Some q-analogues of supercongruences of Rodriguez-Villegas

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Abstract. We study different q-analogues and generalizations of the ex-conjectures of Rodriguez-Villegas. For example, for any odd prime p, we show that the known congruence

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\equiv\left(\frac{-1}{p}\right)\pmod{p^2},$$

where $(\frac{\cdot}{p})$ is the Legendre symbol, has the following two nice q-analogues:

$$\sum_{k=0}^{p-1}\frac{(q;q^2)_k^2}{(q^2;q^2)_k^2}q^{(1+\varepsilon)k}\equiv\left(\frac{-1}{p}\right)q^{\frac{(p^2-1)\varepsilon}{4}}\pmod{(1+q+\cdots+q^{p-1})^2},$$

where $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ and $\varepsilon=\pm1$. Several related conjectures are also proposed.

Keywords: congruences, least nonnegative residue, little q-Legendre polynomials, q-binomial theorem, q-Chu-Vandermonde formula

2000 Mathematics Subject Classifications: Primary 11B65, Secondary 05A10, 05A30

1. Introduction

Rodriguez-Villegas [14] discovered numerically some remarkable supercongruences between a truncated hypergeometric function associated to a Calabi-Yau manifold at a prime p and the number of its $\mathbb{F}_p$-points. In particular, Rodriguez-Villegas recorded four such supercongruences associated to elliptic curves. Following a strategy developed by Ahlgren and Ono [1], by using the Gross-Koblitz formula to write the Gaussian hypergeometric series in terms of the p-adic Γ-function, Mortenson [11,12] first proved the following four conjectured supercongruences of Rodriguez-Villegas [14, (36)].

Theorem 1.1 (Rodriguez-Villegas-Mortenson). *Let $p \geqslant 5$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2}, \tag{1.1}$$

$$\sum_{k=0}^{p-1} \frac{\binom{3k}{2k}\binom{2k}{k}}{27^k} \equiv \left(\frac{-3}{p}\right) \pmod{p^2}, \tag{1.2}$$

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{2k}\binom{2k}{k}}{64^k} \equiv \left(\frac{-2}{p}\right) \pmod{p^2}, \tag{1.3}$$

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k}\binom{3k}{k}}{432^k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2}, \tag{1.4}$$

where $(\frac{\cdot}{p})$ denotes the Legendre symbol modulo p.

Elementary proof of Theorem 1.1 has been given by Z.-H. Sun [18]. See also [7,17,22,23] for several simple proofs of (1.1). A generalization of (1.1) to the modulus p^3 case was obtained by Z.-W. Sun [19]. Note that van Hamme [26] and McCarthy and Osburn [10] have studied some related interesting supercongruences.

Recall that the *q-shifted factorials* are defined by $(a;q)_0 = 1$ and

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}) \text{ for } n = 1,2,\ldots,$$

and the *q-integer* is defined as $[p] = 1 + q + \cdots + q^{p-1}$. The starting point of this paper is the observation of the following striking q-analogue of Theorem 1.1.

Conjecture 1.2. *Let $p \geqslant 5$ be a prime and let $(\frac{\cdot}{p})$ be the Legendre symbol modulo p. Then*

$$\sum_{k=0}^{p-1} \frac{(q;q^2)_k^2}{(q^2;q^2)_k^2} \equiv \left(\frac{-1}{p}\right) q^{\frac{1-p^2}{4}} \pmod{[p]^2},$$

$$\sum_{k=0}^{p-1} \frac{(q;q^3)_k(q^2;q^3)_k}{(q^3;q^3)_k^2} \equiv \left(\frac{-3}{p}\right) q^{\frac{1-p^2}{3}} \pmod{[p]^2},$$

$$\sum_{k=0}^{p-1} \frac{(q;q^4)_k(q^3;q^4)_k}{(q^4;q^4)_k^2} \equiv \left(\frac{-2}{p}\right) q^{\frac{3(1-p^2)}{8}} \pmod{[p]^2},$$

$$\sum_{k=0}^{p-1} \frac{(q;q^6)_k(q^5;q^6)_k}{(q^6;q^6)_k^2} \equiv \left(\frac{-1}{p}\right) q^{\frac{5(1-p^2)}{12}} \pmod{[p]^2}.$$

Congruences modulo $[p]$ or $[p]^2$ (even $[p]^3$) have been studied by different authors (see [5, 13, 15, 16, 24]). Throughout the paper we will tacitly use the fact that when p is a prime the q-integer $[p]$ is an irreducible polynomial in $\mathbb{Q}[q]$. Therefore $\mathbb{Q}[q]/[p]$

is a field. Furthermore, rational functions $t(q)/s(q)$ are well defined modulo $[p]$ or $[p]^r$ ($r \geqslant 1$) provided that $s(q)$ is relatively prime to $[p]$. For any two polynomials $A(x,q) = \sum_{k=0}^{n} a_k(q)x^k$ and $B(x,q) = \sum_{k=0}^{n} b_k(q)x^k$ in x with coefficients being rational functions $t(q)/s(q)$ such that $s(q)$ is relatively prime to $[p]$, we use the convention that

$$A(x,q) \equiv B(x,q) \pmod{[p]^r} \Longleftrightarrow a_k(q) \equiv b_k(q) \pmod{[p]^r} \quad \text{for} \quad k = 0, 1, \ldots, n.$$

There are several generalizations and variations of (1.1)–(1.4) in the literature, but no q-analogues seem to be investigated hitherto. Indeed, Tauraso [22] proved the following generalization of (1.1).

