined by

$$
{}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, \dots, b_r\end{matrix}; q, z\right] := \sum_{k=0}^{\infty} \frac{(a_1;q)_k(a_2;q)_k\dots(a_{r+1};q)_k}{(q;q)_k(b_1;q)_k\cdots(b_r;q)_k} z^k.
$$

We can write the left-hand side of (2.2) with $m \geqslant 0$ as a terminating ${}_8\phi_7$ series:

$$
{}_8\phi_7\left[\begin{matrix} q^2, & q^5, & -q^5, & q^2, & q^3, & q^2, & q^{4+(4m+2)n}, & q^{2-(4m+2)n} \\ & q, & -q, & q^4, & q^3, & q^4, & q^{2-(4m+2)n}, & q^{4+(4m+2)n}\end{matrix}; q^4,\ q^{-1}\right]. \tag{2.4}
$$

Performing the parameter substitutions $q \mapsto q^4$, $a = b = d = q^2$, $c = q^3$, $e = q^{4+(4m+2)n}$, and $n \mapsto mn + (n-1)/2$ in Watson's ${}_8\phi_7$ transformation (2.3), we see that (2.4) is equal to

$$
\begin{aligned}
&\frac{(q^6;q^4)_{mn+(n-1)/2}(q^{-(4m+2)n};q^4)_{mn+(n-1)/2}}{(q^4;q^4)_{mn+(n-1)/2}(q^{2-(4m+2)n};q^4)_{mn+(n-1)/2}} \\
&\quad\times\ {}_4\phi_3\left[\begin{matrix} q,\ q^2,\ q^{4+(4m+2)n},\ q^{2-(4m+2)n} \\ q^4,\quad q^3,\quad q^6\end{matrix}; q^4,\ q^4\right].
\end{aligned}
\tag{2.5}
$$

It is not hard to see that there are just $m+1$ factors of the form $1-q^{an}$ with a being an integer in the $mn+(n-1)/2$ factors of $(q^6;q^4)_{mn+(n-1)/2}$. The q-shifted factorial $(q^{-(4m+2)n};q^4)_{mn+(n-1)/2}$ has the same property. However, there are merely m factors of the form $1-q^{an}$ with integral a in each of $(q^4;q^4)_{mn+(n-1)/2}$ and $(q^{2-(4m+2)n};q^4)_{mn+(n-1)/2}$. Note that $\Phi_n(q)$ is a factor of $1-q^N$ if and only if N is divisible by n. Hence, the fraction in front of the ${}_4\phi_3$ series is congruent to 0 modulo $\Phi_n(q)^2$. For any integer x, let $f_n(x)$ denote the minimum positive integer k such that $(q^x;q^4)_k \equiv 0$ modulo $\Phi_n(q)$. In view of $n \equiv 1 \pmod 4$ and $n > 1$, we have $f_n(1) = (n+3)/4$, $f_n(2) = (n+1)/2$, $f_n(3) = (3n+1)/4$, $f_n(4) = n$, and $f_n(6) = (n-1)/2$. This implies that the denominator of the reduced form of the fraction

$$
\frac{(q;q^4)_k(q^2;q^4)_k(q^{4+(4m+2)n};q^4)_k(q^{2-(4m+2)n};q^4)_k}{(q^4;q^4)_k^2(q^3;q^4)_k(q^6;q^4)_k} q^{4k}
$$

is always coprime with $\Phi_n(q)$ for non-negative integers k. Therefore, the expression (2.5) (namely, (2.4)) is congruent to 0 modulo $\Phi_n(q)^2$, confirming (2.2) for $m \geqslant 0$.

It is obvious that $(q^2;q^4)_k^3/(q^4;q^4)_k^3$ is congruent to 0 modulo $\Phi_n(q)^3$ for $mn+(n-1)/2 < k \leqslant (m+1)n-1$. Thus, the q-congruence (2.1) after replacing m by $m+1$ follows from (2.2) immediately. □

From Theorem 2.1 we can easily deduce the following q-analogues of (1.3) and (1.8).

