Some q-supercongruences from Singh's quadratic transformation

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Abstract. Applying Singh's quadratic transformation and the 'creative microscoping' method (introduced by the author and Zudilin in 2019), we prove several new q-supercongruences for truncated ${}_4\phi_3$ series. Some related conjectures on q-supercongruences are also presented.

Keywords: cyclotomic polynomials; q-supercongruences; Singh's quadratic transformation; creative microscoping

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1. Introduction

Let p always stand for a prime throughout this paper. In 1997, Van Hamme [17, (H.2)] proved the following interesting supercongruence:

$$\sum_{k=0}^{(p-1)/2} \frac{(\frac{1}{2})_k^3}{k!^3} \equiv \begin{cases} -\Gamma_p(1/4)^4 \pmod{p^2}, & \text{if } p \equiv 1 \pmod 4, \\ 0 \pmod{p^2}, & \text{if } p \equiv 3 \pmod 4, \end{cases} \tag{1.1}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ denotes the Pochhammer symbol and $\Gamma_p(x)$ is the p-adic Gamma function.

In the past few years, q-analogues of supercongruences have been widely studied by a number of authors (see [1,3–5,7,9–13,15,18–20]). In particular, the author and Zudilin [7] developed a method, which they called 'creative microscoping', to establish q-analogues of many classical supercongruences. They [9, Theorem 2] also utilized the method of creative microscoping to give a q-analogue of (1.1) as follows: modulo $\Phi_n(q)^2$,

$$\sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k^2 (q^2;q^4)_k}{(q^2;q^2)_k^2 (q^4;q^4)_k} q^{2k} \equiv \begin{cases} \dfrac{(q^2;q^4)_{(n-1)/4}^2}{(q^4;q^4)_{(n-1)/4}^2} q^{(n-1)/2} & \text{if } n \equiv 1 \pmod 4, \\ 0 & \text{if } n \equiv 3 \pmod 4. \end{cases} \tag{1.2}$$

Here $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ is the *q-shifted factorial*, $[n] = (1-q^n)/(1-q)$ denotes the *q-integer*, and $\Phi_n(q)$ is the n-th *cyclotomic polynomial* in q, which may be given by

$$\Phi_n(q) = \prod_{\substack{1 \leqslant k \leqslant n \\ \gcd(n,k)=1}} (q - \zeta^k),$$

where ζ denotes an n-th primitive root of unity.

The author and Schlosser [5] investigated q-congruences and q-supercongruences more systematically by employing transformation formulas for basic hypergeometric series, together with all kinds of techniques such as suitably combining terms, and the creative microscoping method.

The purpose of this note is to give some q-supercongruences using Singh's quadratic transformation [2, Appendix (III.21)]. Our first result is closely related to the aforementioned q-supercongruence (1.2), and can be stated as follows.

Theorem 1.1. *Let $d \geqslant 2$ be an integer and $d \neq 3$. Let $n \equiv 1 \pmod{2d}$ be a positive integer. Then, modulo $\Phi_n(q)^2$,*

$$\sum_{k=0}^{(n-1)/d} \frac{(q;q^d)_k^2 (q^{2d-2};q^{2d})_k q^{dk}}{(q^d;q^d)_k (q^{2d-2};q^d)_k (q^{d+2};q^{2d})_k} \equiv \sum_{k=0}^{(n-1)/(2d)} \frac{(q;q^{2d})_k^2 (q^{2d-2};q^{2d})_k q^{2dk}}{(q^{2d};q^{2d})_k (q^{3d-2};q^{2d})_k (q^{d+2};q^{2d})_k}. \tag{1.3}$$

Letting $n = p$ and $q \to 1$ in (1.3), we get the following result: for $d \geqslant 2$, $d \neq 3$, and $p \equiv 1 \pmod{2d}$,

$$\sum_{k=0}^{(p-1)/d} \frac{(\frac{1}{d})_k^2 (\frac{d-1}{d})_k}{k! (\frac{2d-2}{d})_k (\frac{d+2}{2d})_k} \equiv \sum_{k=0}^{(p-1)/(2d)} \frac{(\frac{1}{2d})_k^2 (\frac{d-1}{d})_k}{k! (\frac{3d-2}{2d})_k (\frac{d+2}{2d})_k} \pmod{p^2}. \tag{1.4}$$

