# DWORK-TYPE $q$-CONGRUENCES THROUGH THE $q$-LUCAS THEOREM

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Abstract. Employing the $q$-Lucas theorem and some known $q$-supercongruences, we give some Dwork-type $q$-congruences, confirming three conjectures in [J. Combin. Theory, Ser. A 178 (2021), Art. 105362]. As conclusions, we obtain the following supercongruences: for any prime $p\equiv 1 \pmod 4$ and positive integer $r$,

$$\sum_{k=0}^{(p^r-1)/2}\frac{(\frac{1}{2})_k^3}{k!^3}\equiv -\Gamma_p(\tfrac{1}{4})^4\sum_{k=0}^{(p^{r-1}-1)/2}\frac{(\frac{1}{2})_k^3}{k!^3}\pmod{p^{r+1}},$$

$$\sum_{k=0}^{p^r-1}\frac{(\frac{1}{2})_k^3}{k!^3}\equiv -\Gamma_p(\tfrac{1}{4})^4\sum_{k=0}^{p^{r-1}-1}\frac{(\frac{1}{2})_k^3}{k!^3}\pmod{p^{r+1}},$$

where $(x)_n=\Gamma(x+n)/\Gamma(x)$, and $\Gamma_p(x)$ stands for the $p$-adic Gamma function. The first one confirms a weaker form of Swisher's (H.3) conjecture for $p\equiv 1 \pmod 4$, which originally predicts that the supercongruence is true modulo $p^{3r}$.

## 1. Introduction

In 1914, Ramanujan [28] listed quite a few hypergeometric series representations of $1/\pi$, including

$$\sum_{k=0}^{\infty}(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 4^k}=\frac{4}{\pi},$$

where $(a)_0=1$ and $(a)_n=a(a+1)\cdots(a+n-1)$ for $n\geqslant 1$. In 1997, Van Hamme [35] numerically discovered 13 remarkable $p$-adic analogues of Ramanujan-type formulas, such as

$$\sum_{k=0}^{(p-1)/2}\frac{(\frac{1}{2})_k^3}{k!^3}\equiv\begin{cases}-\Gamma_p(\frac{1}{4})^4 \pmod{p^2}, & \text{if } p\equiv 1 \pmod 4,\\ 0 \pmod{p^2}, & \text{if } p\equiv 3 \pmod 4,\end{cases}\tag{1.1}$$

$$\sum_{k=0}^{(p-1)/2}(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 4^k}\equiv(-1)^{(p-1)/2}p \pmod{p^4}\quad \text{if } p>3,\tag{1.2}$$

where $p$ is an odd prime and $\Gamma_p(x)$ is the $p$-adic Gamma function. Three of them were proved by Van Hamme himself in [35]. For generalizations of (1.1) modulo $p^3$ and $p^4$, see [20,23]. The supercongruence (1.2) was first confirmed by Long [22]. It was not until 2016 that Osburn and Zudilin [27] proved the last remaining case of

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Van Hamme's conjectural supercongruences. In 2019, the author and Zudilin [16] obtained a $q$-analogue of (1.1) as follows: modulo $\Phi_n(q)^2$,

$$\sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k^2 (q^2;q^4)_k}{(q^2;q^2)_k^2 (q^4;q^4)_k} q^{2k} \equiv \begin{cases} \dfrac{(q^2;q^4)_{(n-1)/4}^2}{(q^4;q^4)_{(n-1)/4}^2} q^{(n-1)/2} & \text{if } n\equiv 1 \pmod 4, \\ 0 & \text{if } n\equiv 3 \pmod 4, \end{cases} \tag{1.3}$$

which extends an early result due to the author and Zeng [14, Corollary 1.2]. Further generalizations of (1.3) modulo $\Phi_n(q)^3$ can be found in the literature [10, 11, 36, 37]. At the moment we need to be familiar with the standard $q$-notation. The *$q$-shifted factorial* is defined by $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n\geqslant 1$, $(a;q)_0=1$, and $\Phi_n(q)$ denotes the $n$-th *cyclotomic polynomial* in $q$, which can be written as

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(n,k)=1}} (q-e^{2\pi ik/n}),$$

where $i^2=-1$. For simplicity, we will often adopt the abbreviated notation $(a_1,a_2,\dots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n$ for $n\geqslant 0$. Furthermore, the *$q$-integer* is defined as $[n]_q=(1-q^n)/(1-q)$.

Let $A(q)$ and $B(q)$ be two rational functions in $q$ and $P(q)$ a polynomial in $q$. We call $A(q)$ and $B(q)$ congruent modulo $P(q)$, denoted by $A(q)\equiv B(q) \pmod{P(q)}$, if the numerator of the reduced fraction $A(q)-B(q)$ is divisible by $P(q)$ in the polynomial ring $\mathbb{Z}[q]$.