Corollary 2.2. *Let n and r be positive integers with $n \equiv 1 \pmod 4$ and $n > 1$. Then*

$$\sum_{k=0}^{(n^r-1)/2} [4k+1]_{q^2} \frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3} q^{-k} \equiv 0 \pmod{\prod_{j=1}^{r} \Phi_{n^j}(q)^2}, \tag{2.6}$$

$$\sum_{k=0}^{n^r-1} [4k+1]_{q^2} \frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3} q^{-k} \equiv 0 \pmod{\prod_{j=1}^{r} \Phi_{n^j}(q)^2}. \tag{2.7}$$

Proof. For $r = 1$, the q-congruences (2.6) and (2.7) follow from Theorem 2.1 immediately, since (2.2) is also true for $m = 0$. For $r \geqslant 2$ and $1 \leqslant j \leqslant r$, letting $m = n^{r-j}$ and $n \mapsto n^j$ in (2.1), one sees that (2.7) holds modulo $\Phi_{n^j}(q)^2$, Meanwhile, writing $(n^r - 1)/2 = (n^{r-j} - 1)n^j/2 + (n^j - 1)/2$, one sees that (2.6) also holds modulo $\Phi_{n^j}(q)^2$. Since $\Phi_n(q)^2, \Phi_{n^2}(q)^2, \dots, \Phi_{n^r}(q)^2$ are pairwise coprime polynomials, we conclude that the q-congruences (2.6) and (2.7) hold. ☐

Let $n = p$ be a prime in Corollary 2.2. Taking the limits as $q \to 1$ on both sides of (2.6) and (2.7), and employing the fact that $\Phi_{p^j}(1) = p$ for all positive integers j, we get (1.8) and (1.3), respectively.

3. q-Analogues of (1.6) and (1.7)

Recall that a very-well-poised ${}_6\phi_5$ summation (see [4, Appendix (II.21)]) can be stated as follows:

$${}_6\phi_5\left[\begin{matrix} a, & qa^{\frac12}, & -qa^{\frac12}, & b, & c, & q^{-n} \\ & a^{\frac12}, & -a^{\frac12}, & aq/b, & aq/c, & aq^{n+1} \end{matrix} ; q, \frac{aq^{n+1}}{bc}\right] = \frac{(aq;q)_n (aq/bc;q)_n}{(aq/b;q)_n (aq/c;q)_n}. \tag{3.1}$$

We first use (3.1) and the creative microscoping method [7] to establish the following parametric q-congruence.

Theorem 3.1. *Let $n \equiv 1 \pmod 4$ be a positive integer. Let a be an indeterminate. Then, modulo $\Phi_n(q^2)(1 - aq^{2n})(a - q^{2n})$,*

$$\sum_{k=0}^{(n-1)/2} [4k+1]_{q^2} \frac{(aq^2;q^4)_k (q^2/a;q^4)_k (q^2;q^4)_k}{(aq^4;q^4)_k (q^4/a;q^4)_k (q^4;q^4)_k} q^{-k} \equiv q^{-3(n-1)/2} [n]_{q^2} \frac{(q^5;q^4)_{(n-1)/2}}{(q^3;q^4)_{(n-1)/2}}. \tag{3.2}$$

Proof. Making the substitutions $q \mapsto q^4$, $a = q^2$, $b = q^3$, $c = q^{2+2n}$, and $n \mapsto (n-1)/2$ in (3.1), we obtain

$$\sum_{k=0}^{(n-1)/2} [4k+1]_{q^2} \frac{(q^2;q^4)_k (q^{2+2n};q^4)_k (q^{2-2n};q^4)_k}{(q^4;q^4)_k (q^{4-2n};q^4)_k (q^{4+2n};q^4)_k} q^{-k}$$
$$= \frac{(q^6;q^4)_{(n-1)/2} (q^{1-2n};q^4)_{(n-1)/2}}{(q^3;q^4)_{(n-1)/2} (q^{4-2n};q^4)_{(n-1)/2}}$$

$$= q^{-3(n-1)/2}[n]_{q^2}\frac{(q^5;q^4)_{(n-1)/2}}{(q^3;q^4)_{(n-1)/2}}.$$

Namely, the two sides of (3.2) are equal when $a=q^{\pm 2n}$. Therefore, the q-congruence (3.2) holds modulo $1-aq^{2n}$ and $a-q^{2n}$.