Theorem 1.3 (Tauraso [22])**.** *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k}^2 \frac{x^k}{16^k} \equiv \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 (-x)^k (1-x)^{\frac{p-1}{2}-k} \pmod{p^2}. \tag{1.5}$$

Recently, Z.-H. Sun [18] introduced the generalized Legendre polynomials

$$P_n(a,x) = \sum_{k=0}^{n} \binom{a}{k}\binom{-1-a}{k}\frac{(1-x)^k}{2^k} = \sum_{k=0}^{n} \binom{a}{k}\binom{a+k}{k}\frac{(x-1)^k}{2^k}, \tag{1.6}$$

and proved many supercongruences related to $P_{p-1}(a,x)$. In particular, he obtained the following result.

Theorem 1.4 (Z.-H. Sun [18])**.** *Let p be an odd prime and let a be a p-adic integer. Then*

$$P_{p-1}(a,x) \equiv (-1)^{\langle a\rangle_p} P_{p-1}(a,-x) \pmod{p^2}, \tag{1.7}$$

and so

$$\sum_{k=0}^{p-1} \binom{a}{k}\binom{-1-a}{k}(x^k - (-1)^{\langle a\rangle_p}(1-x)^k) \equiv 0 \pmod{p^2}, \tag{1.8}$$

where $\langle a\rangle_p$ denotes the least nonnegative residue of a modulo p.

It is easy to see that the Rodriguez-Villegas-Mortenson congruences (1.1)–(1.4) immediately follows from the congruence (1.8) by taking $x = 1$ and $a = -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$. Note that the congruence (1.8) with $a = -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$ was first established by Z.-W. Sun [21]. The aim of this paper is to give q-analogues of (1.5)–(1.8). It turns out that a complete q-analogue of (1.5) is easily given. However, for a general p-adic integer a, we can only give q-analogues of (1.7) and (1.8) in the modulus p case. On the other hand, for $a = -\frac{1}{2}$, we shall give complete q-analogues of them. Thus, the first congruence in Conjecture 1.2 is proved, while the other three congruences are still open. Some further related unsolved problems will also be presented in this paper.

2. Results, I: supercongruences modulo $[p]^2$

Recall that the *q-binomial coefficients* ${n \brack k}$ are defined by

$$
{n \brack k} = {n \brack k}_q = \begin{cases} \dfrac{(q^{n-k+1};q)_k}{(q;q)_k}, & \text{if } 0 \leqslant k \leqslant n, \\ 0, & \text{otherwise.} \end{cases}
$$

We first give a q-analogue of Tauraso's congruence (1.5).

Theorem 2.1. *Let p be an odd prime. Then*

$$
\sum_{k=0}^{p-1} \frac{(q;q^2)_k^2}{(q^2;q^2)_k^2} x^k \equiv \sum_{k=0}^{\frac{p-1}{2}} {\frac{p-1}{2} \brack k}_{q^2}^2 q^{-pk+k^2} (-x)^k (x;q^2)_{\frac{p-1}{2}-k} \pmod{[p]^2}. \tag{2.1}
$$

Since

$$
\lim_{q\to 1} \frac{(q;q^2)_k^2}{(q^2;q^2)_k^2} = \left(\prod_{j=1}^{k} \frac{2j-1}{2j} \right)^2 = \binom{2k}{k}^2 16^{-k},
$$

letting $q \to 1$ in (2.1), we obtain (1.5). Moreover, setting $x = 1$ in (2.1) yields the following q-analogue of (1.1).

Corollary 2.2. *Let p be an odd prime. Then*

$$
\sum_{k=0}^{p-1} \frac{(q;q^2)_k^2}{(q^2;q^2)_k^2} \equiv \left(\frac{-1}{p} \right) q^{\frac{1-p^2}{4}} \pmod{[p]^2}. \tag{2.2}
$$

Remark. Corollary 2.2 confirms the first congruence in Conjecture 1.2.

Our second result is another generalization of (2.2).

Theorem 2.3. *Let p be an odd prime. Then*

$$
\sum_{k=0}^{p-1} \frac{(q;q^2)_k^2}{(q^2;q^2)_k^2} x^k \equiv \left(\frac{-1}{p} \right) q^{\frac{1-p^2}{4}} \sum_{k=0}^{p-1} \frac{(q;q^2)_k^2}{(q^2;q^2)_k^2} q^{2k} (x;q^2)_k \pmod{[p]^2}. \tag{2.3}
$$

It is clear that, when $x = 1$, the congruence (2.3) reduces to (2.2). On the other hand, setting $x = 0$ in (2.3), we obtain the following dual form of (2.2).

Corollary 2.4. *Let p be an odd prime. Then*

$$
\sum_{k=0}^{p-1} \frac{(q;q^2)_k^2}{(q^2;q^2)_k^2} q^{2k} \equiv \left(\frac{-1}{p} \right) q^{\frac{p^2-1}{4}} \pmod{[p]^2}.
$$

Our third result is a q-analogue of the $a = -\frac{1}{2}$ case of (1.7).

