Three families of q-supercongruences from a quadratic transformation of Rahman

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Abstract. We present three families of q-supercongruences modulo the square and cube of a cyclotomic polynomial from a quadratic transformation by Rahman. In particular, as a limiting case we obtain the following supercongruence: for $0 < r < d$, $\gcd(d, r) = 1$, $d + r$ odd, and any prime $p \equiv d + r \pmod{2d}$,

$$\sum_{k=0}^{(dp+p-r)/d} (3dk + r)\frac{(\frac{r}{2d})_k(\frac{r}{d})_k^2(\frac{d-r}{d})_k}{k!^3(\frac{d+2r}{2d})_k 4^k} \equiv 0 \pmod{p^3},$$

where $(x)_n = x(x + 1)\cdots(x + n - 1)$ is the rising factorial. We also put forward four related conjectures on q-supercongruences.

Keywords: q-supercongruences; p-adic Gamma function; Rahman's transformation; creative microscoping

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1. Introduction

For any odd prime p, Morita's p-adic Gamma function $\Gamma_p(x)$ is defined as follows [15]: $\Gamma_p(0) = 1$, for any positive integer x,

$$\Gamma_p(n) = (-1)^n \prod_{\substack{1<j<n \\ p\nmid j}} j,$$

and for any p-adic integer x,

$$\Gamma_p(x) = \lim_{x_n \to x} \Gamma_p(x_n)$$

where x_n is any sequence of positive integers p-adically approaching x. Some interesting properties of the p-adic Gamma function can be found in [14].

Following the work of Long [13] and Long and Ramakrishna [14], employing a ${}_7F_6$ summation of Gessel and Stanton [3], He [8] proved the following supercongruence:

$$\sum_{k=0}^{(p-1)/2}(6k+1)\frac{(\frac{1}{2})_k^3(\frac{1}{4})_k}{k!^4 4^k} \equiv \begin{cases} (-1)^{(p+3)/4}p\Gamma_p(\frac{1}{2})\Gamma_p(\frac{1}{4})^2 \pmod{p^2}, & \text{if } p\equiv 1 \pmod 4,\\ 0 \pmod{p^2}, & \text{if } p\equiv 3 \pmod 4, \end{cases} \tag{1.1}$$

where $(x)_n=x(x+1)\cdots(x+n-1)$ is the rising factorial. We need to point out that the factor $(-1)^{(p+3)/4}$ was neglected by He in his original result. Shortly afterwards, Liu [10] further showed that (1.1) holds modulo p^3 applying another ${}_7F_6$ summation in [3].

Very recently, using a quadratic transformation of Rahman (see (1.6))) and the 'creative microscoping' method developed by the author and Zudilin [7], Liu and Wang [12] obtained a q-analogue of Liu's generalization of (1.1) modulo p^3: for any positive odd integer n, modulo $[n]\Phi_n(q)^2$,

$$\sum_{k=0}^{M}[6k+1]\frac{(q;q^4)_k(q;q^2)_k^3}{(q^2;q^2)_k(q^4;q^4)_k^3}q^{k^2+k} \equiv \begin{cases} \dfrac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}[n]q^{(1-n)/4}, & \text{if } n\equiv 1 \pmod 4,\\ 0, & \text{if } n\equiv 3 \pmod 4, \end{cases} \tag{1.2}$$

where $M=n-1$ or $(n-1)/2$. However, it should be pointed out that (1.2) follows directly from the $b=q$ case of [7, Theorem 4.5]. Here we need to familiarize ourselves with the standard q-notation. The *q-shifted factorial* is defined as $(a;q)_0=1$ and $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n\geqslant 1$ or $n=\infty$. For simplicity, we will often use the condensed notation $(a_1,a_2,\dots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n$ for $n=0,1,\dots,$ or $n=\infty$. The *q-integer* is defined as $[n]=(1-q^n)/(1-q)$. Moreover, $\Phi_n(q)$ denotes the n-th *cyclotomic polynomial*, which may be written as