On the other hand, form (1.2) and $d = 2$ case of (1.3) we deduce that, for any positive integer $n \equiv 1 \pmod 4$,

$$\sum_{k=0}^{(n-1)/4} \frac{(q;q^4)_k^2 (q^2;q^4)_k}{(q^4;q^4)_k^3} q^{4k} \equiv \frac{(q^2;q^4)_{(n-1)/4}^2}{(q^4;q^4)_{(n-1)/4}^2} q^{(n-1)/2} \pmod{\Phi_n(q)^2}. \tag{1.5}$$

Our second result is the following q-supercongruence similar to (1.3).

Theorem 1.2. *Let $d \geqslant 4$ be an integer. Let $n \equiv -1 \pmod{2d}$ be a positive integer and $n \neq 7$. Then, modulo $\Phi_n(q)^2$,*

$$\sum_{k=0}^{(n+1)/d} \frac{(q^{-1};q^d)_k^2 (q^{2d+2};q^{2d})_k q^{dk}}{(q^d;q^d)_k (q^{2d+2};q^d)_k (q^{d-2};q^{2d})_k} \equiv \sum_{k=0}^{(n+1)/(2d)} \frac{(q^{-1};q^{2d})_k^2 (q^{2d+2};q^{2d})_k q^{2dk}}{(q^{2d};q^{2d})_k (q^{3d+2};q^{2d})_k (q^{d-2};q^{2d})_k}. \tag{1.6}$$

Our third result is a generalization of (1.3) for $d = 2$.

Theorem 1.3. *Let $n \equiv 1 \pmod 4$ be a positive integer. Then*

$$\sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k^2 (q^2;q^4)_k}{(q^2;q^2)_k^2 (q^4;q^4)_k} q^{2k} \equiv \sum_{k=0}^{(n-1)/2} \frac{(q;q^4)_k^2 (q^2;q^4)_k}{(q^4;q^4)_k^3} q^{4k} \pmod{\Phi_n(q)^3}. \tag{1.7}$$

Letting $n=p$ and $q\to 1$ in (1.7), we are led to

$$\sum_{k=0}^{(p-1)/2}\frac{(\frac{1}{2})_k^3}{k!^3}\equiv\sum_{k=0}^{(p-1)/2}\frac{(\frac{1}{4})_k^2(\frac{1}{2})_k}{k!^3}\pmod{p^3}\quad\text{for } p\equiv 1\pmod 4. \tag{1.8}$$

Note that Long and Ramakrishna [14, Theorem 3] gave a generalization of (1.1):

$$\sum_{k=0}^{(p-1)/2}\frac{(\frac{1}{2})_k^3}{k!^3}\equiv\begin{cases}-\Gamma_p(1/4)^4 \pmod{p^3}, & \text{if } p\equiv 1\pmod 4,\\[2mm] -\dfrac{p^2}{16}\Gamma_p(1/4)^4 \pmod{p^3}, & \text{if } p\equiv 3\pmod 4.\end{cases} \tag{1.9}$$

Combining (1.8) and (1.9) and using Sun's result [16, (4.3)]:

$$-\Gamma_p(1/4)^4\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3}\quad\text{for}\quad p=x^2+4y^2\equiv 1\pmod 4,$$

we have the following conclusion.

Corollary 1.4. *Let $p=x^2+4y^2\equiv 1 \pmod 4$, where x and y are integers. Then*

$$\sum_{k=0}^{(p-1)/2}\frac{(\frac{1}{4})_k^2(\frac{1}{2})_k}{k!^3}\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3}.$$

It is also worth mentioning that Wang [18] and Wei [20] have given two different q-supercongruences on the left-hand side of (1.7) modulo $\Phi_n(q)^3$, which are both q-analogues of (1.9) for $p\equiv 1 \pmod 4$. By the $(d,r)=(2,1)$ case of [4, Theorem 9], we know that the right-hand side of (1.7) is congruent to 0 modulo $\Phi_n(q)^2$, and thus by (1.2) the q-supercongruence (1.7) also holds modulo $\Phi_n(q)^2$ for $n\equiv 3 \pmod 4$.

