More q-congruences from Singh's quadratic transformation

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Abstract. In a recent paper, the first author obtained some q-congruences for truncated ${}_4\phi_3$ series form Singh's quadratic transformation. In this paper, by applying Singh's quadratic transformation again, we give some new q-congruences for truncated ${}_3\phi_2$ series. We also propose several related conjectures on q-congruences for further study.

Keywords: cyclotomic polynomials; q-supercongruences; Singh's quadratic transformation; creative microscoping

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1. Introduction

In 1997, Van Hamme [24, (H.2)] established the following amazing congruence: for any odd prime p,

$$\sum_{k=0}^{(p-1)/2}\frac{(\frac{1}{2})_k^3}{k!^3}\equiv\begin{cases}-\Gamma_p(1/4)^4 \pmod{p^2}, & \text{if } p\equiv 1 \pmod 4,\\ 0 \pmod{p^2}, & \text{if } p\equiv 3 \pmod 4,\end{cases}\tag{1.1}$$

where $(a)_n=a(a+1)\cdots(a+n-1)$ stands for the Pochhammer symbol and $\Gamma_p(x)$ denotes the p-adic Gamma function. A number of distinct generalizations of (1.1) have been given in [7,9,11,12,15,18–20]. For example, the first author and Zudilin [9, Theorem 2] utilized the "creative microscoping" method introduced in [8] to give the following q-analogue of (1.1): modulo $\Phi_n(q)^2$,

$$\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{2k}\equiv\begin{cases}\dfrac{(q^2;q^4)_{(n-1)/4}^2}{(q^4;q^4)_{(n-1)/4}^2}q^{(n-1)/2} & \text{if } n\equiv 1 \pmod 4,\\ 0 & \text{if } n\equiv 3 \pmod 4.\end{cases}$$

Here and in what follows, $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ denotes the *q-shifted factorial*, $[n]=(1-q^n)/(1-q)$ is the *q-integer*, and $\Phi_n(q)$ represents the n-th *cyclotomic polynomial* in q, which can be factorized as follows:

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(n,k)=1}}(q-\zeta^k),$$

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where ζ is an n-th primitive root of unity.

For more q-congruences derived from transformations for basic hypergeometric series, we refer the reader to [2, 5]. Recently, the first author [4] applied Singh's quadratic transformation [1, Appendix (III.21)] to obtain some q-congruences, such as: for any positive integer $n\equiv 1 \pmod 4$,

$$\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{2k}\equiv\sum_{k=0}^{(n-1)/2}\frac{(q;q^4)_k^2(q^2;q^4)_k}{(q^4;q^4)_k^3}q^{4k}\pmod{\Phi_n(q)^3}. \tag{1.2}$$

In this note, we shall deduce more q-congruences from Singh's quadratic transformation. Our first result can be stated as follows.

Theorem 1.1. *Let $d\geqslant 2$ be an integer and x an indeterminate. Let $n\equiv 1 \pmod{2d}$ be a positive integer. Then*

$$\sum_{k=0}^{(n-1)/d}\frac{(q;q^d)_k^2(x;q^d)_k}{(q^d;q^d)_k(q^{d+2};q^{2d})_k}q^{dk}\equiv\sum_{k=0}^{(n-1)/(2d)}\frac{(q;q^{2d})_k^2(x^2;q^{2d})_k}{(q^{2d};q^{2d})_k(q^{d+2};q^{2d})_k}q^{2dk}\pmod{\Phi_n(q)^2}.$$

For $d=2$, we obtain the following conclusion from Theorem 1.1.