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}}(q-\zeta^k),$$

with ζ being any n-th primitive root of unity. Let $A(q)$ and $B(q)$ be two rational functions in q and $P(q)$ a polynomial in q with integer coefficients. We say that $A(q)$ is congruent to $B(q)$ modulo $P(q)$, denoted $A(q)\equiv B(q) \pmod{P(q)}$, if $P(q)$ divides the numerator of the reduced fraction $A(q)-B(q)$ in the polynomial ring $\mathbb{Z}[q]$. If $A(q)\equiv 0 \pmod{P(q)}$, sometimes we will also say that $A(q)$ is divisible by $P(q)$. It should be mentioned that there are plenty of papers on q-supercongruences during the past few years. See, for instance, [1, 5, 6, 9, 11, 12, 16–19, 21].

In this paper, we shall give some generalizations of (1.2) modulo $\Phi_n(q)^3$. Our first result can be stated as follows.

Theorem 1.1. *Let d and r be positive integers such that $d+r$ is odd, $d>r$ and* $\gcd(d,r)=1$. *Let n be a positive integer satisfying $n\equiv d+r \pmod{2d}$. Then*

$$\sum_{k=0}^{(dn+n-r)/(2d)}[3dk+r]\frac{(q^r;q^{2d})_k(q^r,q^r,q^{d-r};q^d)_k}{(q^d;q^d)_k(q^{2d},q^{2d},q^{d+2r};q^{2d})_k}q^{d(k^2+k)/2}\equiv 0 \pmod{\Phi_n(q)^3}. \tag{1.3}$$

It is easy to see that the $(d,r)=(2,1)$ case of (1.3) reduces to the second part of (1.2) modulo $\Phi_n(q)^3$, since $(q;q^2)_k$ is congruent to 0 modulo $\Phi_n(q)^3$ for $(n-1)/2<k\leqslant n-1$. Besides, the $r=1$ case of (1.3) can also be deduced from [4, Theorem 1.1] by letting $e\to 0$ or ∞. Moreover, taking $n=p$ to be a prime and $q\to 1$ in (1.3), we are led to the following result: for $0<r<d$ with $\gcd(d,r)=1$ and $d+r$ odd, and any prime $p\equiv d+r \pmod{2d}$,

$$\sum_{k=0}^{(dp+p-r)/(2d)}(3dk+r)\frac{(\frac{r}{2d})_k(\frac{r}{d})_k^2(\frac{d-r}{d})_k}{k!^3(\frac{d+2r}{2d})_k4^k}\equiv 0 \pmod{p^3},$$

which is a generalization of (1.1) for $p\equiv 3 \pmod 4$.

We also have another generalization of the $n\equiv 3 \pmod 4$ case of (1.2) modulo $\Phi_n(q)^2$.

Theorem 1.2. *Let d and r be positive integers such that r is odd, $d>r$ and* $\gcd(d,r)=1$. *Let n be a positive integer satisfying $n\equiv -r \pmod{2d}$. Suppose that $(d,r)\notin\{(3,1),(4,3)\}$. Then*

$$\sum_{k=0}^{(dn-n-r)/d}[3dk+r]\frac{(q^r;q^{2d})_k(q^r,q^r,q^{d-r};q^d)_k}{(q^d;q^d)_k(q^{2d},q^{2d},q^{d+2r};q^{2d})_k}q^{d(k^2+k)/2}\equiv 0 \pmod{\Phi_n(q)^2}. \tag{1.4}$$

Letting $n=p$ and $q\to 1$ in (1.3), we get the following result: for the same (d,r) in Theorem 1.2 and any prime $p\equiv -r \pmod{2d}$,

$$\sum_{k=0}^{(dp-p-r)/d}(3dk+r)\frac{(\frac{r}{2d})_k(\frac{r}{d})_k^2(\frac{d-r}{d})_k}{k!^3(\frac{d+2r}{2d})_k4^k}\equiv 0 \pmod{p^2},$$

which is another generalization of (1.1) modulo p^2 for $p\equiv 3 \pmod 4$.

The last aim of this paper is to give a generalization of the $n\equiv 1 \pmod 4$ case of (1.2) as follows.