In 2015, Swisher [33] proved several supercongruences of Van Hamme by utilizing Long's method. Meanwhile, she proposed some conjectures on supercongruences that generalize the (A.2)–(L.2) supercongruences of Van Hamme. For example, Swisher's conjectural (C.3) and (H.3) supercongruences can be respectively stated as follows: for any prime $p>3$ and positive integer $r$,

$$\sum_{k=0}^{(p^r-1)/2} (6k+1)\frac{(\frac12)_k^3}{k!^3 4^k} \equiv (-1)^{(p-1)/2} p \sum_{k=0}^{(p^{r-1}-1)/2} (6k+1)\frac{(\frac12)_k^3}{k!^3 4^k} \pmod{p^{4r}}, \tag{1.4}$$

$$\sum_{k=0}^{(p^r-1)/2} \frac{(\frac12)_k^3}{k!^3} \equiv \begin{cases} -\Gamma_p(\frac14)^4 \displaystyle\sum_{k=0}^{(p^{r-1}-1)/2} \frac{(\frac12)_k^3}{k!^3} \pmod{p^{3r}}, & \text{if } p\equiv 1 \pmod 4, \\ p^2 \displaystyle\sum_{k=0}^{(p^{r-2}-1)/2} \frac{(\frac12)_k^3}{k!^3} \pmod{p^{3r-1}}, & \text{if } p\equiv 3 \pmod 4 \text{ and } r\geqslant 2. \end{cases} \tag{1.5}$$

Given a prime $p$, we say that a power series $f(z)=\sum_{k=0}^\infty A_k z^k$ satisfies the Dwork congruence [3, 24] if

$$\frac{f_{r+1}(z)}{f_r(z^p)} \equiv \frac{f_r(z)}{f_{r-1}(z^p)} \pmod{p^r\mathbb{Z}_p[[z]]} \quad \text{for } r=1,2,\dots, \tag{1.6}$$

where

$$f_r(z)=\sum_{k=0}^{p^r-1}A_kz^k$$

is a truncation of $f(z)$. Further, if we can replace the modulus in (1.6) by $p^s\mathbb{Z}_p[[z]]$ for $s=s_r>r$, then we will also say that $f(z)$ satisfies a Dwork supercongruence. Formally, we require the condition $f_1(z^p)=\sum_{k=0}^{p-1}A_kz^{pk}\not\equiv 0 \pmod{p\mathbb{Z}_p[[z]]}$ so that (1.6) is well-defined. But this may be weakened to $f_1(z^p)\not\equiv 0 \pmod{p^m\mathbb{Z}_p[[z]]}$ provided that the congruences (1.6) hold modulo $p^{mr}\mathbb{Z}_p[[z]]$ for certain $m>1$. Thus, it is reasonable to call Swisher's conjectures in [33] Dwork-type supercongruences.

The author [9] proved that (1.4) is true modulo $p^{3r}$ by establishing its $q$-analogue, and he [6, Corollary 4.2] also proved that the second case of (1.5) is true modulo $p^{2r+2}$. Recently, the author and Zudilin [18] proved more Dwork-type supercongruences, including Swisher's supercongruences (B.3) and (L.3), and partial cases of Swisher's supercongruences (E.3) and (F.3).

In this paper, we shall give some Dwork-type $q$-congruences ($q$-analogues of Dwork-type congruences) by using the $q$-Lucas theorem (see Section 2) and some known $q$-supercongruences. Our first result can be stated as follows.

**Theorem 1.1.** *Let $n$ and $r$ be positive integers with $n\equiv 1 \pmod 4$. Then, modulo $\Phi_{n^r}(q)\prod_{j=1}^{r}\Phi_{n^j}(q)$,*

$$\sum_{k=0}^{(n^r-1)/d}\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{2k}\equiv [n]\frac{(q^3;q^4)_{(n^r-1)/2}(q^{5n};q^{4n})_{(n^{r-1}-1)/2}}{(q^5;q^4)_{(n^r-1)/2}(q^{3n};q^{4n})_{(n^{r-1}-1)/2}} \times\sum_{k=0}^{(n^{r-1}-1)/d}\frac{(q^n;q^{2n})_k^2(q^{2n};q^{4n})_k}{(q^{2n};q^{2n})_k^2(q^{4n};q^{4n})_k}q^{2nk}, \tag{1.7}$$

*where $d=1,2$.*

Note that, in an early paper [10], the author conjectured that (1.7) holds modulo $\prod_{j=1}^{r}\Phi_{n^j}(q)^2$.

In order to simplify the $q\to 1$ case of (1.7), we shall also prove the following supercongruence.

**Theorem 1.2.** *Let $p\equiv 1 \pmod 4$ be a prime and $r$ a positive integer. Then*

$$p\frac{(\frac34)_{(p^r-1)/2}(\frac54)_{(p^{r-1}-1)/2}}{(\frac54)_{(p^r-1)/2}(\frac34)_{(p^{r-1}-1)/2}}\equiv -\Gamma_p(\tfrac14)^4 \pmod{p^{2r}}. \tag{1.8}$$

For $n$ prime, letting $q\to 1$ in (1.7), we are led to the following supercongruences: for any prime $p\equiv 1 \pmod 4$ and positive integer $r$,

$$\sum_{k=0}^{(p^r-1)/2}\frac{(\frac12)_k^3}{k!^3}\equiv -\Gamma_p(\tfrac14)^4\sum_{k=0}^{(p^{r-1}-1)/2}\frac{(\frac12)_k^3}{k!^3}\pmod{p^{r+1}}, \tag{1.9}$$

$$\sum_{k=0}^{p^r-1}\frac{(\frac12)_k^3}{k!^3}\equiv -\Gamma_p(\tfrac14)^4\sum_{k=0}^{p^{r-1}-1}\frac{(\frac12)_k^3}{k!^3}\pmod{p^{r+1}}. \tag{1.10}$$

Although the supercongruence (1.9) is much weaker than Swisher's original conjecture (H.3) (the first part of (1.5)), it is the best result on this conjecture so far. Besides, the author [10] has conjectured that (1.10) is true modulo $p^{3r}$.

We have the following different $q$-analogue of (1.9) and (1.10).