Corollary 1.2. *Let $n\equiv 1 \pmod 4$ be a positive integer and x an indeterminate. Then*

$$\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k^2(x;q^2)_k}{(q^2;q^2)_k(q^4;q^4)_k}q^{2k}\equiv\sum_{k=0}^{(n-1)/4}\frac{(q;q^4)_k^2(x^2;q^4)_k}{(q^4;q^4)_k(q^4;q^4)_k}q^{4k}\pmod{\Phi_n(q)^2}. \tag{1.3}$$

Note that the first author and Zeng [6, Theorem 2.5] have already obtained a related q-congruence: for any odd prime p,

$$\sum_{k=0}^{(p-1)/2}\frac{(q;q^2)_k^2(x;q^2)_k}{(q^2;q^2)_k(q^4;q^4)_k}q^{2k}\equiv(-1)^{(p-1)/2}\sum_{k=0}^{(p-1)/2}\frac{(q;q^2)_k^2(-x;q^2)_k}{(q^2;q^2)_k(q^4;q^4)_k}q^{2k}\pmod{[p]^2}. \tag{1.4}$$

For some generalizations of (1.4), see [3]. If we take $x=-q$ or $x=-q^2$ in (1.3), then we are led to the following two q-congruences: for $n\equiv 1 \pmod 4$,

$$\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k(q^2;q^4)_k}{(q^2;q^2)_k(q^4;q^4)_k}q^{2k}\equiv\sum_{k=0}^{(n-1)/4}\frac{(q;q^4)_k^2(q^2;q^4)_k}{(q^4;q^4)_k(q^4;q^4)_k}q^{4k}\pmod{\Phi_n(q)^2}, \tag{1.5}$$

$$\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k^2}{(q^2;q^2)_k^2}q^{2k}\equiv\sum_{k=0}^{(n-1)/4}\frac{(q;q^4)_k^2}{(q^4;q^4)_k}q^{4k}\pmod{\Phi_n(q)^2}. \tag{1.6}$$

Letting n be a prime and $q\to 1$, then both (1.5) and (1.6) imply the following congruence: for any prime $p\equiv 1 \pmod 4$,

$$\sum_{k=0}^{(p-1)/2}\frac{(\frac12)_k^2}{k!^2}\equiv 1 \pmod{p^2},$$

which was conjectured by Rodriguez-Villegas [17] in his study of hypergeometric families of Calabi–Yau manifolds and was first proved by Mortenson [16].

Letting $x=0$ in (1.3), we immediately get the following corollary.

Corollary 1.3. *Let $d\geqslant 2$ be an integer. Let $n\equiv 1 \pmod{2d}$ be a positive integer. Then*

$$\sum_{k=0}^{(n-1)/d}\frac{(q;q^d)_k^2 q^{dk}}{(q^d;q^d)_k(q^{d+2};q^{2d})_k}\equiv\sum_{k=0}^{(n-1)/(2d)}\frac{(q;q^{2d})_k^2 q^{2dk}}{(q^{2d};q^{2d})_k(q^{d+2};q^{2d})_k}\pmod{\Phi_n(q)^2}. \tag{1.7}$$

Letting $n=p$ be a prime and $q\to 1$ in (1.7), we obtain the result: for any integer $d\geqslant 2$, and prime $p\equiv 1 \pmod{2d}$,

$$\sum_{k=0}^{(p-1)/d}\frac{(\frac{1}{d})_k^2}{k!(\frac{d+2}{2d})_k 2^k}\equiv\sum_{k=0}^{(p-1)/(2d)}\frac{(\frac{1}{2d})_k^2}{k!(\frac{d+2}{2d})_k}\pmod{p^2}.$$

Besides, the $d=2$ case of (1.7) leads to the following q-congruence: for any positive integer $n\equiv 1 \pmod 4$,

$$\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k^2 q^{2k}}{(q^2;q^2)_k(q^4;q^4)_k}\equiv\sum_{k=0}^{(n-1)/4}\frac{(q;q^4)_k^2}{(q^4;q^4)_k^2}q^{4k}\pmod{\Phi_n(q)^2}. \tag{1.8}$$

By the proof of [9, Theorem 1], we know that the right-hand side of (1.8) is congruent to

$$q^{(n^2-1)/4}\frac{(q^{3-n};q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}\equiv(-1)^{(n-1)/4}q^{(n-1)(n+3)/8}\frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}\pmod{\Phi_n(q)^2}.$$

Namely, we obtain the following conclusion due to Liu and Liu [13].