Theorem 1.3. *Let d and r be positive integers such that r is odd, $d>r$ and* $\gcd(r,d)=1$. *Let n be a positive integer satisfying $n\equiv r \pmod{2d}$. Then*

$$\begin{aligned}\sum_{k=0}^{(n-r)/d}&[3dk+r]\frac{(q^r;q^{2d})_k(q^r,q^r,q^{d-r};q^d)_k}{(q^d;q^d)_k(q^{2d},q^{2d},q^{d+2r};q^{2d})_k}q^{d(k^2+k)/2}\\&\equiv[n]\frac{(q^d;q^{2d})_{(n-r)/(2d)}}{(q^{d+2r};q^{2d})_{(n-r)/(2d)}}q^{(r-d)(n-r)/(2d)} \pmod{\Phi_n(q)^3}.\end{aligned} \tag{1.5}$$

It is clear that the $(d,r)=(2,1)$ case of (1.5) reduces to the first part of (1.2) modulo $\Phi_n(q)^3$. Moreover, letting $n=p$ be a prime and $q\to 1$ in (1.5), we arrive at the following supercongruence: for $0<r<d$ with $\gcd(d,r)=1$ and r odd, and any prime $p\equiv r \pmod{2d}$,

$$\sum_{k=0}^{p-1}(3dk+r)\frac{(\frac{r}{2d})_k(\frac{r}{d})_k^2(\frac{d-r}{d})_k}{k!^3(\frac{d+2r}{2d})_k 4^k}\equiv p\frac{(\frac{1}{2})_{(p-r)/(2d)}}{(\frac{d+2r}{2d})_{(p-r)/(2d)}} \pmod{p^2}.$$

Recall that the *basic hypergeometric series* ${}_{r+1}\phi_r$ (see [2]) is defined as

$${}_{r+1}\phi_r\left[\begin{matrix} a_1,a_2,\ldots,a_{r+1} \\ b_1,b_2,\ldots,b_r \end{matrix};q,\,z\right]=\sum_{k=0}^{\infty}\frac{(a_1,a_2,\ldots,a_{r+1};q)_k z^k}{(q,b_1,\ldots,b_r;q)_k}.$$

A quadratic transformation of Rahman [2, (3.8.13)] may be stated as follows:

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(1-aq^{3k})(a,d,aq/d;q^2)_k(b,c,aq/bc;q)_k}{(1-a)(aq/d,d,q;q)_k(aq^2/b,aq^2/c,bcq;q^2)_k}q^k\\ &\quad=\frac{(aq^2,bq,cq,aq^2/bc;q^2)_\infty}{(q,aq^2/b,aq^2/c,bcq;q^2)_\infty}\,{}_3\phi_2\left[\begin{matrix} b,c,aq/bc \\ dq,aq^2/d \end{matrix};q^2,q^2\right],\end{aligned}\tag{1.6}$$

provided that d and aq/d are not of the form q^{-2n} (n is a non-negative integer).

We shall prove Theorems 1.1–1.3 by employing the method of 'creative microscoping' and Rahman's transformation (1.6) again.

2. Proof of Theorem 1.1

We first give a generalization of Theorem 1.1 with an additional parameter a. It should be pointed out that, in most cases of using the creative microscoping method, the way to insert the new parameter a is natural and easy. This is also the case here.

Theorem 2.1. *Let d and r be positive integers such that $d+r$ is odd, $d>r$ and $\gcd(d,r)=1$. Let n be a positive integer satisfying $n\equiv d+r \pmod{2d}$ and a an indeterminate. Then, modulo $\Phi_n(q)(1-aq^n)(a-q^n)$,*

$$\sum_{k=0}^{(dn+n-r)/(2d)}[3dk+r]\frac{(q^r;q^{2d})_k(aq^r,q^r/a,q^{d-r};q^d)_k}{(q^d;q^d)_k(aq^{2d},q^{2d}/a,q^{d+2r};q^{2d})_k}q^{d(k^2+k)/2}\equiv 0.\tag{2.1}$$