The paper is organized as follows. We shall prove Theorems 1.1–1.3 in Sections 2–4, respectively. Finally, in Section 5, we propose some conjectural q-supercongruences, including a generalization of (1.7) modulo $\Phi_n(q)^4$.

2. Proof of Theorem 1.1

Recall that the *basic hypergeometric series* ${}_{r+1}\phi_r$ is defined as

$${}_{r+1}\phi_r\left[\begin{matrix}a_1,a_2,\dots,a_{r+1}\\ b_1,b_2,\dots,b_r\end{matrix};q,\,z\right]=\sum_{k=0}^{\infty}\frac{(a_1,a_2,\dots,a_{r+1};q)_k z^k}{(q,b_1,\dots,b_r;q)_k}.$$

We need Singh's quadratic transformation [2, Appendix (III.21)]:

$${}_4\phi_3\left[\begin{matrix}a^2,b^2,c,d\\ ab\sqrt{q},-ab\sqrt{q},-cd\end{matrix};q,q\right]={}_4\phi_3\left[\begin{matrix}a^2,b^2,c^2,d^2\\ a^2b^2q,-cd,-cdq\end{matrix};q^2,q^2\right], \tag{2.1}$$

provided that both series terminate. For an elementary proof of this transformation, see [6, Section 5].

We first give the following parametric generalization of Theorem 1.1.

Theorem 2.1. *Let $d \geqslant 2$ be an integer and a an indeterminate. Let $n \equiv 1 \pmod{2d}$ be a positive integer. Then, modulo $(1-aq^n)(a-q^n)$,*

$$\sum_{k=0}^{(n-1)/d} \frac{(aq;q^d)_k (q/a;q^d)_k (q^{2d-2};q^{2d})_k q^{dk}}{(q^d;q^d)_k (q^{2d-2};q^d)_k (q^{d+2};q^{2d})_k} \equiv \sum_{k=0}^{(n-1)/(2d)} \frac{(aq;q^{2d})_k (q/a;q^{2d})_k (q^{2d-2};q^{2d})_k q^{2dk}}{(q^{2d};q^{2d})_k (q^{3d-2};q^{2d})_k (q^{d+2};q^{2d})_k}. \tag{2.2}$$

Proof. Putting $q \mapsto q^d$, $a = q^{(1-n)/2}$, $b = q^{(1+n)/2}$, $c = q^{d-1}$, and $d = -q^{d-1}$ in Singh's transformation (2.1), for $n \equiv 1 \pmod{2d}$ we have

$${}_4\phi_3\left[\begin{matrix} q^{1-n}, q^{1+n}, q^{d-1}, -q^{d-1} \\ q^{(d+2)/2}, -q^{(d+2)/2}, q^{2d-2} \end{matrix}; q^d, q^d\right] = {}_4\phi_3\left[\begin{matrix} q^{1-n}, q^{1+n}, q^{2d-2}, q^{2d-2} \\ q^{d+2}, q^{2d-2}, q^{3d-2} \end{matrix}; q^{2d}, q^{2d}\right].$$

Namely, both sides of (2.2) are equal for $a = q^{-n}$ and $a = q^n$. Therefore, the q-congruence (2.2) is true modulo $1 - aq^n$ and $a - q^n$. Since $1 - aq^n$ and $a - q^n$ are relatively prime polynomials in q, we complete the proof. □

Proof of Theorem 1.1. Since $n \equiv 1 \pmod{2d}$, we have $\gcd(2d, n) = 1$. Hence, for $d \geqslant 2$ and $d \neq 3$, the polynomials

$$(q^d;q^d)_k (q^{2d-2};q^d)_k (q^{d+2};q^{2d})_k \quad (0 \leqslant k \leqslant (n-1)/d))$$

and

$$(q^{2d};q^{2d})_k (q^{3d-2};q^{2d})_k (q^{d+2};q^{2d})_k \quad (0 \leqslant k \leqslant (n-1)/(2d)))$$

are relatively prime to $\Phi_n(q)$. Moreover, the polynomial $1 - q^n$ has the factor $\Phi_n(q)$. The proof of (1.3) then follows from the congruence (2.2) by letting $a = 1$. □

3. Proof of Theorem 1.2

Similarly as before, we first establish a parametric generalization of Theorem 1.2.