**Theorem 1.3.** *Let $n$ and $r$ be positive integers with $n\equiv 1 \pmod 4$. Then, modulo $\Phi_{n^r}(q)\prod_{j=1}^r\Phi_{n^j}(q)$,*

$$\sum_{k=0}^{(n^r-1)/d}\frac{(1+q^{4k+1})(q^2;q^4)_k^3}{(1+q)(q^4;q^4)_k^3}q^k\equiv\frac{[n]_{q^2}(q^3;q^4)_{(n^r-1)/2}(q^{5n};q^{4n})_{(n^{r-1}-1)/2}}{(q^5;q^4)_{(n^r-1)/2}(q^{3n};q^{4n})_{(n^{r-1}-1)/2}}\,q^{(1-n)/2}$$
$$\times\sum_{k=0}^{(n^{r-1}-1)/d}\frac{(1+q^{(4k+1)n})(q^{2n};q^{4n})_k^3}{(1+q^n)(q^{4n};q^{4n})_k^3}q^{nk}, \tag{1.11}$$

*where $d=1,2$.*

Note that the author and Zudilin [18, Conjecture 4.3] ever conjectured (1.11) also holds modulo $\prod_{j=1}^r\Phi_{n^j}(q)^2$.

The paper is organized as follows. We shall prove Theorems 1.1–1.3 in Sections 2–4, respectively. In Section 5, we shall prove three more Dwork-type $q$-congruences, which were previously conjectured by the author and Zudilin [18]. Finally, in Section 6, we put forward some related conjectures on $q$-supercongruences, most of which (if true) can be utilized to confirm the corresponding complicated conjectures in [18].

## 2. Proof of Theorem 1.1

The *$q$-binomial coefficient* ${M\brack N}$ can be defined by

$${M\brack N}={M\brack N}_q=\begin{cases}\dfrac{(q;q)_M}{(q;q)_N(q;q)_{M-N}}, & \text{if } 0\le N\le M,\\ 0, & \text{otherwise.}\end{cases}$$

The so-called *$q$-Lucas theorem* (see Olive [26] and Désarménien [2, Proposition 2.2]) can be stated as follows: Let $n$ be a positive integer, and let $a,b,r,s$ be nonnegative integers with $b,s\le n-1$. Then

$${an+b\brack rn+s}\equiv\binom{a}{r}{b\brack s}\pmod{\Phi_n(q)}.$$

In order to prove Theorem 1.1, we need the following two lemmas. The proof of the first one is easy and can be found in [13, Lemma 3.1].

**Lemma 2.1.** *Let $n$ be a positive odd integer. Let $r$ and $s$ be nonnegative integers with $s \leqslant n-1$. Then*

$$(-q;q)_{rn+s} \equiv 2^r(-q;q)_s \pmod{\Phi_n(q)}.$$

**Lemma 2.2.** *Let $m$ and $n$ be positive integers with $n \equiv 1 \pmod 4$. Then, modulo $\Phi_n(q)$,*

$$\sum_{k=0}^{mn-1} \frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k} q^{2k} \equiv [n]\frac{(q^3;q^4)_{(n-1)/2}}{(q^5;q^4)_{(n-1)/2}} \sum_{k=0}^{m-1} \frac{(\frac{1}{2})_k^3}{k!^3}. \tag{2.1}$$

*Proof.* It is easy to see that

$$\frac{(q;q^2)_k}{(q^2;q^2)_k} = \frac{1}{(-q;q)_k^2}\begin{bmatrix}2k\\k\end{bmatrix}.$$

Thus, the left-hand side of (2.1) can be written as

$$\begin{aligned}\sum_{k=0}^{mn-1} \frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k} q^{2k} &= \sum_{k=0}^{mn-1} \frac{q^{2k}}{(-q;q)_k^4(-q^2;q^2)_k^2}\begin{bmatrix}2k\\k\end{bmatrix}^2\begin{bmatrix}2k\\k\end{bmatrix}_{q^2}\\ &= \sum_{r=0}^{m-1}\sum_{s=0}^{n-1} \frac{q^{2rn+2s}}{(-q;q)_{rn+s}^4(-q^2;q^2)_{rn+s}^2}\begin{bmatrix}2rn+2s\\rn+s\end{bmatrix}^2\begin{bmatrix}2rn+2s\\rn+s\end{bmatrix}_{q^2}.\end{aligned}$$

For odd $n$, we have $\Phi_n(q^2)=\Phi_n(q)\Phi_n(-q)$. By the $q$-Lucas theorem and Lemma 2.1, we get

$$\begin{aligned}&\sum_{s=0}^{n-1} \frac{q^{2rn+2s}}{(-q;q)_{rn+s}^4(-q^2;q^2)_{rn+s}^2}\begin{bmatrix}2rn+2s\\rn+s\end{bmatrix}^2\begin{bmatrix}2rn+2s\\rn+s\end{bmatrix}_{q^2}\\ &\quad\equiv \frac{\binom{2r}{r}^3}{2^{6r}} \sum_{s=0}^{n-1} \frac{q^{2s}}{(-q;q)_s^4(-q^2;q^2)_s^2}\begin{bmatrix}2s\\s\end{bmatrix}^2\begin{bmatrix}2s\\s\end{bmatrix}_{q^2}\\ &\quad\equiv \frac{\binom{2r}{r}^3}{2^{6r}} \sum_{s=0}^{(n-1)/2} \frac{(q;q^2)_s^2(q^2;q^4)_s}{(q^2;q^2)_s^2(q^4;q^4)_s} q^{2s} \pmod{\Phi_n(q)}.\end{aligned}$$