In view of [6, Lemma 3.1], for $0\leqslant k\leqslant (n-1)/2$, we have

$$\frac{(aq^2;q^4)_{(n-1)/2-k}}{(q^4/a;q^4)_{(n-1)/2-k}} \equiv (-a)^{(n-1)/2-2k}\frac{(aq^2;q^4)_k}{(q^4/a;q^4)_k}q^{(n-1)^2/2+2k} \pmod{\Phi_n(q^2)},$$

and so

$$\begin{aligned}
&[4(m-k)+1]_{q^2}\frac{(aq^2;q^4)_{m-k}(q^2/a;q^4)_{m-k}(q^2;q^4)_{m-k}}{(aq^4;q^4)_{m-k}(q^4/a;q^4)_{m-k}(q^4;q^4)_{m-k}}q^{-(m-k)} \\
&\equiv -[4k+1]_{q^2}\frac{(aq^2;q^4)_k(q^2/a;q^4)_k(q^2;q^4)_k}{(aq^4;q^4)_k(q^4/a;q^4)_k(q^4;q^4)_k}q^{-k} \pmod{\Phi_n(q^2)},
\end{aligned}$$

where $m=(n-1)/2$ and we have used the fact $q^{2n}\equiv 1 \pmod{\Phi_n(q^2)}$ and the condition $n\equiv 1 \pmod 4$. This means that the k-th and $(m-k)$-th summands on the left-hand side of (3.2) cancel each other modulo $\Phi_n(q)$ and so the left-hand side of (3.2) is congruent to 0 modulo $\Phi_n(q^2)$. Since the right-hand side of (3.2) is also congruent to 0 modulo $\Phi_n(q^2)$, we conclude that (3.2) holds modulo $\Phi_n(q^2)$.

The proof of (3.2) then follows from the fact that $\Phi_n(q^2)$, $1-aq^{2n}$, and $a-q^{2n}$ are pairwise coprime polynomials in q. □

We are now able to give q-analogues of (1.6) and (1.7) as follows.

Theorem 3.2. *Let n and r be positive integers with $n\equiv 1 \pmod 4$ and $n>1$. Then, modulo $\Phi_{n^r}(q)\prod_{j=1}^{r}\Phi_{n^j}(q)^2$, we have*

$$\sum_{k=0}^{(n^r-1)/2}[4k+1]_{q^2}\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3}q^{-k} \equiv q^{2-2n^r}[n^r]_{q^2}[n^r]\frac{(q^4;q^4)_{(n^r-1)/4}^2}{(q^2;q^4)_{(n^r-1)/4}^2}, \tag{3.3}$$

$$\sum_{k=0}^{n^r-1}[4k+1]_{q^2}\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3}q^{-k} \equiv q^{2-2n^r}[n^r]_{q^2}[n^r]\frac{(q^4;q^4)_{(n^r-1)/4}^2}{(q^2;q^4)_{(n^r-1)/4}^2}. \tag{3.4}$$

Proof. Letting $a=1$ in (3.2) and noticing that $1-q^{2n}$ contains the factor $\Phi_n(q^2)$, which is coprime with $(q^3;q^4)_{(n-1)/2}$, we obtain

$$\sum_{k=0}^{(n-1)/2}[4k+1]_{q^2}\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3}q^{-k} \equiv q^{-3(n-1)/2}[n]_{q^2}\frac{(q^5;q^4)_{(n-1)/2}}{(q^3;q^4)_{(n-1)/2}} \pmod{\Phi_n(q^2)^3}, \tag{3.5}$$

and

$$\sum_{k=0}^{n-1}[4k+1]_{q^2}\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3}q^{-k} \equiv q^{-3(n-1)/2}[n]_{q^2}\frac{(q^5;q^4)_{(n-1)/2}}{(q^3;q^4)_{(n-1)/2}} \pmod{\Phi_n(q^2)^3}, \tag{3.6}$$

since $(q^2;q^4)_k^3/(q^4;q^4)_k^3 \equiv 0 \pmod{\Phi_n(q)^3}$ for $(n-1)/2 < k \leqslant n-1$. It is easy to see that

$$\begin{aligned}
\frac{(q^5;q^4)_{(n-1)/2}}{(q^3;q^4)_{(n-1)/2}} &= [n]\frac{(q;q^4)_{(n-1)/4}(q^{n+4};q^4)_{(n-1)/4}}{(q^3;q^4)_{(n-1)/4}(q^{n+2};q^4)_{(n-1)/4}} \\
&\equiv [n]\frac{(q^{1-n};q^4)_{(n-1)/4}(q^4;q^4)_{(n-1)/4}}{(q^{3-n};q^4)_{(n-1)/4}(q^2;q^4)_{(n-1)/4}} \\
&= q^{(1-n)/2}[n]\frac{(q^4;q^4)_{(n-1)/4}^2}{(q^2;q^4)_{(n-1)/4}^2} \pmod{\Phi_n(q)^2}. \qquad (3.7)
\end{aligned}$$