Proof. It is easy to see that the $d\to 0$ case of (1.6) reduces to

$$\sum_{k=0}^{\infty}\frac{(1-aq^{3k})(a;q^2)_k(b,c,aq/bc;q)_k}{(1-a)(q;q)_k(aq^2/b,aq^2/c,bcq;q^2)_k}q^{(k^2+k)/2}=\frac{(aq^2,bq,cq,aq^2/bc;q^2)_\infty}{(q,aq^2/b,aq^2/c,bcq;q^2)_\infty}.\tag{2.2}$$

Letting $q\mapsto q^d$, $a=q^{r-dn-n}$, $b=aq^r$, $c=q^r/a$ in the above formula, we get

$$\begin{aligned}
&\sum_{k=0}^{(dn+n-r)/(2d)} \frac{(1-q^{3dk+r-dn-n})(q^{r-dn-n};q^{2d})_k(aq^r,q^r/a,q^{d-r-dn-n};q^d)_k}{(1-q^{r-dn-n})(q^d;q^d)_k(q^{2d-dn-n}/a,aq^{2d-dn-n},q^{d+2r};q^{2d})_k} q^{d(k^2+k)/2}\\
&=\frac{(q^{2d+r-dn-n},aq^{d+r},q^{d+r}/a,q^{2d-r-dn-n};q^{2d})_\infty}{(q^d,q^{2d-dn-n}/a,aq^{2d-dn-n},q^{d+2r};q^{2d})_\infty}\\
&=0, \qquad (2.3)
\end{aligned}$$

where we have used $(q^{r-dn-n};q^{2d})_k=0$ for $k>(dn+n-r)/(2d)$ and $(q^{2d+r-dn-n};q^{2d})_\infty=0$. Since $n\equiv d+r \pmod{2d}$, $d+r$ is odd, and $\gcd(d,r)=1$, we have $\gcd(2d,n)=1$. Note that $1-q^m\equiv 0 \pmod{\Phi_n(q)}$ if and only if n divides m. Thus, the smallest positive integer k such that $(q^{d+2r};q^{2d})_k\equiv 0 \pmod{\Phi_n(q)}$ is $(dn+2n-d-2r)/(2d)+1$. This indicates that the polynomial $(q^{d+2r};q^{2d})_k$ is coprime with $\Phi_n(q)$ for $0\leqslant k\leqslant (dn+n-r)/(2d)$ (since $0<dn+n-r\leqslant dn+2n-d-2r$). In light of $q^n\equiv 1 \pmod{\Phi_n(q)}$, from (2.3) we deduce that

$$\sum_{k=0}^{(dn+n-r)/(2d)} [3dk+r]\frac{(q^r;q^{2d})_k(aq^r,q^r/a,q^{d-r};q^d)_k}{(q^d;q^d)_k(aq^{2d},q^{2d}/a,q^{d+2r};q^{2d})_k} q^{d(k^2+k)/2}\equiv 0 \pmod{\Phi_n(q)}. \qquad (2.4)$$

On the other hand, performing the substitutions $q\mapsto q^d$, $a=q^r$, $b=q^{r+n}$, $c=q^{r-n}$ in (2.2), we obtain

$$\begin{aligned}
&\sum_{k=0}^{(n-r)/d} \frac{(1-q^{3dk+r})(q^r;q^{2d})_k(q^{r+n},q^{r-n},q^{d-r};q^d)_k}{(1-q^r)(q^d;q^d)_k(q^{2d-n},q^{2d+n},q^{d+2r};q^{2d})_k} q^{d(k^2+k)/2}\\
&=\frac{(q^{2d+r},q^{d+r+n},q^{d+r-n},q^{2d-r};q^{2d})_\infty}{(q^d,q^{2d-n},q^{2d+n},q^{d+2r};q^{2d})_\infty}\\
&=0, \qquad (2.5)
\end{aligned}$$

where we have used $(q^{r-n};q^d)_k=0$ for $k>(n-r)/d$ and $(q^{d+r-n};q^{2d})_\infty=0$. Clearly, $(dn+n-r)/(2d)\geqslant (n-r)/d$, and so by (2.5) the left-hand side of (2.1) is also equal to 0 for $a=q^{-n}$ or $a=q^n$. In other words, the q-congruence (2.1) is true modulo $1-aq^n$ and $a-q^n$.