Theorem 2.5. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}\frac{(q;q^2)_k^2(x;q^2)_k q^{2k}}{(q^2;q^2)_k^2(-q^2;q^2)_k}\equiv\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{(q;q^2)_k^2(-x;q^2)_k q^{2k}}{(q^2;q^2)_k^2(-q^2;q^2)_k}\pmod{[p]^2}.\tag{2.4}$$

Letting $x=-1$ in (2.4), and noticing that $\frac{(-1;q^2)_k}{(-q^2;q^2)_k}=\frac{2}{1+q^{2k}}$, we are led to another q-analogue of (1.1).

Corollary 2.6. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}\frac{2(q;q^2)_k^2 q^{2k}}{(q^2;q^2)_k^2(1+q^{2k})}\equiv\left(\frac{-1}{p}\right)\pmod{[p]^2}.\tag{2.5}$$

If $\left(\frac{-1}{p}\right)=-1$ and $x=0$, then we immediately deduce that both sides of (2.4) are congruent to 0 modulo $[p]^2$, which may be restated as follows.

Corollary 2.7. *Let p be a prime of the form $4k+3$. Then*

$$\sum_{k=0}^{p-1}\frac{(q;q^2)_k^2 q^{2k}}{(q^2;q^2)_k^2(-q^2;q^2)_k}\equiv 0\pmod{[p]^2}.\tag{2.6}$$

Note that, when $q=1$, the congruence (2.6) can be written as

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{32^k}\equiv 0\pmod{p^2}\quad\text{for}\quad p\equiv 3\pmod 4,$$

which was conjecture by Z.-W. Sun [20] and proved by Tauraso [22] and Z.-H. Sun [17,18].

3. Results, II: congruences modulo $[p]$

In this section, we first give q-analogues of (1.2)–(1.4). Actually we shall prove the following more general results.

Theorem 3.1. *Let p be an odd prime and m, r two positive integers with $p\nmid m$. Then*

$$\sum_{k=0}^{p-1}\frac{(q^r;q^m)_k(q^{m-r};q^m)_k x^k}{(q^m;q^m)_k^2}$$
$$\equiv(-1)^{\langle -\frac{r}{m}\rangle_p}q^{\frac{-m\langle -\frac{r}{m}\rangle_p(\langle -\frac{r}{m}\rangle_p+1)}{2}}\sum_{k=0}^{p-1}\frac{(q^r;q^m)_k(q^{m-r};q^m)_k(x;q^m)_k q^{mk}}{(q^m;q^m)_k^2}\pmod{[p]}.\tag{3.1}$$

In particular, if $p\equiv \pm 1 \pmod{m}$, then

$$\begin{aligned}
&\sum_{k=0}^{p-1}\frac{(q^r;q^m)_k(q^{m-r};q^m)_k x^k}{(q^m;q^m)_k^2}\\
&\quad\equiv (-1)^{\langle -\frac{r}{m}\rangle_p} q^{\frac{r(m-r)(1-p^2)}{2m}}\sum_{k=0}^{p-1}\frac{(q^r;q^m)_k(q^{m-r};q^m)_k(x;q^m)_k q^{mk}}{(q^m;q^m)_k^2} \pmod{[p]}. \qquad (3.2)
\end{aligned}$$

Note that, for $p\geqslant 5$, we have

$$(-1)^{\langle -\frac{1}{3}\rangle_p}=\begin{cases}(-1)^{\frac{p-1}{3}}=1, & \text{if } p\equiv 1 \pmod{3}\\ (-1)^{\frac{2p-1}{3}}=-1, & \text{if } p\equiv 2 \pmod{3}\end{cases}=\left(\frac{-3}{p}\right),$$

$$(-1)^{\langle -\frac{1}{4}\rangle_p}=\begin{cases}(-1)^{\frac{p-1}{4}}, & \text{if } p\equiv 1 \pmod{4}\\ (-1)^{\frac{3p-1}{4}}, & \text{if } p\equiv 3 \pmod{4}\end{cases}=\left(\frac{-2}{p}\right),$$

$$(-1)^{\langle -\frac{1}{6}\rangle_p}=\begin{cases}(-1)^{\frac{p-1}{6}}, & \text{if } p\equiv 1 \pmod{6}\\ (-1)^{\frac{5p-1}{6}}, & \text{if } p\equiv 5 \pmod{6}\end{cases}=(-1)^{\frac{p-1}{2}}=\left(\frac{-1}{p}\right).$$

Letting $x=1$, $r=1$, $m=3,4,6$ in (3.2) with $p\geqslant 5$, we obtain the following result.

Corollary 3.2. *Let $p\geqslant 5$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{(q;q^3)_k(q^2;q^3)_k}{(q^3;q^3)_k^2}\equiv\left(\frac{-3}{p}\right)q^{\frac{1-p^2}{3}} \pmod{[p]},$$

$$\sum_{k=0}^{p-1}\frac{(q;q^4)_k(q^3;q^4)_k}{(q^4;q^4)_k^2}\equiv\left(\frac{-2}{p}\right)q^{\frac{3(1-p^2)}{8}} \pmod{[p]},$$

$$\sum_{k=0}^{p-1}\frac{(q;q^6)_k(q^5;q^6)_k}{(q^6;q^6)_k^2}\equiv\left(\frac{-1}{p}\right)q^{\frac{5(1-p^2)}{12}} \pmod{[p]}.$$

Remark. The congruences in Corollary 3.2 confirm the remaining three congruences in Conjecture 1.2 modulo $[p]$.