The proof then follows from the $n \equiv 1 \pmod 4$ case of (1.3) and the easily checked $q$-congruence:

$$\begin{aligned}\frac{(q^2;q^4)_{(n-1)/4}^2}{(q^4;q^4)_{(n-1)/4}^2} q^{(n-1)/2} &\equiv \frac{(q^{2-n},q^{2+n};q^4)_{(n-1)/4}}{(q^{4-n},q^{4+n};q^4)_{(n-1)/4}} q^{(n-1)/2}\\ &= [n]\frac{(q^3;q^4)_{(n-1)/2}}{(q^5;q^4)_{(n-1)/2}} \pmod{\Phi_n(q)^2} \end{aligned}\tag{2.2}$$

for $n \equiv 1 \pmod 4$. ☐

**Lemma 2.3.** *Let $n \equiv 1 \pmod 4$ be an integer greater than 1 and let $r, s$ be positive integers with $r > s$. Then, modulo $\Phi_{n^s}(q)$,*

$$[n]\frac{(q^3;q^4)_{(n^r-1)/2}(q^{5n};q^{4n})_{(n^{r-1}-1)/2}}{(q^5;q^4)_{(n^r-1)/2}(q^{3n};q^{4n})_{(n^{r-1}-1)/2}} \equiv [n]\frac{(q^3;q^4)_{(n^s-1)/2}(q^{5n};q^{4n})_{(n^{s-1}-1)/2}}{(q^5;q^4)_{(n^s-1)/2}(q^{3n};q^{4n})_{(n^{s-1}-1)/2}}. \tag{2.3}$$

*Proof.* For $n\equiv 1\pmod 4$, we have

$$[n]\frac{(q^3;q^4)_{(n-1)/2}}{(q^5;q^4)_{(n-1)/2}}=\frac{(q^{2-n},q^{2+n};q^4)_{(n-1)/4}}{(q^{4-n},q^{4+n};q^4)_{(n-1)/4}}q^{(n-1)/2},$$

and so

$$\begin{aligned}[n]&\frac{(q^3;q^4)_{(n^r-1)/2}(q^{5n};q^{4n})_{(n^{r-1}-1)/2}}{(q^5;q^4)_{(n^r-1)/2}(q^{3n};q^{4n})_{(n^{r-1}-1)/2}}\\ &=\frac{(q^{2-n^r},q^{2+n^r};q^4)_{(n^r-1)/4}(q^{4n-n^r},q^{4n+n^r};q^{4n})_{(n^{r-1}-1)/4}}{(q^{4-n^r},q^{4+n^r};q^4)_{(n^r-1)/4}(q^{2n-n^r},q^{2n+n^r};q^{4n})_{(n^{r-1}-1)/4}}q^{(n-1)/2}.\end{aligned}\tag{2.4}$$

Note that $(q^{2n-n^r},q^{2n+n^r};q^{4n})_{(n^{r-1}-1)/4}$ (respectively, $(q^{4n-n^r},q^{4n+n^r};q^{4n})_{(n^{r-1}-1)/4}$) is the product of all the factors of the form $1-q^{an}$ in $(q^{2-n^r},q^{2+n^r};q^4)_{(n^r-1)/4}$ (respectively, $(q^{4-n^r},q^{4+n^r};q^4)_{(n^r-1)/4}$). Using the following easily checked $q$-congruence:

$$(1-q^{m-n^r})(1-q^{m+n^r})\equiv(1-q^m)^2\pmod{\Phi_N(q)^2},$$

where $N$ divides $n^r$, we see that, modulo $\Phi_{n^s}(q)^2$, the right-hand side of (2.4) is congruent to

$$\begin{aligned}&\frac{(q^2;q^4)_{(n^r-1)/4}^2(q^{4n};q^{4n})_{(n^{r-1}-1)/4}^2}{(q^4;q^4)_{(n^r-1)/4}^2(q^{2n};q^{4n})_{(n^{r-1}-1)/4}^2}q^{(n-1)/2}\\ &\quad=\frac{(q^2;q^4)_{(n^s-1)/4}^2(q^{4n};q^{4n})_{(n^{s-1}-1)/4}^2}{(q^4;q^4)_{(n^s-1)/4}^2(q^{2n};q^{4n})_{(n^{s-1}-1)/4}^2}\frac{(q^{n^s+1};q^4)_{(n^r-n^s)/4}^2(q^{n^s+3n};q^{4n})_{(n^{r-1}-n^{s-1})/4}^2}{(q^{n^s+3};q^4)_{(n^r-n^s)/4}^2(q^{n^s+n};q^{4n})_{(n^{r-1}-n^{s-1})/4}^2}q^{(n-1)/2}.\end{aligned}\tag{2.5}$$

Furthermore, the polynomial $(q^{n^s+1};q^4)_{(n^r-n^s)/4}$ is divisible by $(q^{n^s+n};q^{4n})_{(n^{r-1}-n^{s-1})/4}$, and the quotient

$$\begin{aligned}\frac{(q^{n^s+1};q^4)_{(n^r-n^s)/4}}{(q^{n^s+n};q^{4n})_{(n^{r-1}-n^{s-1})/4}}&=\frac{(1-q^{n^s+1})(1-q^{n^s+5})\cdots(1-q^{n^r-3})}{(1-q^{n^s+n})(1-q^{n^s+5n})\cdots(1-q^{n^r-3n})}\\ &\equiv\frac{(1-q^{1-n^r})(1-q^{5-n^r})\cdots(1-q^{-n^s-3})}{(1-q^{n-n^r})(1-q^{5n-n^r})\cdots(1-q^{-n^s-3n})}\\ &=\frac{(q^{n^s+3};q^4)_{(n^r-n^s)/4}}{(q^{n^s+3n};q^{4n})_{(n^{r-1}-n^{s-1})/4}}q^{(n^r-n^s)(n^{r-1}+n^{s-1}-n^r-n^s)/8}\\ &\equiv\frac{(q^{n^s+3};q^4)_{(n^r-n^s)/4}}{(q^{n^s+3n};q^{4n})_{(n^{r-1}-n^{s-1})/4}}\not\equiv 0\pmod{\Phi_{n^s}(q)}.\end{aligned}$$