Since $1-aq^n$, $a-q^n$ and $\Phi_n(q)$ are pairwise coprime polynomials in q, we accomplish the proof of the theorem. □

Proof of Theorem 1.1. Since $\gcd(2d,n)=1$, the polynomial $(q^{2d};q^{2d})_k$ is coprime with $\Phi_n(q)$ for any $0\leqslant k\leqslant n-1$. Further, the polynomial $(1-q^n)^2$ contains the factor $\Phi_n(q)^2$. The proof of (1.3) then follows from (2.1) by taking $a=1$. □

3. Proof of Theorem 1.2

Similarly as before, we first give the following parametric generalization of Theorem 1.2.

Theorem 3.1. *Let d and r be positive integers such that r is odd, $d>r$ and $\gcd(d,r)=1$. Let n be a positive integer satisfying $n\equiv -r \pmod{2d}$ and a an indeterminate. Then, modulo $(1-aq^{(d-1)n})(a-q^{(d-1)n})$,*

$$\sum_{k=0}^{(dn-n-r)/d}[3dk+r]\frac{(q^r;q^{2d})_k(aq^r,q^r/a,q^{d-r};q^d)_k}{(q^d;q^d)_k(aq^{2d},q^{2d}/a,q^{d+2r};q^{2d})_k}q^{d(k^2+k)/2}\equiv 0. \tag{3.1}$$

Proof. The proof is similar to that of Theorem 3.1. This time we take $q\mapsto q^d$, $a=q^r$, $b=q^{r+(d-1)n}$, $c=q^{r-(d-1)n}$ in (2.2) to get

$$\begin{aligned}&\sum_{k=0}^{(dn-n-r)/d}\frac{(1-q^{3dk+r})(q^r;q^{2d})_k(q^{r+(d-1)n},q^{r-(d-1)n},q^{d-r};q^d)_k}{(1-q^r)(q^d;q^d)_k(q^{2d-(d-1)n},q^{2d+(d-1)n},q^{d+2r};q^{2d})_k}q^{d(k^2+k)/2}\\ &=\frac{(q^{2d+r},q^{d+r+(d-1)n},q^{d+r-(d-1)n},q^{2d-r};q^{2d})_\infty}{(q^d,q^{2d-(d-1)n},q^{2d+(d-1)n},q^{d+2r};q^{2d})_\infty}\\ &=0,\end{aligned}$$

where we have applied $(q^{r-(d-1)n};q^d)_k=0$ for $k>(dn-n-r)/d$ and $(q^{d+r-(d-1)n};q^{2d})_\infty=0$. This means that the left-hand side of (3.1) is equal to 0 for $a=q^{-(d-1)n}$ or $a=q^{(d-1)n}$. Namely, the q-congruence (3.1) is true modulo $1-aq^{(d-1)n}$ and $a-q^{(d-1)n}$. □

Proof of Theorem 1.2. For $d=2$ (and so $r=1$), the q-supercongruence (1.4) follows from (1.2) immediately. We now consider the $d\geqslant 3$ case. Since r is odd, $\gcd(d,r)=1$, and $n\equiv -r \pmod{2d}$, we have $\gcd(2d,n)=1$. Thus, the smallest positive integer k such that $(q^{d-r};q^d)_k\equiv 0 \pmod{\Phi_n(q)}$ is $(n+r-d)/d+1$, while the smallest positive integer k such that $(q^{d+2r};q^{2d})_k\equiv 0 \pmod{\Phi_n(q)}$ is $(dn-2n-d-2r)/(2d)+1$ (We can check that $dn-2n-d-2r\geqslant 0$ for $d=3$, 4, and $d\geqslant 5$, respectively).