In view of $\Phi_n(q^2)=\Phi_n(q)\Phi_n(-q)$ for odd n, from (3.5)–(3.7) we deduce that

$$\sum_{k=0}^{(n-1)/2} [4k+1]_{q^2}\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3}q^{-k} \equiv q^{2-2n}[n]_{q^2}[n]\frac{(q^4;q^4)_{(n-1)/4}^2}{(q^2;q^4)_{(n-1)/4}^2} \pmod{\Phi_n(q)^3}, \qquad (3.8)$$

and its companion:

$$\sum_{k=0}^{n-1}[4k+1]_{q^2}\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3}q^{-k} \equiv q^{2-2n}[n]_{q^2}[n]\frac{(q^4;q^4)_{(n-1)/4}^2}{(q^2;q^4)_{(n-1)/4}^2} \pmod{\Phi_n(q)^3}. \qquad (3.9)$$

Replacing n by n^r in (3.8) and (3.9), we see that (3.3) and (3.4) hold modulo $\Phi_{n^r}(q)^3$. Since both $[n^r]_{q^2}$ and $[n^r]$ are divisible by $\prod_{j=1}^{r-1}\Phi_{n^j}(q)$, and the denominator of the reduced form of $(q^4;q^4)_{(n^r-1)/4}^2/(q^2;q^4)_{(n^r-1)/4}^2$ is coprime with $\prod_{j=1}^{r-1}\Phi_{n^j}(q)$, in light of (2.6) and (2.7), we see that (3.3) and (3.4) also hold modulo $\prod_{j=1}^{r-1}\Phi_{n^j}(q)^2$. This completes the proof of the theorem. □

Further, similarly to the proof of (3.5) and [7, Theorem 1.4], we can prove the following q-analogue of (1.1).

Theorem 3.3. *Let $n>1$ be an odd integer. Then, modulo $[n]\Phi_n(q)^2$,*

$$\sum_{k=0}^{(n-1)/d} (-1)^k[4k+1]_{q^2}\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3}q^{-k} \equiv (-q)^{-3(n-1)/2}[n]_{q^2}\frac{(-q^5;q^4)_{(n-1)/2}}{(-q^3;q^4)_{(n-1)/2}}, \qquad (3.10)$$

where $d=1,2$.

4. Proof of Theorem 1.1

We need the following congruence modulo a prime p.

Proposition 4.1. *Let $p\equiv 1 \pmod 4$ be a prime and r a positive integer. Then*

$$\frac{1}{2^{(p^r-1)/2}}\binom{(p^r-1)/2}{(p^r-1)/4} \equiv (-1)^{(p-1)r/4}\frac{\Gamma_p(\frac14)^{2r}}{\Gamma_p(\frac12)^r} \pmod{p}. \qquad (4.1)$$

Proof. By Fermat's little theorem, we have $2^{p-1}\equiv 1 \pmod p$, and so

$$2^{(p^r-1)/2} = 2^{((p-1+1)^r-1)/2} \equiv 2^{(p-1)r/2} \pmod{p}.$$

Since $(p^r-1)/2=(p-1)p^{r-1}/2+(p-1)p^{r-2}/2+\cdots+(p-1)/2$ and $(p^r-1)/4=(p-1)p^{r-1}/4+(p-1)p^{r-2}/4+\cdots+(p-1)/4$, by the Lucas theorem, we have

$$\binom{(p^r-1)/2}{(p^r-1)/4}=\binom{(p-1)/2}{(p-1)/4}^r \pmod p.$$

Applying Van Hamme's result [19, Theorem 3]:

$$\frac{1}{2^{(p-1)/2}}\binom{(p-1)/2}{(p-1)/4}\equiv(-1)^{(p-1)/4}\frac{\Gamma_p(\frac14)^2}{\Gamma_p(\frac12)} \pmod p, \tag{4.2}$$