Corollary 1.4. *Let $n\equiv 1 \pmod 4$ be a positive integer. Then*

$$\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k^2 q^{2k}}{(q^2;q^2)_k(q^4;q^4)_k}\equiv(-1)^{(n-1)/4}q^{(n-1)(n+3)/8}\frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}\pmod{\Phi_n(q)^2}. \tag{1.9}$$

Note that the q-congruence (1.9) may be regarded as a q-analogue of the first part of the following congruence conjecture by Z.-W. Sun [22, Conjecture 5.5]: modulo p^2,

$$\sum_{k=0}^{(p-1)/2}\frac{1}{32^k}\binom{2k}{k}^2\equiv\begin{cases}2x-\dfrac{p}{2x}, & \text{if } p\equiv 1 \pmod 4\\ & \text{and } p=x^2+y^2 \text{ with } x\equiv 1 \pmod 4,\\ 0, & \text{if } p\equiv 3 \pmod 4,\end{cases}$$

which was confirmed by Tauraso [23] and Z.-H. Sun [21].

Our second result is the following q-congruence analogous to (1.3).

Theorem 1.5. *Let $d \geqslant 3$ be an integer and x an indeterminate. Let $n \equiv -1 \pmod{2d}$ be a positive integer. Then*

$$\sum_{k=0}^{(n+1)/d} \frac{(q^{-1};q^d)_k^2 (x;q^d)_k q^{dk}}{(q^d;q^d)_k (q^{d-2};q^{2d})_k} \equiv \sum_{k=0}^{(n+1)/(2d)} \frac{(q^{-1};q^{2d})_k^2 (x^2;q^{2d})_k q^{2dk}}{(q^{2d};q^{2d})_k (q^{d-2};q^{2d})_k} \pmod{\Phi_n(q)^2}. \tag{1.10}$$

Letting $x = 0$ in the above theorem, we get the following corollary.

Corollary 1.6. *Let $d \geqslant 3$ be an integer. Let $n \equiv -1 \pmod{2d}$ be a positive integer. Then*

$$\sum_{k=0}^{(n+1)/d} \frac{(q^{-1};q^d)_k^2 q^{dk}}{(q^d;q^d)_k (q^{d-2};q^{2d})_k} \equiv \sum_{k=0}^{(n+1)/(2d)} \frac{(q^{-1};q^{2d})_k^2 q^{2dk}}{(q^{2d};q^{2d})_k (q^{d-2};q^{2d})_k} \pmod{\Phi_n(q)^2}. \tag{1.11}$$

Similarly as before, when $n = p$ is a prime we may take $q \to 1$ to obtain the following result: for any integer $d \geqslant 3$, and prime $p \equiv -1 \pmod{2d}$,

$$\sum_{k=0}^{(p+1)/d} \frac{(-\frac{1}{d})_k^2}{k!(\frac{d-2}{2d})_k 2^k} \equiv \sum_{k=0}^{(p+1)/(2d)} \frac{(-\frac{1}{2d})_k^2}{k!(\frac{d-2}{2d})_k} \pmod{p^2}.$$

Besides, taking $x = -q$ and $d = 4$ in Theorem 1.5, we arrive at the following conclusion.

Corollary 1.7. *Let $n \equiv 7 \pmod{8}$ be a positive integer. Then*

$$\sum_{k=0}^{(n+1)/4} \frac{(q^{-1};q^4)_k^2 q^{4k}}{(q^4;q^4)_k (q;q^4)_k} \equiv \sum_{k=0}^{(n+1)/8} \frac{(q^{-1};q^8)_k^2 q^{8k}}{(q^8;q^8)_k} \pmod{\Phi_n(q)^2}. \tag{1.12}$$

Letting n be a prime and $q \to 1$ in (1.12) yields the following congruence: for any prime $p \equiv 7 \pmod{8}$,

$$\sum_{k=0}^{(p+1)/4} \frac{(-\frac{1}{4})_k^2}{k!(\frac{1}{4})_k} \equiv 1 \pmod{p^2}. \tag{1.13}$$

Our last result is a q-congruence modulo $\Phi_n(q)^3$, which is a generalization of (1.3) for $d = 2$ and $x = q$.