This means that the right-hand side of (2.5) reduces to

$$
\begin{aligned}
&\frac{(q^2;q^4)^2_{(n^s-1)/4}(q^{4n};q^{4n})^2_{(n^{s-1}-1)/4}}{(q^4;q^4)^2_{(n^s-1)/4}(q^{2n};q^{4n})^2_{(n^{s-1}-1)/4}}q^{(n-1)/2}\\
&\equiv \frac{(q^{2-n^s},q^{2+n^s};q^4)_{(n^s-1)/4}(q^{4n-n^s},q^{4n+n^s};q^{4n})_{(n^{s-1}-1)/4}}{(q^{4-n^s},q^{4+n^s};q^4)_{(n^s-1)/4}(q^{2n-n^s},q^{2n+n^s};q^{4n})_{(n^{s-1}-1)/4}}q^{(n-1)/2}\\
&=[n]\frac{(q^3;q^4)_{(n^s-1)/2}(q^{5n};q^{4n})_{(n^{s-1}-1)/2}}{(q^5;q^4)_{(n^s-1)/2}(q^{3n};q^{4n})_{(n^{s-1}-1)/2}} \pmod{\Phi_{n^s}(q)},
\end{aligned}
$$

as desired. ☐

Now we can prove Theorem 1.1.

*Proof of Theorem 1.1.* By (1.3) and (2.2), we obtain

$$
\sum_{k=0}^{(n^r-1)/d}\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{2k}\equiv[n^r]\frac{(q^3;q^4)_{(n^r-1)/2}}{(q^5;q^4)_{(n^r-1)/2}} \pmod{\Phi_{n^r}(q)^2},
$$

$$
\sum_{k=0}^{(n^{r-1}-1)/d}\frac{(q^n;q^{2n})_k^2(q^{2n};q^{4n})_k}{(q^{2n};q^{2n})_k^2(q^{4n};q^{4n})_k}q^{2nk}\equiv[n^{r-1}]_{q^n}\frac{(q^{3n};q^{4n})_{(n^{r-1}-1)/2}}{(q^{5n};q^{4n})_{(n^{r-1}-1)/2}} \pmod{\Phi_{n^{r-1}}(q^n)^2},
$$

where $d=1,2$. Since $\Phi_{n^{r-1}}(q^n)$ is divisible by $\Phi_{n^r}(q)$, from the above two $q$-congruences we conclude that (1.7) holds modulo $\Phi_{n^r}(q)^2$.

By Lemma 2.2, for $1\leqslant j\leqslant r-1$, there hold

$$
\sum_{k=0}^{n^r-1}\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{2k}\equiv[n^j]\frac{(q^3;q^4)_{(n^j-1)/2}}{(q^5;q^4)_{(n^j-1)/2}}\sum_{k=0}^{n^{r-j}-1}\frac{(\frac12)_k^3}{k!^3} \pmod{\Phi_{n^j}(q)},
$$

$$
\begin{aligned}
&\sum_{k=0}^{n^{r-1}-1}\frac{(q^n;q^{2n})_k^2(q^{2n};q^{4n})_k}{(q^{2n};q^{2n})_k^2(q^{4n};q^{4n})_k}q^{2nk}\\
&\equiv[n^{j-1}]_{q^n}\frac{(q^{3n};q^{4n})_{(n^{j-1}-1)/2}}{(q^{5n};q^{4n})_{(n^{j-1}-1)/2}}\sum_{k=0}^{n^{r-j}-1}\frac{(\frac12)_k^3}{k!^3} \pmod{\Phi_{n^{j-1}}(q^n)}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\sum_{k=0}^{n^r-1}\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{2k}&\equiv[n]\frac{(q^3;q^4)_{(n^j-1)/2}(q^{5n};q^{4n})_{(n^{j-1}-1)/2}}{(q^5;q^4)_{(n^j-1)/2}(q^{3n};q^{4n})_{(n^{j-1}-1)/2}}\\
&\quad\times\sum_{k=0}^{n^{r-1}-1}\frac{(q^n;q^{2n})_k^2(q^{2n};q^{4n})_k}{(q^{2n};q^{2n})_k^2(q^{4n};q^{4n})_k}q^{2nk} \pmod{\Phi_{n^j}(q)}.
\end{aligned}
$$

In view of Lemma 2.3, we see that the $d=1$ case of (1.7) holds modulo $\Phi_{n^j}(q)$ for $1\leqslant j\leqslant r-1$. Since $\Phi_n(q),\Phi_{n^2}(q),\dots,\Phi_{n^{r-1}}(q),\Phi_{n^r}(q)^2$ are pairwise coprime polynomials in $q$, we complete the proof of the $d=1$ case of (1.7).