we get the desired congruence (4.1). □

Proof of Theorem 1.1. Letting $n=p$ be a prime and taking the limits as $q\to 1$ in (3.3) and (3.4), we are led to the following supercongruences:

$$\sum_{k=0}^{(p^r-1)/2}\frac{4k+1}{64^k}\binom{2k}{k}^3\equiv p^{2r}\frac{2^{p^r-1}}{\binom{(p^r-1)/2}{(p^r-1)/4}^2} \pmod{p^{2r+1}}, \tag{4.3}$$

$$\sum_{k=0}^{p^r-1}\frac{4k+1}{64^k}\binom{2k}{k}^3\equiv p^{2r}\frac{2^{p^r-1}}{\binom{(p^r-1)/2}{(p^r-1)/4}^2} \pmod{p^{2r+1}}. \tag{4.4}$$

In light of Proposition 4.1, the right-hand sides of (4.3) and (4.4) is congruent to

$$p^{2r}\frac{\Gamma_p(\frac12)^{2r}}{\Gamma_p(\frac14)^{4r}} \pmod{p^{2r+1}}.$$

Noticing that $\Gamma_p(\frac12)^2=-1$ and $\Gamma_p(\frac14)^2\Gamma_p(\frac34)^2=1$, we finish the proof of the theorem. □

5. Proof of Theorem 1.2

Let p be an odd prime. We first recall some fundamental properties of the p-adic Gamma function [1, 13]. For any positive integer n, the p-adic Gamma function is defined as

$$\Gamma_p(n):=(-1)^n\prod_{\substack{0<k<n\\ p\nmid k}}k.$$

Moreover, put $\Gamma_p(0)=1$. Let $\mathbb{Z}_p$ stand for the ring of all p-adic integers. Then Γ_p can be extended to all $x\in\mathbb{Z}_p$ by defining

$$\Gamma_p(x)=\lim_{x_n\to x}\Gamma_p(x_n),$$

where x_n denotes any sequence of positive integers p-adically approximating x. From the definition of p-adic Gamma function, we can easily deduce that

$$\frac{\Gamma_p(x+1)}{\Gamma_p(x)}=\begin{cases}-x, & p\nmid x,\\ -1, & p\mid x.\end{cases} \tag{5.1}$$

We also need the following properties: for any $x\in\mathbb{Z}_p$,

$$\Gamma_p(x)\Gamma_p(1-x)=(-1)^{p-\langle -x\rangle_p}, \tag{5.2}$$

and for any $a, m \in \mathbb{Z}_p$,

$$\Gamma_p(a+mp) \equiv \Gamma_p(a) + \Gamma_p'(a)mp \pmod{p^2} \tag{5.3}$$

(see [11, Theorem 14]).

In order to prove Theorem 1.2, we first give two congruences modulo p^2, though for the first one we only need it modulo p.

Lemma 5.1. *Let $p \equiv 3 \pmod 4$ be a prime and let $r \geqslant 1$. Then*

$$p^r \frac{(1)_{(p^{2r}-1)/4}}{(\frac{1}{2})_{(p^{2r}-1)/4}} \equiv (-1)^r \pmod{p^2}, \tag{5.4}$$

$$p \frac{(\frac{5}{4})_{(p^{2r-1}-1)/2}}{(\frac{3}{4})_{(p^{2r-1}-1)/2}} \equiv -16\Gamma_p(\tfrac{3}{4})^4 \pmod{p^2}. \tag{5.5}$$

Proof. Let $\Gamma(x)$ be the classical Gamma function. We prove (5.4) by induction on r. For $r=1$, in view of (5.1), we have