In the same vein, letting $x=0$, $r=1$, and $m=3,4,6$ in (3.1), we obtain

Corollary 3.3. *Let $p\geqslant 5$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{(q;q^3)_k(q^2;q^3)_k}{(q^3;q^3)_k^2}q^{3k}\equiv\left(\frac{-3}{p}\right)q^{\frac{p^2-1}{3}} \pmod{[p]},$$

$$\sum_{k=0}^{p-1}\frac{(q;q^4)_k(q^3;q^4)_k}{(q^4;q^4)_k^2}q^{4k}\equiv\left(\frac{-2}{p}\right)q^{\frac{3(p^2-1)}{8}} \pmod{[p]},$$

$$\sum_{k=0}^{p-1}\frac{(q;q^6)_k(q^5;q^6)_k}{(q^6;q^6)_k^2}q^{6k}\equiv\left(\frac{-1}{p}\right)q^{\frac{5(p^2-1)}{12}} \pmod{[p]}.$$

It seems that we have the following stronger result (see Conjecture 7.2 for a further generalization).

Conjecture 3.4. *The congruences in Corollary* 3.3 *hold modulo* $[p]^2$.

Theorem 3.5. *Let p be an odd prime and m, r two positive integers with $p \nmid m$. Then*

$$\sum_{k=0}^{p-1} \frac{(q^r;q^m)_k (q^{m-r};q^m)_k x^k}{(q^m;q^m)_k^2}$$
$$\equiv \sum_{k=0}^{\langle -\frac{r}{m}\rangle_p} \begin{bmatrix} \langle -\frac{r}{m}\rangle_p \\ k \end{bmatrix}_{q^m}^2 q^{\frac{mk(k-1)}{2} - mk\langle -\frac{r}{m}\rangle_p} (-x)^k (x;q^m)_{\langle -\frac{r}{m}\rangle_p - k} \pmod{[p]}.$$

Next, we give q-analogues of (1.7)–(1.8) in the modulus p case. For this end, we introduce the following polynomial

$$P_{n,m,r}(q,x) = \sum_{k=0}^{n} \frac{(q^r;q^m)_k (q^{m-r};q^m)_k (x;q^m)_k q^{mk}}{(q^m;q^m)_k^2 (-q^m;q^m)_k}.$$

Note that $P_{n,m,r}(1,x)$ is the generalized Legendre polynomial $P_n(a,x)$ in (1.6) with $a = -\frac{r}{m}$.

Theorem 3.6. *Let p be an odd prime and m, r two positive integers with $p \nmid m$. Then*

$$P_{p-1,m,r}(q,x) \equiv (-1)^{\langle -\frac{r}{m}\rangle_p} P_{p-1,m,r}(q,-x) \pmod{[p]}. \tag{3.3}$$

Letting $x = 0$ in (3.3), we obtain

Corollary 3.7. *Let p be an odd prime and m, r two integers with $p \nmid m$ and $\langle -r/m\rangle_p \equiv 1 \pmod 2$. Then*

$$\sum_{k=0}^{p-1} \frac{(q^r;q^m)_k (q^{m-r};q^m)_k q^{mk}}{(q^m;q^m)_k^2 (-q^m;q^m)_k} \equiv 0 \pmod{[p]}. \tag{3.4}$$

Taking $(m,r) = (3,1), (4,1), (6,1)$ in (3.4), we get the following congruences.

Corollary 3.8. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} \frac{(q;q^3)_k (q^2;q^3)_k q^{3k}}{(q^3;q^3)_k^2 (-q^3;q^3)_k} \equiv 0 \pmod{[p]}, \quad \textit{for} \quad p \equiv 2 \pmod 3, \tag{3.5}$$

$$\sum_{k=0}^{p-1} \frac{(q;q^4)_k (q^3;q^4)_k q^{4k}}{(q^4;q^4)_k^2 (-q^4;q^4)_k} \equiv 0 \pmod{[p]}, \quad \textit{for} \quad p \equiv 5,7 \pmod 8, \tag{3.6}$$

$$\sum_{k=0}^{p-1} \frac{(q;q^6)_k (q^5;q^6)_k q^{6k}}{(q^6;q^6)_k^2 (-q^6;q^6)_k} \equiv 0 \pmod{[p]}, \quad \textit{for} \quad p \equiv 3 \pmod 4. \tag{3.7}$$

Letting $x=-1$ in (3.3), we obtain

$$P_{p-1,m,r}(q,-1) \equiv (-1)^{\langle -\frac{r}{m}\rangle_p} \pmod{[p]}. \tag{3.8}$$

Taking $(m,r)=(3,1),(4,1),(6,1)$, we get the following result.

Corollary 3.9. *Let $p \geqslant 5$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{2(q;q^3)_k(q^2;q^3)_k q^{3k}}{(q^3;q^3)_k^2(1+q^{3k})} \equiv \left(\frac{-3}{p}\right) \pmod{[p]}, \tag{3.9}$$

$$\sum_{k=0}^{p-1} \frac{2(q;q^4)_k(q^3;q^4)_k q^{4k}}{(q^4;q^4)_k^2(1+q^{4k})} \equiv \left(\frac{-2}{p}\right) \pmod{[p]}, \tag{3.10}$$

$$\sum_{k=0}^{p-1} \frac{2(q;q^6)_k(q^5;q^6)_k q^{6k}}{(q^6;q^6)_k^2(1+q^{6k})} \equiv \left(\frac{-1}{p}\right) \pmod{[p]}. \tag{3.11}$$

4. Proofs of Theorems 2.1 and 2.3

Recall that the little q-Legendre polynomials are defined by

$$P_n(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n+k \\ k \end{bmatrix} q^{\frac{k(k+1)}{2}-nk}(-x)^k. \tag{4.1}$$

They can also be written as (see [25])

$$P_n(x|q) = (-1)^n q^{-\frac{n(n+1)}{2}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n+k \\ k \end{bmatrix} (-1)^k q^{\frac{k(k+1)}{2}-nk}(xq;q)_k. \tag{4.2}$$

We now give a new expansion for the little q-Legendre polynomials.