Theorem 1.8. *Let $n \equiv 1 \pmod{4}$ be a positive integer. Then*

$$\sum_{k=0}^{n-1} \frac{(q;q^2)_k^3}{(q^2;q^2)_k (q^4;q^4)_k} q^{2k} \equiv \sum_{k=0}^{n-1} \frac{(q;q^4)_k^2 (q^2;q^4)_k}{(q^4;q^4)_k^2} q^{4k} \pmod{\Phi_n(q)^3}. \tag{1.14}$$

However, we cannot obtain anything interesting when we let n be a prime and take $q \to 1$ in (1.14).

The paper is organized as follows. We shall prove Theorems 1.1, 1.5, and 1.8 in Sections 2–4, respectively. In the final Section 5, we propose three conjectures on q-congruences, which are generalizations of (1.3) modulo $\Phi_n(q)^3$.

2. Proof of Theorem 1.1

Recall that the *basic hypergeometric series* ${}_{r+1}\phi_r$ is defined by

$$ {}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix}; q,\ z\right] = \sum_{k=0}^{\infty} \frac{(a_1;q)_k (a_2;q)_k \cdots (a_{r+1};q)_k z^k}{(q;q)_k (b_1;q)_k \cdots (b_r;q)_k}. $$

Then Singh's quadratic transformation [1, Appendix (III.21)] can be stated as follows:

$$ {}_4\phi_3\left[\begin{matrix} a^2, b^2, c, d \\ ab\sqrt{q}, -ab\sqrt{q}, -cd \end{matrix}; q, q\right] = {}_4\phi_3\left[\begin{matrix} a^2, b^2, c^2, d^2 \\ a^2b^2q, -cd, -cdq \end{matrix}; q^2, q^2\right], \tag{2.1} $$

provided that the two series terminate. It is clear that the $d=0$ case of (2.1) reduces to

$$ {}_3\phi_2\left[\begin{matrix} a^2, b^2, c \\ ab\sqrt{q}, -ab\sqrt{q} \end{matrix}; q, q\right] = {}_3\phi_2\left[\begin{matrix} a^2, b^2, c^2 \\ a^2b^2q, 0 \end{matrix}; q^2, q^2\right], \tag{2.2} $$

provided that the two ${}_3\phi_2$ series terminate. The transformation (2.2) may be considered as a q-analogue of Gauss' quadratic transformation

$$ {}_2F_1(2a, 2b; a+b+\tfrac{1}{2}; z) = {}_2F_1(a, b; a+b+\tfrac{1}{2}; 4z(1-z)) $$

when the two series terminate.

We first use (2.2) to build a parametric generalization of Theorem 1.1.

Theorem 2.1. *Let $d \geqslant 2$ be an integer and let a and x be indeterminates. Let $n \equiv 1 \pmod{2d}$ be a positive integer. Then, modulo $(1-aq^n)(a-q^n)$,*

$$ \sum_{k=0}^{(n-1)/d} \frac{(aq;q^d)_k (q/a;q^d)_k (x;q^d)_k q^{dk}}{(q^d;q^d)_k (q^{d+2};q^{2d})_k} \equiv \sum_{k=0}^{(n-1)/(2d)} \frac{(aq;q^{2d})_k (q/a;q^{2d})_k (x^2;q^{2d})_k q^{2dk}}{(q^{2d};q^{2d})_k (q^{d+2};q^{2d})_k}. \tag{2.3} $$

Proof. Making the parameter substitutions $q \mapsto q^d$, $a = q^{(1-n)/2}$, $b = q^{(1+n)/2}$, and $c = x$ in Singh's transformation (2.1), we obtain

$$ {}_3\phi_2\left[\begin{matrix} q^{1-n}, q^{1+n}, x \\ q^{(d+2)/2}, -q^{(d+2)/2} \end{matrix}; q^d, q^d\right] = {}_3\phi_2\left[\begin{matrix} q^{1-n}, q^{1+n}, x^2 \\ q^{d+2}, 0 \end{matrix}; q^{2d}, q^{2d}\right]. $$