It is easily seen that $0<(n+r-d)/d\leqslant (dn-2n-d-2r)/(2d)$ by the condition in the theorem. Therefore, the denominator of reduced form of $(q^{d-r};q^d)_k/(q^{d+2r};q^{2d})_k$ is always coprime with $\Phi_n(q)$ for $0\leqslant k\leqslant (dn-n-r)/d$. Meanwhile, the polynomial $(q^{2d};q^{2d})_k$ is also coprime with $\Phi_n(q)$ for any $0\leqslant k\leqslant n-1$. Thus, specializing $a=1$ in (3.1), we immediately obtain (1.4). □

4. Proof of Theorem 1.3

Likewise, we have the following parametric generalization of Theorem 1.3.

Theorem 4.1. *Let d and r be positive integers such that r is odd, $d>r$ and $\gcd(r,d)=1$. Let n be a positive integer with $n\equiv r \pmod{2d}$ and a an indeterminate. Then, modulo $\Phi_n(q)(1-aq^n)(a-q^n)$,*

$$\sum_{k=0}^{(n-r)/d}[3dk+r]\frac{(q^r;q^{2d})_k(aq^r,q^r/a,q^{d-r};q^d)_k}{(q^d;q^d)_k(aq^{2d},q^{2d}/a,q^{d+2r};q^{2d})_k}q^{d(k^2+k)/2}$$
$$\equiv [n]\frac{(q^d;q^{2d})_{(n-r)/(2d)}}{(q^{d+2r};q^{2d})_{(n-r)/(2d)}}q^{(r-d)(n-r)/(2d)}. \tag{4.1}$$

Proof. For $n=r$, both sides of (4.1) are equal to $[r]$ and the theorem is true. We now assume that $n\geqslant 2d+r$. Letting $q\mapsto q^d, a=q^{r-n}, b=aq^r$, $c=q^r/a$ in (2.2) leads to

$$\sum_{k=0}^{(n-r)/(2d)}\frac{(1-q^{3dk+r-n})(q^{r-n};q^{2d})_k(aq^r,q^r/a,q^{d-r-n};q^d)_k}{(1-q^{r-n})(q^d;q^d)_k(q^{2d-n}/a,aq^{2d-n},q^{d+2r};q^{2d})_k}q^{d(k^2+k)/2}$$
$$=\frac{(q^{2d+r-n},aq^{d+r},q^{d+r}/a,q^{2d-r-n};q^{2d})_\infty}{(q^d,q^{2d-n}/a,aq^{2d-n},q^{d+2r};q^{2d})_\infty}$$
$$=0. \tag{4.2}$$

By the condition in the theorem, we have $\gcd(2d,n)=1$. Like the proof of (2.3), this time the smallest positive integer k such that $(q^{d+2r};q^{2d})_k\equiv 0 \pmod{\Phi_n(q)}$ is again $(dn+2n-d-2r)/(2d)+1$, which is clearly greater than $(n-r)/(2d)+1$. This means that $(q^{d+2r};q^{2d})_k$ is coprime with $\Phi_n(q)$ for $0\leqslant k\leqslant (n-r)/(2d)$. In view of $q^n\equiv 1 \pmod{\Phi_n(q)}$, from (4.2) it follows that

$$\sum_{k=0}^{(n-r)/(2d)}[3dk+r]\frac{(q^r;q^{2d})_k(aq^r,q^r/a,q^{d-r};q^d)_k}{(q^d;q^d)_k(aq^{2d},q^{2d}/a,q^{d+2r};q^{2d})_k}q^{d(k^2+k)/2}\equiv 0 \pmod{\Phi_n(q)}. \tag{4.3}$$

Moreover, since $(n-r)/2d<(n-r)/d<(dn+2n-d-2r)/(2d)$, the k-th summand on the left-hand side of (4.1) is divisible by $\Phi_n(q)$ for $(n-r)/2d<k\leqslant (n-r)/d$. This proves that the left-hand side of (4.1) is congruent to 0 modulo $\Phi_n(q)$, and so (4.1) holds modulo $\Phi_n(q)$.