Lemma 4.1. *Let n be a nonnegative integer. Then*

$$P_n(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}^2 q^{\frac{k(k+1)}{2}-nk}(-x)^k(xq;q)_{n-k}. \tag{4.3}$$

Proof. By the q-binomial theorem (see [4, p. 36, Theorem 3.3]):

$$(x;q)_N = \sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix} (-x)^k q^{\frac{k(k-1)}{2}}, \tag{4.4}$$

one sees that, for $0 \leqslant m \leqslant n$, the coefficient of x^m in the right-hand side of (4.3) is given by

$$\begin{aligned}
&\sum_{k=0}^{m} \begin{bmatrix} n \\ k \end{bmatrix}^2 q^{\frac{k(k+1)}{2}-nk}(-1)^k \begin{bmatrix} n-k \\ m-k \end{bmatrix}(-1)^{m-k} q^{\frac{(m-k)(m-k+1)}{2}} \\
&= (-1)^m \begin{bmatrix} n \\ m \end{bmatrix} \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}\begin{bmatrix} n \\ k \end{bmatrix} q^{(m-k)(n-k)-mn+\frac{m(m+1)}{2}} \\
&= (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}\begin{bmatrix} n+m \\ m \end{bmatrix} q^{-mn+\frac{m(m+1)}{2}},
\end{aligned}$$

where the last step follows from the q-Chu-Vandermonde formula (see [4, p. 37, Theorem 3.4]). This completes the proof. □

We also need the following result.

Lemma 4.2. *Let p be an odd prime and $0 \leqslant k \leqslant p-1$. Then*

$$\frac{(q;q^2)_k^2}{(q^2;q^2)_k^2} \equiv (-1)^k \begin{bmatrix} \frac{p-1}{2} \\ k \end{bmatrix}_{q^2} \begin{bmatrix} \frac{p-1}{2}+k \\ k \end{bmatrix}_{q^2} q^{k^2-kp} \pmod{[p]^2}. \tag{4.5}$$

Proof. Observing that

$$(1-q^{2j-1})^2 + (1-q^{p-2j+1})(1-q^{p+2j-1})q^{2j-1-p} = (1-q^p)^2 q^{2j-1-p},$$

we have

$$(1-q^{2j-1})^2 \equiv -(1-q^{p-2j+1})(1-q^{p+2j-1})q^{2j-1-p} \pmod{[p]^2}.$$

It follows that

$$\begin{aligned}
\frac{(q;q^2)_k^2}{(q^2;q^2)_k^2} = \prod_{j=1}^{k} \frac{(1-q^{2j-1})^2}{(1-q^{2j})^2} &\equiv (-1)^k \prod_{j=1}^{k} \frac{(1-q^{p-2j+1})(1-q^{p+2j-1})q^{2j-1-p}}{(1-q^{2j})^2} \\
&= (-1)^k \begin{bmatrix} \frac{p-1}{2} \\ k \end{bmatrix}_{q^2} \begin{bmatrix} \frac{p-1}{2}+k \\ k \end{bmatrix}_{q^2} q^{k^2-kp} \pmod{[p]^2},
\end{aligned}$$

as desired. □

Proof of Theorem 2.1. Letting $n = \frac{p-1}{2}$, replacing q and x by q^2 and xq^{-2} respectively in (4.1) and (4.3), and then applying (4.5), we obtain (2.1). □

Proof of Theorem 2.3. The proof is similar to that of Theorem 2.1 by just comparing (4.1) and (4.2). □

5. Proof of Theorem 2.5

We fist establish two lemmas.

Lemma 5.1. *Let n be a positive integer and $0 \leqslant j \leqslant n$. Then*

$$\sum_{k=j}^{n}(-1)^k \begin{bmatrix} n+k \\ k \end{bmatrix} \begin{bmatrix} n-j \\ k-j \end{bmatrix} \frac{q^{\frac{k(k+1)}{2}-nk}}{(-q;q)_k}$$
$$= \begin{cases} (-1)^{\frac{n-j}{2}} \begin{bmatrix} n \\ \frac{n}{2} \end{bmatrix}_{q^2} \dfrac{(q^{n+1};q^2)_{\frac{j}{2}}\, q^{\frac{(n-j)(n-j+2)}{4}-\frac{j(j-1)}{2}}}{(-q;q)_n (q^{n-j+1};q^2)_{\frac{j}{2}}}, & \text{if } n \equiv j \equiv 0 \pmod 2, \\ (-1)^{\frac{n-j}{2}-1} \begin{bmatrix} n-1 \\ \frac{n-1}{2} \end{bmatrix}_{q^2} \dfrac{(q^{n+2};q^2)_{\frac{j-1}{2}}\, q^{\frac{(n-j)(n-j+2)}{4}-\frac{j(j-1)}{2}}}{(-q;q)_{n-1} (q^{n-j+1};q^2)_{\frac{j-1}{2}}}, & \text{if } n \equiv j \equiv 1 \pmod 2, \\ 0, & \text{otherwise.} \end{cases} \tag{5.1}$$