Namely, the two sides of (2.3) are equal for $a = q^{\pm n}$. Hence, the q-congruence (2.3) holds modulo $1-aq^n$ and $a-q^n$. Since $1-aq^n$ and $a-q^n$ are coprime polynomials in q, we complete the proof. ☐

Proof of Theorem 1.1. Since $n \equiv 1 \pmod{2d}$, we know that $\gcd(2d, n) = 1$. Therefore, when $d \geqslant 2$ the polynomials

$$ (q^d;q^d)_k (q^{d+2};q^{2d})_k \quad (0 \leqslant k \leqslant (n-1)/d) $$

and

$$ (q^{2d};q^{2d})_k (q^{d+2};q^{2d})_k \quad (0 \leqslant k \leqslant (n-1)/(2d)) $$

are all coprime with $\Phi_n(q)$. Furthermore, the polynomial $1-q^n$ contains the factor $\Phi_n(q)$. The proof of (1.3) then follows from the q-congruence (2.3) by setting $a=1$. ☐

3. Proof of Theorem 1.5

Similarly as before, we need to establish the following parametric generalization of Theorem 1.5.

Theorem 3.1. *Let $d \geqslant 3$ be an integer and a an indeterminate. Let $n \equiv -1 \pmod{2d}$ be a positive integer. Then, modulo $(1-aq^n)(a-q^n)$,*

$$\sum_{k=0}^{(n+1)/d} \frac{(aq^{-1};q^d)_k(q^{-1}/a;q^d)_k(q^{d+1};q^{2d})_k q^{dk}}{(q^d;q^d)_k(q^{d-2};q^{2d})_k}$$
$$\equiv \sum_{k=0}^{(n+1)/(2d)} \frac{(aq^{-1};q^{2d})_k(q^{-1}/a;q^{2d})_k(q^{2d+2};q^{2d})_k q^{2dk}}{(q^{2d};q^{2d})_k(q^{d-2};q^{2d})_k}. \tag{3.1}$$

Proof. Performing the parameter substitutions $q \mapsto q^d$, $a = q^{-(n+1)/2}$, $b = q^{(n-1)/2}$, and $c = q^{d+1}$ in (2.1), we get

$${}_3\phi_2\left[\begin{matrix} q^{-n-1}, q^{n-1}, q^{d+1} \\ q^{(d-2)/2}, -q^{(d-2)/2} \end{matrix}; q^d, q^d\right] = {}_3\phi_2\left[\begin{matrix} q^{-n-1}, q^{n-1}, q^{2d+2} \\ q^{d-2}, 0 \end{matrix}; q^{2d}, q^{2d}\right].$$

Namely, both sides of (3.1) are equal for $a = q^{\pm n}$. This proves that (3.1) is true modulo $1-aq^n$ and $a-q^n$. □

Proof of Theorem 1.5. Since $d \geqslant 3$, $n \equiv -1 \pmod{2d}$, we conclude that $\gcd(2d,n)=1$ and the polynomials

$$(q^d;q^d)_k(q^{d-2};q^{2d})_k \quad (0 \leqslant k \leqslant (n+1)/d))$$

and

$$(q^{2d};q^{2d})_k(q^{d-2};q^{2d})_k \quad (0 \leqslant k \leqslant (n+1)/(2d))$$

are all coprime with $\Phi_n(q)$. The proof then follows from the q-congruence (3.1) by taking $a=1$. □

4. Proof of Theorem 1.8

Likewise, we first give a q-congruence with an extra parameter a.