Note that the $k$-th summand on the left-hand side of (1.7) is congruent to 0 modulo $\Phi_{n^j}(q)$ for $k$ in the range $(n^r-1)/2 < k \leqslant n^j(n^{r-j}+1)/2$. By Lemma 2.2 again, for $1 \leqslant j \leqslant r-1$, there hold

$$\sum_{k=0}^{(n^r-1)/2} \frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k} q^{2k} \equiv \sum_{k=0}^{n^j(n^{r-j}+1)/2} \frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k} q^{2k}$$
$$\equiv [n^j] \frac{(q^3;q^4)_{(n^j-1)/2}}{(q^5;q^4)_{(n^j-1)/2}} \sum_{k=0}^{(n^{r-j}+1)/2} \frac{(\frac{1}{2})_k^3}{k!^3} \pmod{\Phi_{n^j}(q)},$$

and

$$\sum_{k=0}^{(n^{r-1}-1)/2} \frac{(q^n;q^{2n})_k^2(q^{2n};q^{4n})_k}{(q^{2n};q^{2n})_k^2(q^{4n};q^{4n})_k} q^{2nk}$$
$$\equiv [n^{j-1}]_{q^n} \frac{(q^{3n};q^{4n})_{(n^{j-1}-1)/2}}{(q^{5n};q^{4n})_{(n^{j-1}-1)/2}} \sum_{k=0}^{(n^{r-j}+1)/2} \frac{(\frac{1}{2})_k^3}{k!^3} \pmod{\Phi_{n^{j-1}}(q^n)}.$$

Similarly as before, we can show that the $d=2$ case of (1.7) is true. □

## 3. Proof of Theorem 1.2

We first recall some basic properties of Morita's $p$-adic Gamma function [1, 29]. For any odd prime $p$, the $p$-adic Gamma function is defined by $\Gamma_p(0)=1$, and

$$\Gamma_p(n) = (-1)^n \prod_{\substack{0<k<n \\ p\nmid k}} k$$

for integers $n \geqslant 1$. Let $\mathbb{Z}_p$ denote the ring of all $p$-adic integers. We can extend $\Gamma_p$ to all $x \in \mathbb{Z}_p$ by the limit:

$$\Gamma_p(x) = \lim_{x_n \to x} \Gamma_p(x_n),$$

where $x_n$ is any positive integer sequence that $p$-adically tends to $x$. By the definition, one has

$$\frac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} -x, & p \nmid x, \\ -1, & p \mid x. \end{cases} \tag{3.1}$$

It is also known that, for $x \in \mathbb{Z}_p$, there holds

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{a_0(x)}, \tag{3.2}$$

where $a_0(x)$ is the smallest positive integer such that $a_0(x) \equiv x \pmod{p}$.

To prove Theorem 1.3, we also need the following result (see [23, Theorem 14]).

**Lemma 3.1.** *Let $p$ be an odd prime and $r$ a positive integer. Then, for $a, m \in \mathbb{Z}_p$,*

$$\Gamma_p(a+mp^r) \equiv \Gamma_p(a) + \Gamma_p'(a)mp^r \pmod{p^{2r}}. \tag{3.3}$$

*Proof of Theorem* 1.2. Let $\Gamma(x)$ be the classical Gamma function. The $r=1$ case is already known. This can also be deduced from comparing (1.1) and (1.3) and noticing that, for $n\equiv 1 \pmod 4$,

$$\frac{(q^2;q^4)_{(n-1)/4}^2}{(q^4;q^4)_{(n-1)/4}^2}q^{(n-1)/2} \equiv \frac{(q^{2-n},q^{2+n};q^4)_{(n-1)/4}}{(q^{4-n},q^{4+n};q^4)_{(n-1)/4}}q^{(n-1)/2}$$
$$=[n]\frac{(q^3;q^4)_{(n-1)/2}}{(q^5;q^4)_{(n-1)/2}} \pmod{\Phi_n(q)^2}.$$

We now assume that $r>1$. In view of (3.1), there hold

$$\frac{(\frac{3}{4})_{\frac{p^r-1}{2}}}{(\frac{5}{4})_{\frac{p^r-1}{2}}} = \frac{\Gamma(\frac{2p^r+1}{4})\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})\Gamma(\frac{2p^r+3}{4})} = \frac{\frac{3p}{4}\cdot\frac{7p}{4}\cdots\frac{2p^r-3p}{4}}{\frac{p}{4}\cdot\frac{5p}{4}\cdots\frac{2p^r-p}{4}}\cdot\frac{\Gamma_p(\frac{2p^r+1}{4})\Gamma_p(\frac{5}{4})}{\Gamma_p(\frac{3}{4})\Gamma_p(\frac{2p^r+3}{4})}$$
$$=\frac{(\frac{3}{4})_{\frac{p^{r-1}-1}{2}}}{\frac{p}{4}(\frac{5}{4})_{\frac{p^{r-1}-1}{2}}}\cdot\frac{\Gamma_p(\frac{2p^r+1}{4})\Gamma_p(\frac{5}{4})}{\Gamma_p(\frac{3}{4})\Gamma_p(\frac{2p^r+3}{4})}.$$

It follows that

$$p\frac{(\frac{3}{4})_{(p^r-1)/2}(\frac{5}{4})_{(p^{r-1}-1)/2}}{(\frac{5}{4})_{(p^r-1)/2}(\frac{3}{4})_{(p^{r-1}-1)/2}} = 4\frac{\Gamma_p(\frac{2p^r+1}{4})\Gamma_p(\frac{5}{4})}{\Gamma_p(\frac{3}{4})\Gamma_p(\frac{2p^r+3}{4})} = -\frac{\Gamma_p(\frac{2p^r+1}{4})\Gamma_p(\frac{1}{4})}{\Gamma_p(\frac{3}{4})\Gamma_p(\frac{2p^r+3}{4})}.$$