Proof. Replacing k by $k+j$, we can write the left-hand side of (5.1) as

$$\begin{aligned} &\sum_{k=0}^{n-j}(-1)^{k+j} \begin{bmatrix} n+k+j \\ n \end{bmatrix} \begin{bmatrix} n-j \\ k \end{bmatrix} \frac{q^{\frac{(k+j)(k+j+1)}{2}-n(k+j)}}{(-q;q)_{k+j}} \\ &= \sum_{k=0}^{n-j}(-1)^{k+j} \frac{(q;q)_{n+k+j}(q;q)_{n-j}\, q^{\frac{(k+j)(k+j+1)}{2}-n(k+j)}}{(q;q)_n (q;q)_{k+j} (q;q)_k (q;q)_{n-j-k} (-q;q)_{k+j}} \\ &= \sum_{k=0}^{n-j}(-1)^{j} \frac{(q;q)_{n+j}(q^{n+j+1};q)_k (q^{-n+j};q)_k\, q^{\frac{j(j+1)}{2}-nj}}{(q;q)_n (q;q)_j (q^{j+1};q)_k (q;q)_k (-q;q)_j (-q^{j+1};q)_k} \\ &= (-1)^j \begin{bmatrix} n+j \\ j \end{bmatrix} \frac{q^{\frac{j(j+1)}{2}-nj}}{(-q;q)_j} \sum_{k=0}^{n-j} \frac{(q^{n+j+1};q)_k (q^{-n+j};q)_k q^k}{(q;q)_k (q^{2j+2};q^2)_k}, \end{aligned} \tag{5.2}$$

where we have used the relation

$$\frac{(q;q)_{n-j}}{(q;q)_{n-j-k}} = (-1)^k (q^{-n+j};q)_k q^{\frac{k(2n-2j-k+1)}{2}}.$$

Taking $a = q^{n+j+1}$ and $b = q^{-n+j}$ in Andrews' q-analogue of Gauss' ${}_2F_1(-1)$ sum (see [2,3] or [8, Appendix (II.11)]):

$$\sum_{k=0}^{\infty} \frac{(a;q)_k (b;q)_k q^{\frac{k(k+1)}{2}}}{(q;q)_k (abq;q^2)_k} = \frac{(aq;q^2)_\infty (bq;q^2)_\infty}{(q;q^2)_\infty (abq;q^2)_\infty}, \tag{5.3}$$

where $(x;q)_\infty=\lim_{n\to\infty}(x;q)_n$, we have

$$\sum_{k=0}^{n-j}\frac{(q^{n+j+1};q)_k(q^{-n+j};q)_k q^{\frac{k(k+1)}{2}}}{(q;q)_k(q^{2j+2};q^2)_k}=\frac{(q^{n+j+2};q^2)_\infty(q^{-n+j+1};q^2)_\infty}{(q;q^2)_\infty(q^{2j+2};q^2)_\infty}$$
$$=\begin{cases}\dfrac{(q^{-n+j+1};q^2)_{\frac{n-j}{2}}}{(q^{2j+2};q^2)_{\frac{n-j}{2}}}, & \text{if } n\equiv j \pmod 2,\\ 0, & \text{otherwise.}\end{cases} \tag{5.4}$$

Replacing q by q^{-1} in (5.4) and noticing that $(q^{-m};q^{-1})_k=(-1)^k q^{-mk-\frac{k(k-1)}{2}}(q^m;q)_k$, we get

$$\sum_{k=0}^{n-j}\frac{(q^{n+j+1};q)_k(q^{-n+j};q)_k q^k}{(q;q)_k(q^{2j+2};q^2)_k}=\begin{cases}\dfrac{(q^{-n+j+1};q^2)_{\frac{n-j}{2}}\,q^{(n+j+1)(n-j)}}{(q^{2j+2};q^2)_{\frac{n-j}{2}}}, & \text{if } n\equiv j \pmod 2,\\ 0, & \text{otherwise.}\end{cases} \tag{5.5}$$

Substituting (5.5) into (5.2) and making some simplifications, we obtain the desired identity (5.1). □

Lemma 5.2. *Let n be a positive integer and*

$$F_n(x,q)=\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}n+k\\k\end{bmatrix}\frac{(x;q)_k q^{\frac{k(k+1)}{2}-nk}}{(-q;q)_k}. \tag{5.6}$$

Then

$$F_n(x,q)=(-1)^nF_n(-x,q). \tag{5.7}$$

Proof. By the q-binomial theorem (4.4), the coefficient of x^j $(0\leqslant j\leqslant n)$ in the right-hand side of (5.6) is given by

$$q^{\frac{j(j-1)}{2}}\sum_{k=j}^{n}(-1)^{k-j}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}n+k\\k\end{bmatrix}\begin{bmatrix}k\\j\end{bmatrix}\frac{q^{\frac{k(k+1)}{2}-nk}}{(-q;q)_k}$$
$$=q^{\frac{j(j-1)}{2}}\begin{bmatrix}n\\j\end{bmatrix}\sum_{k=j}^{n}(-1)^{k-j}\begin{bmatrix}n+k\\k\end{bmatrix}\begin{bmatrix}n-j\\k-j\end{bmatrix}\frac{q^{\frac{k(k+1)}{2}-nk}}{(-q;q)_k}$$

which, by Lemma 5.1, is equal to 0 if $n-j\equiv 1 \pmod 2$. This proves (5.7). □