Theorem 3.1. *Let $d \geqslant 4$ be an integer and a an indeterminate. Let $n \equiv -1 \pmod{2d}$ be a positive integer. Then, modulo $(1-aq^n)(a-q^n)$,*

$$\begin{aligned} &\sum_{k=0}^{(n+1)/d} \frac{(aq^{-1};q^d)_k (q^{-1}/a;q^d)_k (q^{2d+2};q^{2d})_k q^{dk}}{(q^d;q^d)_k (q^{2d+2};q^d)_k (q^{d-2};q^{2d})_k} \\ &\quad \equiv \sum_{k=0}^{(n+1)/(2d)} \frac{(aq^{-1};q^{2d})_k (q^{-1}/a;q^{2d})_k (q^{2d+2};q^{2d})_k q^{2dk}}{(q^{2d};q^{2d})_k (q^{3d+2};q^{2d})_k (q^{d-2};q^{2d})_k}. \end{aligned} \tag{3.1}$$

Proof. Making the parameter substitutions $q \mapsto q^d$, $a = q^{-(n+1)/2}$, $b = q^{(n-1)/2}$, $c = q^{d+1}$, and $d = -q^{d+1}$ in (2.1), for $n \equiv -1 \pmod{2d}$ we have

$$ {}_4\phi_3\left[\begin{matrix} q^{-n-1}, q^{n-1}, q^{d+1}, -q^{d+1} \\ q^{(d-2)/2}, -q^{(d-2)/2}, q^{2d+2} \end{matrix}; q^d, q^d\right] = {}_4\phi_3\left[\begin{matrix} q^{-n-1}, q^{n-1}, q^{2d+2}, q^{2d+2} \\ q^{d-2}, q^{2d+2}, q^{3d+2} \end{matrix}; q^{2d}, q^{2d}\right]. $$

That is, the two sides of (3.1) are equal for $a = q^{-n}$ and $a = q^n$. This proves that (2.2) holds modulo $1 - aq^n$ and $a - q^n$. □

Proof of Theorem 1.2. Since $d \geqslant 4$, $n \equiv -1 \pmod{2d}$ and $n > 7$, we have $\gcd(2d, n) = 1$ and the polynomials

$$ (q^d; q^d)_k (q^{2d+2}; q^d)_k (q^{d-2}; q^{2d})_k \quad (0 \leqslant k \leqslant (n+1)/d)) $$

and

$$ (q^{2d}; q^{2d})_k (q^{3d+2}; q^{2d})_k (q^{d-2}; q^{2d})_k \quad (0 \leqslant k \leqslant (n+1)/(2d)) $$

are relatively prime to $\Phi_n(q)$. The proof of (1.6) then follows from the congruence (3.1) by setting $a = 1$. □

4. Proof of Theorem 1.3

Likewise, we have a parametric generalization of Theorem 1.3.

Theorem 4.1. *Let $n \equiv 1 \pmod 4$ be a positive integer and a an indeterminate. Then, modulo $\Phi_n(q)(1 - aq^n)(a - q^n)$,*

$$ \sum_{k=0}^{(n-1)/2} \frac{(aq; q^2)_k (q/a; q^2)_k (q^2; q^4)_k}{(q^2; q^2)_k^2 (q^4; q^4)_k} q^{2k} \equiv \sum_{k=0}^{(n-1)/2} \frac{(aq; q^4)_k (q/a; q^4)_k (q^2; q^4)_k}{(q^4; q^4)_k^3} q^{4k}. \tag{4.1} $$