By (3.2) and (3.3), we have

$$\frac{\Gamma_p(\frac{2p^r+1}{4})}{\Gamma_p(\frac{2p^r+3}{4})} = (-1)^{(p+3)/4}\Gamma_p(\tfrac{1+2p^r}{4})\Gamma_p(\tfrac{1-2p^r}{4})$$
$$\equiv (-1)^{(p+3)/4}(\Gamma_p(\tfrac{1}{4})+\tfrac{1}{2}\Gamma_p'(\tfrac{1}{4})p^r)(\Gamma_p(\tfrac{1}{4})-\tfrac{1}{2}\Gamma_p'(\tfrac{1}{4})p^r)$$
$$\equiv (-1)^{(p+3)/4}\Gamma_p(\tfrac{1}{4})^2 \pmod{p^{2r}}.$$

The proof then follows from the identity $\Gamma_p(\frac{1}{4})\Gamma_p(\frac{3}{4})=(-1)^{(p+3)/4}$. □

## 4. Proof of Theorem 1.3

The proof is exactly the same as that of 1.1. Firstly, we have the following different $q$-analogue of (1.1) obtained by the author and Zudilin [17]:

$$\sum_{k=0}^{(n-1)/2}\frac{(1+q^{4k+1})(q^2;q^4)_k^3}{(1+q)(q^4;q^4)_k^3}q^k$$
$$\equiv\begin{cases}\dfrac{[n]_{q^2}(q^3;q^4)_{(n-1)/2}}{(q^5;q^4)_{(n-1)/2}}q^{(1-n)/2} \pmod{\Phi_n(q)^2} & \text{if } n\equiv 1 \pmod 4,\\ 0 \pmod{\Phi_n(q)^2} & \text{if } n\equiv 3 \pmod 4.\end{cases} \tag{4.1}$$

Secondly, we can prove that, for all positive integers $m$ and $n$ with $n\equiv 1 \pmod 4$, modulo $\Phi_n(q)$,

$$\sum_{k=0}^{mn-1}\frac{(1+q^{4k+1})(q^2;q^4)_k^3}{(1+q)(q^4;q^4)_k^3}q^k \equiv \frac{[n]_{q^2}(q^3;q^4)_{(n-1)/2}}{(q^5;q^4)_{(n-1)/2}}q^{(1-n)/2}\sum_{k=0}^{m-1}\frac{(\frac{1}{2})_k^3}{k!^3}. \tag{4.2}$$

## 5. More Dwork-type $q$-congruences

Rodriguez-Villegas [30, (36)] conjectured that, for any odd prime $p$,

$$\sum_{k=0}^{(p-1)/2} \frac{(\frac{1}{2})_k^2}{k!^2} \equiv (-1)^{(p-1)/2} \pmod{p^2}, \tag{5.1}$$

which was later confirmed by Mortenson [25]. The author, Pan, and Zhang [12, Corollary 3.1] gave a $q$-analogue of (5.1) as follows: for any odd integer $n>1$,

$$\sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k^2}{(q^2;q^2)_k^2} \equiv (-1)^{(n-1)/2} q^{(1-n^2)/4} \pmod{\Phi_n(q)^2}. \tag{5.2}$$

In this section, we shall give the following generalization of (5.2), which was conjectured by the author and Zudilin [18, Conjecture 4.6].

**Theorem 5.1.** *Let $n>1$ be an odd integer and let $r \geqslant 1$. Then, modulo $\Phi_{n^r}(q)\prod_{j=1}^r \Phi_{n^j}(q)$,*

$$\sum_{k=0}^{(n^r-1)/d} \frac{(q;q^2)_k^2}{(q^2;q^2)_k^2} \equiv (-1)^{(n-1)/2} q^{(1-n)(1+n^{2r-1})/4} \sum_{k=0}^{(n^{r-1}-1)/d} \frac{(q^n;q^{2n})_k^2}{(q^{2n};q^{2n})_k^2}, \tag{5.3}$$

*where $d=1,2$.*

*Sketch of proof.* In view of (5.2), we have

$$\sum_{k=0}^{(n^r-1)/d} \frac{(q;q^2)_k^2}{(q^2;q^2)_k^2} \equiv (-1)^{(n^r-1)/2} q^{(1-n^{2r})/4} \pmod{\Phi_{n^r}(q)^2},$$

$$\sum_{k=0}^{(n^{r-1}-1)/d} \frac{(q^n;q^{2n})_k^2}{(q^{2n};q^{2n})_k^2} \equiv (-1)^{(n^{r-1}-1)/2} q^{n(1-n^{2r-2})/4} \pmod{\Phi_{n^{r-1}}(q^n)^2},$$

where $d=1,2$. Since $\Phi_{n^{r-1}}(q^n)$ is a multiple of $\Phi_{n^r}(q)$, from the above two $q$-congruences we see that (5.3) holds modulo $\Phi_{n^r}(q)^2$.