On the other hand, making the substitutions $q\mapsto q^d$, $a=q^r$, $b=q^{r+n}$, $c=q^{r-n}$ in (2.2), we obtain

$$\sum_{k=0}^{(n-r)/d}\frac{(1-q^{3dk+r})(q^r;q^{2d})_k(q^{r+n},q^{r-n},q^{d-r};q^d)_k}{(1-q^r)(q^d;q^d)_k(q^{2d-n},q^{2d+n},q^{d+2r};q^{2d})_k}q^{d(k^2+k)/2}$$
$$=\frac{(q^{2d+r},q^{d+r+n},q^{d+r-n},q^{2d-r};q^{2d})_\infty}{(q^d,q^{2d-n},q^{2d+n},q^{d+2r};q^{2d})_\infty}$$
$$=\frac{(q^{2d+r},q^{d+r-n};q^{2d})_{(n-r)/(2d)}}{(q^{d+2r},q^{2d-n};q^{2d})_{(n-r)/(2d)}}$$
$$=\frac{(1-q^n)(q^d;q^{2d})_{(n-1)/(2d)}}{(1-q^r)(q^{d+2r};q^{2d})_{(n-r)/(2d)}}q^{(r-d)(n-r)/(2d)}. \tag{4.4}$$

Multiplying both sides of (4.4) by $[r]$, we see that both sides of (4.1) are equal for $a=q^{-n}$ and $a=q^n$. Namely, the q-congruence (4.1) holds modulo $1-aq^n$ and $a-q^n$. □

Proof of Theorem 1.3. Since $\gcd(2d,n)=1$, the polynomial $(q^{2d};q^{2d})_k$ is coprime with $\Phi_n(q)$ for any $0\leqslant k\leqslant (n-r)/d$. The proof of (1.3) then follows from (4.1) by setting $a=1$. □

5. Concluding remarks

In this section, we propose some open problems for further study. Numerical calculation suggests that we can compute the sum in (1.3) for k up to $n-1$. Namely, we have the following conjecture.

Conjecture 5.1. *Let d and r be positive integers such that $d+r$ is odd, $d>r$ and* $\gcd(d,r)=1$. *Let n be a positive integer satisfying $n\equiv d+r \pmod{2d}$. Then*

$$\sum_{k=0}^{n-1}[3dk+r]\frac{(q^r;q^{2d})_k(q^r,q^r,q^{d-r};q^d)_k}{(q^d;q^d)_k(q^{2d},q^{2d},q^{d+2r};q^{2d})_k}q^{d(k^2+k)/2}\equiv 0 \pmod{\Phi_n(q)^3}. \tag{5.1}$$

In view of (1.2), the above conjecture is true for $d=2$ ($r=1$). We point out that this conjecture is also true for $d=4$. This is because, for $(d,r)=(4,1)$, there holds $(q^{d-r};q^d)_k/(q^{d+2};q^{2d})_k=1/(-q^3;q^4)_k$, and so each k-th summand on the left-hand side of (5.1) is divisible by $\Phi_n(q)^3$ for $(5n-1)/8<k\leqslant n-1$; while for $(d,r)=(4,3)$, there holds $[3dk+r](q^{d-r};q^d)_k/(q^{d+2};q^{2d})_k=[12k+3]/([4k+1](-q^5;q^4)_k)$, and likewise each k-th summand on the left-hand side of (5.1) is divisible by $\Phi_n(q)^3$ for $(5n-3)/8<k\leqslant n-1$. However, the same argument does not work for $d=3$ or $d\geqslant 5$.

For $d=3$ (and so $r=2$), the q-supercongruence in Theorem 1.1 seems to be true modulo $\Phi_n(q)^4$, which can be formulated as follows.

Conjecture 5.2. *Let n be a positive integer satisfying $n\equiv 5 \pmod 6$. Then*

$$\sum_{k=0}^{(2n-1)/3}[9k+2]\frac{(q^2;q^6)_k(q^2,q^2,q;q^3)_k}{(q^3;q^3)_k(q^6,q^6,q^7;q^6)_k}q^{3(k^2+k)/2}\equiv 0 \pmod{\Phi_n(q)^4}. \tag{5.2}$$

In particular, for any prime $p\equiv 5 \pmod 6$,

$$\sum_{k=0}^{(2p-1)/3}(9k+2)\frac{(\frac{1}{3})_k^2(\frac{2}{3})_k^2}{k!^3(\frac{7}{6})_k4^k}\equiv 0 \pmod{p^4}.$$

It is also natural to conjecture that Theorem 1.2 has the following stronger version.