$$\begin{aligned} p\frac{(1)_{(p^2-1)/4}}{(\frac{1}{2})_{(p^2-1)/4}} &= p\frac{\Gamma(\frac{p^2+3}{4})\Gamma(\frac{1}{2})}{\Gamma(\frac{p^2+1}{4})} \\ &= -p\frac{p\cdot 2p\cdot \cdots \cdot \frac{(p-3)p}{4}}{\frac{p}{2}\cdot\frac{3p}{2}\cdot\cdots\cdot\frac{\frac{(p-1)p}{2}}{2}} \cdot \frac{\Gamma_p(\frac{p^2+3}{4})\Gamma_p(\frac{1}{2})}{\Gamma_p(\frac{p^2+1}{4})} \\ &= -\frac{(1)_{(p-3)/4}}{(\frac{1}{2})_{(p+1)/4}} \cdot \frac{\Gamma_p(\frac{p^2+3}{4})\Gamma_p(\frac{1}{2})}{\Gamma_p(\frac{p^2+1}{4})} \\ &\equiv -\frac{\Gamma_p(\frac{p+1}{4})\Gamma_p(\frac{1}{2})}{\Gamma_p(\frac{p+3}{4})} \cdot \frac{\Gamma_p(\frac{3}{4})\Gamma_p(\frac{1}{2})}{\Gamma_p(\frac{1}{4})} \pmod{p^2}. \end{aligned} \tag{5.6}$$

By (5.2) and (5.3), we get

$$\frac{\Gamma_p(\frac{p+1}{4})}{\Gamma_p(\frac{p+3}{4})} = (-1)^{(p+1)/4}\Gamma_p(\tfrac{1+p}{4})\Gamma_p(\tfrac{1-p}{4}) \equiv (-1)^{(p+1)/4}\Gamma_p(\tfrac{1}{4})^2 \pmod{p^2}. \tag{5.7}$$

Substituting (5.7) into (5.6) and using $\Gamma_p(\frac{1}{2})^2 = 1$ and $\Gamma_p(\frac{1}{4})\Gamma_p(\frac{3}{4}) = (-1)^{(p+1)/4}$ for $p \equiv 3 \pmod 4$, we deduce that

$$p\frac{(1)_{(p^2-1)/4}}{(\frac{1}{2})_{(p^2-1)/4}} \equiv -1 \pmod{p^2}.$$

We now assume that the congruence (5.4) holds for some $r-1$ $(r \geqslant 2)$. Then

$$\begin{aligned} p^r\frac{(1)_{(p^{2r}-1)/4}}{(\frac{1}{2})_{(p^{2r}-1)/4}} &= p^r\frac{\Gamma(\frac{p^{2r}+3}{4})\Gamma(\frac{1}{2})}{\Gamma(\frac{p^{2r}+1}{4})} \\ &= -p^r\frac{p\cdot 2p\cdot\cdots\cdot\frac{(p^{2r-1}-3)p}{4}}{\frac{p}{2}\cdot\frac{3p}{2}\cdot\cdots\cdot\frac{\frac{(p^{2r-1}-1)p}{2}}{2}} \cdot \frac{\Gamma_p(\frac{p^{2r}+3}{4})\Gamma_p(\frac{1}{2})}{\Gamma_p(\frac{p^{2r}+1}{4})} \end{aligned}$$

$$
\begin{aligned}
&= -p^{r-1}\frac{(1)_{(p^{2r-1}-3)/4}}{(\frac12)_{(p^{2r-1}+1)/4}}\cdot\frac{\Gamma_p(\frac{p^{2r}+3}{4})\Gamma_p(\frac12)}{\Gamma_p(\frac{p^{2r}+1}{4})}\\
&= -p^{r-1}\frac{p\cdot 2p\cdot\cdots\cdot\frac{(p^{2r-2}-1)p}{4}}{\frac{p}{2}\cdot\frac{3p}{2}\cdot\cdots\cdot\frac{\frac{(p^{2r-2}-3)p}{2}}{2}}\cdot\frac{\Gamma_p(\frac{p^{2r-1}+1}{4})\Gamma_p(\frac12)}{\Gamma_p(\frac{p^{2r-1}+3}{4})}\cdot\frac{\Gamma_p(\frac{p^{2r}+3}{4})\Gamma_p(\frac12)}{\Gamma_p(\frac{p^{2r}+1}{4})}\\
&\equiv -p^{r-1}\frac{(1)_{(p^{2r-2}-1)/4}}{(\frac12)_{(p^{2r-2}-1)/4}} \pmod{p^2}.
\end{aligned}
$$

By the induction hypothesis, we have

$$
p^{r-1}\frac{(1)_{(p^{2r-2}-1)/4}}{(\frac12)_{(p^{2r-2}-1)/4}}\equiv(-1)^{r-1}\pmod{p^2},
$$

and so (5.4) holds for r.