Proof of Theorem 2.5. Note that, for $k>\frac{p-1}{2}$, there holds $(q;q^2)_k^2\equiv 0 \pmod{[p]^2}$. By Lemma 4.2, we have

$$\sum_{k=0}^{p-1}\frac{(q;q^2)_k^2(x;q^2)_k q^{2k}}{(q^2;q^2)_k^2(-q^2;q^2)_k}\equiv\sum_{k=0}^{\frac{p-1}{2}}(-1)^k\begin{bmatrix}\frac{p-1}{2}\\k\end{bmatrix}_{q^2}\begin{bmatrix}\frac{p-1}{2}+k\\k\end{bmatrix}_{q^2}\frac{(x;q^2)_k q^{k^2+2k-kp}}{(-q^2;q^2)_k} \pmod{[p]^2}.$$

The proof then follows from Lemma 5.2. □

Remark. Another application of Andrews's q-analogue of Gauss's ${}_2F_1(-1)$ sum (5.3) to q-congruences can be found in [9].

6. Proofs of Theorems 3.1, 3.5 and 3.6

Proof of Theorems 3.1. Since $q^r \equiv q^{r-p} \pmod{[p]}$, we may assume that $1 \leqslant r \leqslant p$. When $m=1$, we have

$$\sum_{k=0}^{p-1} \frac{(q^r;q)_k (q^{1-r};q)_k x^k}{(q;q)_k^2} = \sum_{k=0}^{r-1} \begin{bmatrix} r-1 \\ k \end{bmatrix} \begin{bmatrix} r-1+k \\ k \end{bmatrix} (-x)^k q^{\frac{k(k-1)}{2}-k(r-1)},$$

and

$$\sum_{k=0}^{p-1} \frac{(q^r;q)_k (q^{1-r};q)_k (x;q)_k q^k}{(q;q)_k^2} = \sum_{k=0}^{r-1} \begin{bmatrix} r-1 \\ k \end{bmatrix} \begin{bmatrix} r-1+k \\ k \end{bmatrix} (-1)^k (x;q)_k q^{\frac{k(k+1)}{2}-k(r-1)}.$$

The proof then follows from (4.1) and (4.2) with $P_n(x|q)$ replaced by $P_{r-1}(xq^{-1}|q)$.

When $m \geqslant 2$, let

$$s = \frac{m\langle -\frac{r}{m}\rangle_p + r}{p}.$$

Then s is a positive integer, $m|ps-r$, and so

$$\begin{aligned}
\frac{(q^r;q^m)_k (q^{m-r};q^m)_k}{(q^m;q^m)_k^2} &= \prod_{j=1}^{k} \frac{(1-q^{mj-r})(1-q^{mj+r-m})}{(1-q^{mj})^2} \\
&\equiv (-1)^k \prod_{j=1}^{k} \frac{(1-q^{ps+mj-r})(1-q^{ps-mj-r+m})q^{mj+r-m}}{(1-q^{mj})^2} \\
&= (-1)^k \begin{bmatrix} \frac{ps-r}{m} \\ k \end{bmatrix}_{q^m} \begin{bmatrix} \frac{ps-r}{m}+k \\ k \end{bmatrix}_{q^m} q^{\frac{mk(k-1)}{2}+kr} \\
&\equiv (-1)^k \begin{bmatrix} \langle -\frac{r}{m}\rangle_p \\ k \end{bmatrix}_{q^m} \begin{bmatrix} \langle -\frac{r}{m}\rangle_p + k \\ k \end{bmatrix}_{q^m} q^{\frac{mk(k-1)}{2}-k(ps-r)} \pmod{[p]}.
\end{aligned} \tag{6.1}$$

It follows that

$$\begin{aligned}
&\sum_{k=0}^{p-1} \frac{(q^r;q^m)_k (q^{m-r};q^m)_k x^k}{(q^m;q^m)_k^2} \\
&\equiv \sum_{k=0}^{\langle -\frac{r}{m}\rangle_p} \begin{bmatrix} \langle -\frac{r}{m}\rangle_p \\ k \end{bmatrix}_{q^m} \begin{bmatrix} \langle -\frac{r}{m}\rangle_p + k \\ k \end{bmatrix}_{q^m} (-x)^k q^{\frac{mk(k-1)}{2}-mk\langle -\frac{r}{m}\rangle_p} \pmod{[p]},
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=0}^{p-1} \frac{(q^r;q^m)_k (q^{m-r};q^m)_k (x;q^m)_k q^{mk}}{(q^m;q^m)_k^2} \\
&\equiv \sum_{k=0}^{\langle -\frac{r}{m}\rangle_p} \begin{bmatrix} \langle -\frac{r}{m}\rangle_p \\ k \end{bmatrix}_{q^m} \begin{bmatrix} \langle -\frac{r}{m}\rangle_p + k \\ k \end{bmatrix}_{q^m} (-1)^k (x;q^m)_k q^{\frac{mk(k+1)}{2}-mk\langle -\frac{r}{m}\rangle_p} \pmod{[p]}.
\end{aligned}$$