Proof. Letting $q \mapsto q^2$, $a \to \sqrt{aq}$, $b \to \sqrt{q/a}$, $c = q^{1-n}$, and $d = -q^{1-n}$ in (2.1), for odd n we have

$$ {}_4\phi_3\left[\begin{matrix} aq, q/a, q^{1-n}, -q^{1-n} \\ q^2, -q^2, q^{2-2n} \end{matrix}; q^2, q^2\right] = {}_4\phi_3\left[\begin{matrix} aq, q/a, q^{2-2n}, q^{2-2n} \\ q^4, q^{2-2n}, q^{4-2n} \end{matrix}; q^4, q^4\right], $$

which can be written as

$$ \sum_{k=0}^{(n-1)/2} \frac{(aq; q^2)_k (q/a; q^2)_k (q^{2-2n}; q^4)_k}{(q^2; q^2)_k (q^{2-2n}; q^2)_k (q^4; q^4)_k} q^{2k} = \sum_{k=0}^{(n-1)/2} \frac{(aq; q^4)_k (q/a; q^4)_k (q^{2-2n}; q^4)_k}{(q^4; q^4)_k^2 (q^{4-2n}; q^4)_k} q^{4k}. \tag{4.2} $$

Note that $q^n \equiv 1 \pmod{\Phi_n(q)}$, and, for $0 \leqslant k \leqslant (n-1)/2$, the polynomials $(q^2; q^2)_k^2 (q^4; q^4)_k$ and $(q^4; q^4)_k^3$ are relatively prime to $\Phi_n(q)$. From (4.2) we deduce that (4.1) holds modulo $\Phi_n(q)$.

On the other hand, by the $d=2$ case of (2.2), modulo $(1-aq^n)(a-q^n)$,

$$\sum_{k=0}^{(n-1)/2}\frac{(aq;q^2)_k(q/a;q^2)_k(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{2k}\equiv\sum_{k=0}^{(n-1)/4}\frac{(aq;q^4)_k(q/a;q^4)_k(q^2;q^4)_k}{(q^4;q^4)_k^3}q^{4k},$$

which is equivalent to (4.1) modulo $(1-aq^n)(a-q^n)$. This is because $(aq;q^4)_k(q/a;q^4)_k$ contains the factor $(1-aq^n)(1-q^n/a)$ for $(n-1)/4<k\leqslant(n-1)/2$. Since $\Phi_n(q)$ is coprime with $(1-aq^n)(a-q^n)$, we complete the proof. $\square$

Proof of Theorem 1.3. Letting $a=1$ in (4.1), we obtain the desired q-supercongruence (1.7). $\square$

5. Concluding remarks and open problems

Numerical evaluation implies that the q-congruence (1.3) does not hold for $d=3$ and the q-congruence (1.6) does not hold for $n=7$. Moreover, when we sum both sides of (1.3) over k up to $n-1$, the q-congruence seems still to be true. This is also the case for $d=3$ (which is more or less surprising). Namely, we have the following conjecture.

Conjecture 5.1. *Let $d\geqslant 2$ be an integer. Let $n\equiv 1 \pmod{2d}$ be a positive integer. Then, modulo $\Phi_n(q)^2$,*

$$\sum_{k=0}^{n-1}\frac{(q;q^d)_k^2(q^{2d-2};q^{2d})_kq^{dk}}{(q^d;q^d)_k(q^{2d-2};q^d)_k(q^{d+2};q^{2d})_k}\equiv\sum_{k=0}^{n-1}\frac{(q;q^{2d})_k^2(q^{2d-2};q^{2d})_kq^{2dk}}{(q^{2d};q^{2d})_k(q^{3d-2};q^{2d})_k(q^{d+2};q^{2d})_k}.$$

Similarly, Theorem 1.2 has such a generalization.

Conjecture 5.2. *Let $d\geqslant 3$ be an integer. Let $n\equiv -1 \pmod{2d}$ be a positive integer. Then, modulo $\Phi_n(q)^2$,*

$$\sum_{k=0}^{n-1}\frac{(q^{-1};q^d)_k^2(q^{2d+2};q^{2d})_kq^{dk}}{(q^d;q^d)_k(q^{2d+2};q^d)_k(q^{d-2};q^{2d})_k}\equiv\sum_{k=0}^{n-1}\frac{(q^{-1};q^{2d})_k^2(q^{2d+2};q^{2d})_kq^{2dk}}{(q^{2d};q^{2d})_k(q^{3d+2};q^{2d})_k(q^{d-2};q^{2d})_k}.$$

Before we propose the third conjecture of this paper, we give the following q-congruence related to Theorem 1.3.