Conjecture 5.3. *Let d and r be positive integers such that r is odd, $d>r$ and* $\gcd(d,r)=1$*. Let n be a positive integer satisfying $n\equiv -r \pmod{2d}$. Then*

$$\sum_{k=0}^{n-1}[3dk+r]\frac{(q^r;q^{2d})_k(q^r,q^r,q^{d-r};q^d)_k}{(q^d;q^d)_k(q^{2d},q^{2d},q^{d+2r};q^{2d})_k}q^{d(k^2+k)/2}\equiv 0 \pmod{\Phi_n(q)^3}, \tag{5.3}$$

Note that, for $(d,r)=(3,1)$ or $(4,3)$, the q-supercongruence (5.3) modulo $\Phi_n(q)^2$ is not yet confirmed. From the proof of Theorem 1.2, we know that the k-th summand on the left-hand side of (5.3) is divisible by $\Phi_n(q)^3$ for k in the range $(2dn-n-r)/(2d)<k\leqslant n-1$. Thus, we may truncated the left-hand side of (5.3) at $k=(2dn-n-r)/(2d)$. To prove Conjecture 5.3, a natural way is to prove the following parametric form: modulo $\Phi_n(q)(1-aq^{(d-1)n})(a-q^{(d-1)n})$,

$$\sum_{k=0}^{(2dn-n-r)/(2d)}[3dk+r]\frac{(q^r;q^{2d})_k(aq^r,q^r/a,q^{d-r};q^d)_k}{(q^d;q^d)_k(aq^{2d},q^{2d}/a,q^{d+2r};q^{2d})_k}q^{d(k^2+k)/2}\equiv 0. \tag{5.4}$$

This q-congruence modulo $(1-aq^{(d-1)n})(a-q^{(d-1)n})$ follows from (3.1). However, the modulus $\Phi_n(q)$ case seems rather difficult. With the assumption that $(d,r)\notin\{(3,1),(4,3)\}$, we can only show that

$$\sum_{k=0}^{(n+r-d)/d}[3dk+r]\frac{(q^r;q^{2d})_k(aq^r,q^r/a,q^{d-r};q^d)_k}{(q^d;q^d)_k(aq^{2d},q^{2d}/a,q^{d+2r};q^{2d})_k}q^{d(k^2+k)/2}\equiv 0,$$

which is far from (5.4).

Recently, Wei [20] has proved that (1.2) is true modulo $\Phi_n(q)^4$ for $M=n-1$ and $n\equiv 3 \pmod 4$ by using a more complicated summation formula of Gasper and Rahman. However, neither (5.1) nor (5.3) holds $\Phi_n(q)^4$ for general r.

Finally, we conjecture that the following variation of Theorem 1.3 should be true.

Conjecture 5.4. *Let d and r be positive integers such that r is odd, $d>r$ and* $\gcd(r,d)=1$*. Let n be a positive integer satisfying $n\equiv r \pmod{2d}$. Then*

$$\sum_{k=0}^{n-1}[3dk+r]\frac{(q^r;q^{2d})_k(q^r,q^r,q^{d-r};q^d)_k}{(q^d;q^d)_k(q^{2d},q^{2d},q^{d+2r};q^{2d})_k}q^{d(k^2+k)/2}$$
$$\equiv [n]\frac{(q^d;q^{2d})_{(n-r)/(2d)}}{(q^{d+2r};q^{2d})_{(n-r)/(2d)}}q^{(r-d)(n-r)/(2d)} \pmod{\Phi_n(q)^3}. \tag{5.5}$$

For the same reason as before, we may truncated the left-hand side of (5.5) at $k=(dn-n-d+r)/d$. It is easy to see that Conjecture 5.4 is true for $d=2,3,4$. Nevertheless, Conjecture 5.4 is still a challenging problem for $d\geqslant 5$.

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