The proof of (3.1) then follows from the two expressions (4.1) and (4.2) for $P_{\langle -\frac{r}{m}\rangle_p}(xq^{-m}|q^m)$. Moreover, if $p\equiv \pm 1 \pmod m$, then $\frac{r(m-r)(1-p^2)}{2m}$ is an integer and

$$\frac{-m\langle -\frac{r}{m}\rangle_p(\langle -\frac{r}{m}\rangle_p+1)}{2}\equiv \frac{r(m-r)(1-p^2)}{2m} \pmod p.$$

This proves (3.2). □

Proof of Theorems 3.5. Apply (4.3) to $P_{\langle -\frac{r}{m}\rangle_p}(xq^{-m}|q^m)$. □

Proof of Theorems 3.6. Similarly as before, we have

$$\begin{aligned}
&P_{p-1,m,r}(q,x)\\
&\quad=\sum_{k=0}^{p-1}\frac{(q^r;q^m)_k(q^{m-r};q^m)_k(x;q^m)_kq^{mk}}{(q^m;q^m)_k^2(-q^m;q^m)_k}\\
&\quad\equiv\sum_{k=0}^{\langle -\frac{r}{m}\rangle_p}(-1)^k\begin{bmatrix}\langle -\frac{r}{m}\rangle_p\\ k\end{bmatrix}_{q^m}\begin{bmatrix}\langle -\frac{r}{m}\rangle_p+k\\ k\end{bmatrix}_{q^m}\frac{(x;q^m)_kq^{\frac{mk(k+1)}{2}-mk\langle -\frac{r}{m}\rangle_p}}{(-q^m;q^m)_k}\pmod{[p]}.
\end{aligned}$$

The proof then follows directly from Lemma 5.2. □

7. Concluding remarks and open problems

We have the following two stronger conjectural results for Theorems 3.1.

Conjecture 7.1. *Let p be an odd prime and m, r two positive integers with $p\nmid m$ and $m\nmid r$. Then there exists a unique integer $f_{p,m,r}$ such that*

$$\begin{aligned}
&\sum_{k=0}^{p-1}\frac{(q^r;q^m)_k(q^{m-r};q^m)_kx^k}{(q^m;q^m)_k^2}\\
&\quad\equiv(-1)^{\langle -\frac{r}{m}\rangle_p}q^{f_{p,m,r}}\sum_{k=0}^{p-1}\frac{(q^r;q^m)_k(q^{m-r};q^m)_k(x;q^m)_kq^{mk}}{(q^m;q^m)_k^2}\pmod{[p]^2}.
\end{aligned}$$

Furthermore, the numbers $f_{p,m,r}$ satisfy the following recurrence relation:

$$f_{p,m,m+r}=\begin{cases}-f_{p,m,r}, & \textit{if } r\equiv 0\pmod p,\\ f_{p,m,r}-r, & \textit{otherwise.}\end{cases}$$

Here are some values of $f_{p,m,r}$:

$f_{7,2,1}=-12,\ f_{7,2,3}=-13,\ f_{7,2,5}=-16,\ f_{7,2,7}=-21,\ f_{7,2,9}=21,\ f_{7,2,11}=12,\ f_{7,2,13}=1,$
$f_{7,2,15}=-12,\ f_{7,2,17}=-27,\ f_{7,2,19}=-44,\ f_{7,2,21}=-63,\ f_{7,2,23}=63,\ f_{7,2,25}=40,$
$f_{3,5,1}=-5,\ f_{3,5,2}=-3,\ f_{3,5,6}=-6,\ f_{3,5,7}=-5,\ f_{3,5,8}=3,\ f_{3,5,9}=-9,$
$f_{7,5,1}=-29,\ f_{7,5,2}=-19,\ f_{7,5,6}=-30,\ f_{7,5,7}=-21,\ f_{7,5,8}=-22,\ f_{7,5,9}=-33,$
$f_{11,7,1}=-86,\ f_{11,7,2}=-103,\ f_{11,7,3}=-51,\ f_{11,7,8}=-87,\ f_{11,7,9}=-105, f_{11,7,10}=-54.$

Conjecture 7.2. *Let p be an odd prime and m, r two positive integers with $r < m$ and $p \equiv \pm 1 \pmod{m}$. Then*

$$f_{p,m,r} = \frac{r(m-r)(1-p^2)}{2m}.$$

Note that Conjecture 7.1 is a q-analogue of (1.8) while Theorem 3.6 is a partial q-analogue of (1.7), of which we speculate the following complete q-analogue.

Conjecture 7.3. *Let p be an odd prime and m, r two positive integers with $p \nmid m$. Then*

$$P_{p-1,m,r}(q,x) \equiv (-1)^{\langle -\frac{r}{m}\rangle_p} P_{p-1,m,r}(q,-x) \pmod{[p]^2}.$$

There are some similar congruences in the literature. For example, van Hamme [26] proved the following variant of a conjecture of Beukers [6]:

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{64^k} \equiv 0 \pmod{p^2}, \quad \text{for} \quad p \equiv 3 \pmod{4}. \tag{7.1}$$

Recently, the authors have obtained a nice q-analogue of (7.1), which will appear in a forthcoming paper.

Acknowledgments. The first author was partially supported by the Fundamental Research Funds for the Central Universities and the National Natural Science Foundation of China (grant 11371144).

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