Theorem 4.1. *Let $n \equiv 1 \pmod 4$ be a positive integer and a an indeterminate. Then, modulo $\Phi_n(q)(1-aq^n)(a-q^n)$,*

$$\sum_{k=0}^{n-1} \frac{(aq;q^2)_k(q/a;q^2)_k(q;q^2)_k}{(q^2;q^2)_k(q^4;q^4)_k} q^{2k} \equiv \sum_{k=0}^{n-1} \frac{(aq;q^4)_k(q/a;q^4)_k(q^2;q^4)_k}{(q^4;q^4)_k^2} q^{4k}. \tag{4.1}$$

Proof. Letting $q \mapsto q^2$, $a \to \sqrt{aq}$, $b \to \sqrt{q/a}$, and $c = q^{1-n}$ in (2.1), we have

$$
{}_3\phi_2\left[\begin{matrix} aq, q/a, q^{1-n} \\ q^2, -q^2 \end{matrix} ; q^2, q^2\right] = {}_3\phi_2\left[\begin{matrix} aq, q/a, q^{2-2n} \\ q^4, 0 \end{matrix} ; q^4, q^4\right],
$$

which can be written as

$$
\sum_{k=0}^{n-1} \frac{(aq;q^2)_k (q/a;q^2)_k (q^{1-n};q^2)_k}{(q^2;q^2)_k (q^4;q^4)_k} q^{2k} = \sum_{k=0}^{n-1} \frac{(aq;q^4)_k (q/a;q^4)_k (q^{2-2n};q^4)_k}{(q^4;q^4)_k^2} q^{4k}. \tag{4.2}
$$

Since $q^n \equiv 1 \pmod{\Phi_n(q)}$, and the polynomials $(q^2;q^2)_k (q^4;q^4)_k$ and $(q^4;q^4)_k^2$ are coprime with $\Phi_n(q)$ for $0 \leqslant k \leqslant n-1$. we deduce from (4.2) that (4.1) is true modulo $\Phi_n(q)$.

On the other hand, the $d = 2$ case of (2.3) implies that, modulo $(1 - aq^n)(a - q^n)$,

$$
\sum_{k=0}^{(n-1)/2} \frac{(aq;q^2)_k (q/a;q^2)_k (q;q^4)_k}{(q^2;q^2)_k (q^4;q^4)_k} q^{2k} \equiv \sum_{k=0}^{(n-1)/4} \frac{(aq;q^4)_k (q/a;q^4)_k (q^2;q^4)_k}{(q^4;q^4)_k^2} q^{4k},
$$

which is equivalent to (4.1) modulo $(1 - aq^n)(a - q^n)$. This is because $(aq;q^4)_k (q/a;q^4)_k$ incorporates the factor $(1 - aq^n)(1 - q^n/a)$ for $(n-1)/4 < k \leqslant (n-1)/2$. Noticing that $\Phi_n(q)$ is coprime with $(1 - aq^n)(a - q^n)$, we accomplish the proof. □

Proof of Theorem 1.8. Letting $a = 1$ in (4.1), we obtain the desired q-supercongruence (1.14). □

5. A generalization of (1.13)

We find that the congruence (1.13) has the following generalization: for any prime $p \equiv 3 \pmod 4$,

$$
\sum_{k=0}^{(p+1)/4} \frac{(-\frac{1}{4})_k^2}{k! (\frac{1}{4})_k} \equiv (-1)^{(p+1)/4} \pmod{p^2}. \tag{5.1}
$$

In fact, we have the following q-analogue of (5.1).

Theorem 5.1. *Let $n \equiv 3 \pmod 4$ be a positive integer. Then modulo $\Phi_n(q)^2$,*

$$
\sum_{k=0}^{(n+1)/4} \frac{(q^{-1};q^4)_k^2 q^{4k}}{(q^4;q^4)_k (q;q^4)_k} \equiv (-1)^{(n+1)/4} q^{(n^2-1)/8}. \tag{5.2}
$$

Proof. We first establish the following q-congruence:

$$
\sum_{k=0}^{(n+1)/4} \frac{(aq^{-1};q^4)_k (a^{-1}q^{-1};q^4)_k q^{4k}}{(q^4;q^4)_k (q;q^4)_k} \equiv (-1)^{(n+1)/4} q^{(n^2-1)/8} \pmod{(1 - aq^n)(a - q^n)}. \tag{5.3}
$$