Wang and Pan [22] proved that

$$
\frac{(\frac34)_{(p^{2r}-1)/2}}{(\frac54)_{(p^{2r}-1)/2}}\equiv 1\pmod{p^2}.
$$

They also gave the following identity:

$$
\frac{(\frac34)_{(p^{2r}-1)/2}}{(\frac54)_{(p^{2r}-1)/2}}=\frac{p(\frac14)_{(p^{2r-1}+1)/2}}{(\frac34)_{(p^{2r-1}-1)/2}}\cdot\frac{\Gamma_p(\frac{p^{2r}+1}{4})\Gamma_p(\frac54)}{\Gamma_p(\frac34)\Gamma_p(\frac{2p^{2r}+3}{4})}.
$$

It follows that

$$
\frac{p}{4}\frac{(\frac54)_{(p^{2r-1}-1)/2}}{(\frac34)_{(p^{2r-1}-1)/2}}=\frac{p(\frac14)_{(p^{2r-1}+1)/2}}{(\frac34)_{(p^{2r-1}-1)/2}}\equiv\frac{\Gamma_p(\frac34)\Gamma_p(\frac{2p^{2r}+3}{4})}{\Gamma_p(\frac{p^{2r}+1}{4})\Gamma_p(\frac54)}\equiv\frac{-4\Gamma_p(\frac34)^2}{\Gamma_p(\frac14)^2}. \pmod{p^2}
$$

Noticing that $\Gamma_p(\frac14)^2\Gamma_p(\frac34)^2=1$, we obtain (5.5). □

We now present a q-analogue of (1.9) as follows.

Theorem 5.2. *Let n and r be positive integers with $n\equiv 3\pmod 4$. Then, modulo $\Phi_{n^{2r}}(q)\prod_{j=1}^{r}\Phi_{n^{2j}}(q)^2$, we have*

$$
\sum_{k=0}^{(n^{2r}-1)/d}[4k+1]_{q^2}\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3}q^{-k}\equiv q^{2-2n^{2r}}[n^{2r}]_{q^2}[n^{2r}]\frac{(q^4;q^4)_{(n^{2r}-1)/4}^2}{(q^2;q^4)_{(n^{2r}-1)/4}^2}, \tag{5.8}
$$

where $d=1,2$.

Proof. It is clear that $n^2\equiv 1\pmod 4$. Replacing n by n^2 in (3.3) and (3.4), we obtain the desired q-congruence (5.8). □

Proof of (1.9). Letting $n=p$ be a prime and taking the limits as $q\to 1$ in (5.8), we arrive at

$$
\sum_{k=0}^{(p^{2r}-1)/d}\frac{4k+1}{64^k}\binom{2k}{k}^3\equiv p^{4r}\frac{(1)_{(p^{2r}-1)/4}^2}{(\frac12)_{(p^{2r}-1)/4}^2}\pmod{p^{2r+1}},
$$

where $d=1,2$. The proof of (1.9) then follows from (5.4). □

Similarly, we have the following q-analogue of (1.10).

Theorem 5.3. *Let n and r be positive integers with $n \equiv 3 \pmod 4$. Then, modulo $\Phi_{n^{2r-1}}^2(q)\prod_{j=1}^{r-1}\Phi_{n^{2j}}(q)^2$, we have*

$$\sum_{k=0}^{(n^{2r-1}-1)/d}[4k+1]_{q^2}\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3}q^{-k} \equiv q^{-3(n^{2r-1}-1)/2}[n^{2r-1}]_{q^2}\frac{(q^5;q^4)_{(n^{2r-1}-1)/2}}{(q^3;q^4)_{(n^{2r-1}-1)/2}}, \tag{5.9}$$

where $d = 1, 2$.