Theorem 5.3. *Let $n\equiv 1 \pmod 4$ be a positive integer. Then*

$$\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{2k}\equiv\sum_{k=0}^{n-1}\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{2k}\pmod{\Phi_n(q)^4}.\qquad(5.1)$$

Proof. By [2, Appendix (I.11)], we have

$$\frac{(a;q)_{n-k}}{(b;q)_{n-k}}=\frac{(a;q)_n(q^{1-n}/b;q)_k}{(b;q)_n(q^{1-n}/a;q)_k}\left(\frac{b}{a}\right)^k\equiv\frac{(a;q)_n(q/b;q)_k}{(b;q)_n(q/a;q)_k}\left(\frac{b}{a}\right)^k\pmod{\Phi_n(q)}.$$

It follows that

$$\begin{aligned}&\sum_{k=(n+1)/2}^{n-1}\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{2k}\\&=\sum_{k=1}^{(n-1)/2}\frac{(q;q^2)_{n-k}^2(q^2;q^4)_{n-k}}{(q^2;q^2)_{n-k}^2(q^4;q^4)_{n-k}}q^{2n-2k}\\&\equiv\frac{(q;q^2)_n^2(q^2;q^4)_nq^{2n}}{(q^2;q^2)_{n-1}^2(q^4;q^4)_{n-1}}\sum_{k=1}^{(n-1)/2}\frac{(q^2;q^2)_{k-1}^2(q^4;q^4)_{k-1}}{(q;q^2)_k^2(q^2;q^4)_k}q^{2k}\pmod{\Phi_n(q)^4},\end{aligned}\tag{5.2}$$

where we have used the q-congruence $(q;q^2)_n^2(q^2;q^4)_n\equiv 0\pmod{\Phi_n(q)^3}$. Similarly to [5, Lemma 3.1], we can prove that

$$\frac{(q^2;q^2)_{(n+1)/2-k-1}}{(q;q^2)_{(n+1)/2-k}}\equiv(-1)^{(n+1)/2}\frac{(q^2;q^2)_{k-1}}{(q;q^2)_k}q^{(n^2-1)/4+k}\pmod{\Phi_n(q)}$$

for $1\leqslant k\leqslant(n-1)/2$, and so the k-th term plus the $((n+1)/2-k)$-th term s in the summation of the right-hand side (5.2) is congruent to 0 modulo $\Phi_n(q)$. Since the fraction before the summation is congruent to 0 modulo $\Phi_n(q)^3$, we conclude that the right-hand side of (5.2) is congruent to 0 modulo $\Phi_n(q)^4$, thus establishing (5.1). □

Conjecture 5.4. *Let $n\equiv 1\pmod 4$ be a positive integer. Then*

$$\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{2k}\equiv\sum_{k=0}^{n-1}\frac{(q;q^4)_k^2(q^2;q^4)_k}{(q^4;q^4)_k^3}q^{4k}\pmod{\Phi_n(q)^4}.$$

Recently, the author and Zudilin [9] proved a number of Dwork-type q-supercongruences. We conjecture that (1.5) can be generalized to the following Dwork-type q-supercongruence.

Conjecture 5.5. *Let $n\equiv 1\pmod 4$ be a positive integer and let $r\geqslant 1$. Then, modulo $\prod_{j=1}^r\Phi_{n^j}(q)^2$,*

$$\begin{aligned}&\sum_{k=0}^{(n^r-1)/d}\frac{(q;q^4)_k^2(q^2;q^4)_k}{(q^4;q^4)_k^3}q^{4k}\\&\equiv\frac{(q^2;q^4)_{(n^r-1)/4}^2(q^{4n};q^{4n})_{(n^{r-1}-1)/4}^2}{(q^4;q^4)_{(n^r-1)/4}^2(q^{2n};q^{4n})_{(n^{r-1}-1)/4}^2}q^{(n-1)/2}\sum_{k=0}^{(n^{r-1}-1)/d}\frac{(q^n;q^{4n})_k^2(q^{2n};q^{4n})_k}{(q^{4n};q^{4n})_k^3}q^{4nk},\end{aligned}$$

where $d=1,2,4$.

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