Recall that the q-Chu-Vandermonde summation [1, Appendix (II.6)] can be written as

$$ {}_2\phi_1\left[\begin{matrix} a, q^{-n} \\ c \end{matrix}; q, q\right] = \frac{(c/a;q)_n a^n}{(c;q)_n}. \tag{5.4} $$

Letting $q \mapsto q^4$, $a \mapsto q^{n-1}$, $n \mapsto (n+1)/4$, and $c \mapsto q$ in (5.4), we have

$$ \sum_{k=0}^{(n+1)/4} \frac{(q^{n-1};q^4)_k (q^{-n-1};q^4)_k q^{4k}}{(q^4;q^4)_k (q;q^4)_k} = (-1)^{(n+1)/4} q^{(n^2-1)/8}. \tag{5.5} $$

which is just the $a = q^{\pm n}$ case of (5.3). It is easy to see that the denominators of (5.3) are all coprime with $\Phi_n(q)$. The proof of (5.2) then immediately follows from the $a = 1$ case of (5.3). $\square$

6. Some open problems

Numerical calculation implies that the q-congruence (1.8) holds modulo $\Phi_n(q)^3$ when both sides are summed over k up to $n-1$. Namely, we have the following conjecture.

Conjecture 6.1. *Let $n \equiv 1 \pmod 4$ be a positive integer. Then*

$$ \sum_{k=0}^{n-1} \frac{(q;q^2)_k^2}{(q^2;q^2)_k (q^4;q^4)_k} q^{2k} \equiv \sum_{k=0}^{n-1} \frac{(q;q^4)_k^2}{(q^4;q^4)_k^2} q^{4k} \pmod{\Phi_n(q)^3}. $$

In particular, if $n \equiv 1 \pmod 4$ is a prime, then

$$ \sum_{k=0}^{p-1} \frac{1}{32^k} \binom{2k}{k}^2 \equiv \sum_{k=0}^{p-1} \frac{(\frac{1}{4})_k^2}{k!^2} \pmod{p^3}. $$

It is natural to believe that Conjecture 6.1 have a parametric version as follows:

Conjecture 6.2. *Let $n \equiv 1 \pmod 4$ be a positive integer and a an indeterminate. Then, modulo $\Phi_n(q)(1-aq^n)(a-q^n)$,*

$$ \sum_{k=0}^{n-1} \frac{(aq;q^2)_k (q/a;q^2)_k}{(q^2;q^2)_k (q^4;q^4)_k} q^{2k} \equiv \sum_{k=0}^{n-1} \frac{(aq;q^4)_k (q/a;q^4)_k}{(q^4;q^4)_k^2} q^{4k}. \tag{6.1} $$

Note that the $d = 2$ and $x = 0$ case of Theorem 2.1 means that the q-congruence (6.1) is true modulo $(1-aq^n)(a-q^n)$. It remains to prove that (6.1) is also true modulo $\Phi_n(q)$. Although we cannot prove it at the moment being, we find the following more general conjecture.

Conjecture 6.3. *Let n be a positive odd integer and x, y indeterminates. Then*

$$\sum_{k=0}^{n-1} \frac{(x;q)_k (y;q^2)_k}{(q;q)_k (xyq;q)_k} q^k \equiv \sum_{k=0}^{n-1} \frac{(x;q^2)_k (y;q^2)_k}{(q^2;q^2)_k (xyq;q^2)_k} q^{2k} \pmod{\Phi_n(q)}. \tag{6.2}$$

It is not difficult to see that the q-congruence (6.1) modulo $\Phi_n(q)$ follows from (6.2) by making suitable parameter replacements. By induction on n, we can show that

$$\sum_{k=0}^{n-1} \frac{(x;q)_k}{(q;q)_k} q^k = \frac{(xq;q)_{n-1}}{(q;q)_{n-1}}.$$

Since $(xq;q)_{n-1}$ is congruent to $(xq^2;q^2)_{n-1}$ modulo $\Phi_n(q)$, we conclude that Conjecture 6.3 is true for $x = 0$ or $y = 0$.

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