Proof. Note that the q-congruence (3.2) also holds modulo $(1-aq^{2n})(a-q^{2n})$ for $n \equiv 3 \pmod 4$. Namely, the q-congruences (3.5) and (3.6) hold modulo $\Phi_n(q^2)^2$ for $n \equiv 3$. Replacing n by n^{2r-1} in (3.5) and (3.6), we are led to the following result: modulo $\Phi_{n^{2r-1}}(q^2)^2$,

$$\sum_{k=0}^{(n^{2r-1}-1)/d}[4k+1]_{q^2}\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3}q^{-k} \equiv q^{-3(n^{2r-1}-1)/2}[n^{2r-1}]_{q^2}\frac{(q^5;q^4)_{(n^{2r-1}-1)/2}}{(q^3;q^4)_{(n^{2r-1}-1)/2}}, \tag{5.10}$$

Since $n^2 \equiv n^4 \equiv \cdots \equiv n^{2r-2} \equiv 1 \pmod 4$, by Theorem 2.1, the left-hand side of (5.10) is congruent to 0 modulo $\prod_{j=1}^{r-1}\Phi_{n^{2j}}(q)^2$. Meanwhile, it is not difficult to see that the right-hand side of (5.10) is also congruent to 0 modulo $\prod_{j=1}^{r-1}\Phi_{n^{2j}}(q)^2$. This completes the proof of (5.10). □

Proof of (1.10). Letting $n = p$ be a prime and taking $q \to 1$ in (5.10), we conclude that

$$\sum_{k=0}^{(p^{2r-1}-1)/d}\frac{4k+1}{64^k}\binom{2k}{k}^3 \equiv p^{2r-1}\frac{(\frac{5}{4})_{(p^{2r-1}-1)/2}}{(\frac{3}{4})_{(p^{2r-1}-1)/2}} \pmod{p^{2r}},$$

where $d = 1, 2$. The proof of (1.10) then follows from (5.5). □

6. Concluding remarks and open problems

By establishing a suitable q-analogue, the author and Zudilin [8, Theorem 3.3] proved the following Dwork-type supercongruence: for any odd prime p and positive integer r,

$$\sum_{k=0}^{(p^r-1)/d}(-1)^k\frac{4k+1}{64^k}\binom{2k}{k}^3 \equiv (-1)^{(p-1)/2}p\sum_{k=0}^{(p^{r-1}-1)/d}(-1)^k\frac{4k+1}{64^k}\binom{2k}{k}^3 \pmod{p^{3r}}, \tag{6.1}$$

where $d = 1, 2$, the $d = 2$ case confirming the (B.3) conjecture of Swisher [18].

It is natural to propose the following new Dwork-type supercongruence conjecture.

Conjecture 6.1. *Let $p \equiv 1 \pmod 4$ be a prime and let $r \geqslant 1$. Then*

$$\sum_{k=0}^{(p^r-1)/d}\frac{4k+1}{64^k}\binom{2k}{k}^3 \equiv -p^2\Gamma_p(\tfrac{3}{4})^4\sum_{k=0}^{(p^{r-1}-1)/d}\frac{4k+1}{64^k}\binom{2k}{k}^3 \pmod{p^{3r}}, \tag{6.2}$$

where $d = 1, 2$.

Although many other Dwork-type supercongruences modulo p^{3r} have been proved by the author and Zudilin [8], none of them are related to p-adic Gamma functions. For this reason, we believe that Conjecture 6.1 is rather challenging.

Numerical evaluations imply that the following generalization of (1.10) should be true.

Conjecture 6.2. *Let $p \equiv 3 \pmod 4$ be a prime with $p > 3$ and let $r \geqslant 1$. Then*

$$\sum_{k=0}^{(p^{2r-1}-1)/d} \frac{4k+1}{64^k}\binom{2k}{k}^3 \equiv -16p^{2r-2}\Gamma_p(\tfrac{3}{4})^4 \pmod{p^{2r+1}}, \tag{6.3}$$

where $d = 1, 2$.

Note that the $r = 1$ case of (6.3) has already been proved by Wang and Sun [23, Theorem 1.2].

In [5, Conjecture 7.2], the author proposed the following curious conjecture: for all positive integers n with $n \equiv 1 \pmod 4$,

$$\sum_{k=0}^{(n-1)/2} [4k+1]\frac{(q;q^2)_k^3}{(q^2;q^2)_k^3} q^{k(n^2-2nk-n-2)/4} \equiv 0 \pmod{\Phi_n(q)^2},$$

which is a q-analogue of (1.3) for $r = 1$. The author and Zudilin [7, Theorem 4.11] have showed that the above q-congruence is true modulo $\Phi_n(q)$. It would be very interesting if the reader can confirm this conjecture completely, though it is not a full q-analogue of Z.-W. Sun's original conjecture (1.3).

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