Moreover, we can also deduce from (5.2) that, for all positive integers $m$ and $n$ with $n$ odd, modulo $\Phi_n(q)$,

$$\sum_{k=0}^{mn-1} \frac{(q;q^2)_k^2}{(q^2;q^2)_k^2} \equiv (-1)^{(n-1)/2} q^{(1-n^2)/4} \sum_{k=0}^{m-1} \frac{(\frac{1}{2})_k^2}{k!^2}.$$

In particular, for $1 \leqslant j \leqslant r-1$, there hold

$$\sum_{k=0}^{n^r-1} \frac{(q;q^2)_k^2}{(q^2;q^2)_k^2} \equiv (-1)^{(n^j-1)/2} q^{(1-n^{2j})/4} \sum_{k=0}^{n^{r-j}-1} \frac{(\frac{1}{2})_k^2}{k!^2} \pmod{\Phi_{n^j}(q)},$$

$$\sum_{k=0}^{n^{r-1}-1} \frac{(q^n;q^{2n})_k^2}{(q^{2n};q^{2n})_k^2} \equiv (-1)^{(n^{j-1}-1)/2} q^{n(1-n^{2j-2})/4} \sum_{k=0}^{n^{r-j}-1} \frac{(\frac{1}{2})_k^3}{k!^3} \pmod{\Phi_{n^{j-1}}(q^n)}.$$

It follows that, for $1 \leqslant j \leqslant r-1$,

$$\sum_{k=0}^{n^r-1} \frac{(q;q^2)_k^2}{(q^2;q^2)_k^2} \equiv (-1)^{(n-1)/2} q^{(1-n)(1+n^{2r-1})/4} \sum_{k=0}^{n^{r-1}-1} \frac{(q^n;q^{2n})_k^2}{(q^{2n};q^{2n})_k^2} \pmod{\Phi_{n^j}(q)},$$

where we have used the fact $q^{(1-n)(1+n^{2j-1})/4} \equiv q^{(1-n)(1+n^{2r-1})/4} \pmod{\Phi_{n^j}(q)}$. This proves (5.3) for $d=1$. Similarly, we can prove it for $d=2$. □

For $n$ prime, letting $q \to 1$ in (5.3), we obtain the following result: for any odd prime $p$ and positive integer $r$,

$$\sum_{k=0}^{(p^r-1)/d} \frac{(\frac{1}{2})_k^2}{k!^2} \equiv (-1)^{(p-1)/2} \sum_{k=0}^{(p^{r-1}-1)/2} \frac{(\frac{1}{2})_k^2}{k!^2} \pmod{p^{r+1}},$$

where $d=1,2$. Note that the author and Zudilin [18, (3.52)] have proved that the above supercongruence holds modulo $p^{2r}$.

The author and Zudilin [15, Theorem 4.14] applied Andrews' $q$-analogue of Gauss' ${}_2F_1(-1)$ summation (see [4, Appendix (II.11)]) to show that, for $n \equiv 3 \pmod 4$,

$$\sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k^2 q^{2k}}{(q^2;q^2)_k (q^4;q^4)_k} \equiv 0 \pmod{\Phi_n(q)^2}.$$

They [18] mentioned the following companion $q$-congruence: for $n \equiv 1 \pmod 4$,

$$\sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k^2 q^{2k}}{(q^2;q^2)_k (q^4;q^4)_k} \equiv \left(\frac{-2}{n}\right) q^{(n-1)(n+3)/8} \frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}} \pmod{\Phi_n(q)^2}, \quad (5.4)$$

where $(\frac{a}{b})$ denotes the Kronecker symbol.

Here we shall prove the following Dwork-type generalization of the above $q$-congruence, which was originally conjectured by the author and Zudilin [18, Conjecture 4.7].

**Theorem 5.2.** *Let $n > 1$ be an integer with $n \equiv 1 \pmod 4$ and let $r \geqslant 1$. Then, modulo $\Phi_{n^r}(q) \prod_{j=1}^{r} \Phi_{n^j}(q)$,*

$$\begin{aligned} \sum_{k=0}^{(n^r-1)/d} & \frac{(q;q^2)_k^2}{(q^2;q^2)_k (q^4;q^4)_k} q^{2k} \\ & \equiv \left(\frac{-2}{n}\right) q^{(n-1)(n^{2r-1}+3)/8} \frac{(q^2;q^4)_{(n^r-1)/4} (q^{4n};q^{4n})_{(n^{r-1}-1)/4}}{(q^4;q^4)_{(n^r-1)/4} (q^{2n};q^{4n})_{(n^{r-1}-1)/4}} \\ & \quad \times \sum_{k=0}^{(n^{r-1}-1)/d} \frac{(q^n;q^{2n})_k^2}{(q^{2n};q^{2n})_k (q^{4n};q^{4n})_k} q^{2nk}, \end{aligned} \quad (5.5)$$

*where $d=1,2$.*

*Sketch of proof.* By (5.4), modulo $\Phi_{n^r}(q)^2$,

$$\sum_{k=0}^{(n^r-1)/d}\frac{(q;q^2)_k^2q^{2k}}{(q^2;q^2)_k(q^4;q^4)_k}\equiv\left(\frac{-2}{n^r}\right)q^{(n^r-1)(n^r+3)/8}\frac{(q^2;q^4)_{(n^r-1)/4}}{(q^4;q^4)_{(n^r-1)/4}},$$

$$\sum_{k=0}^{(n^{r-1}-1)/d}\frac{(q^n;q^{2n})_k^2q^{2nk}}{(q^{2n};q^{2n})_k(q^{4n};q^{4n})_k}\equiv\left(\frac{-2}{n^{r-1}}\right)q^{n(n^{r-1}-1)(n^{r-1}+3)/8}\frac{(q^{2n};q^{4n})_{(n^{r-1}-1)/4}}{(q^{4n};q^{4n})_{(n^{r-1}-1)/4}},$$

where $d=1,2$. Hence, the $q$-congruence (5.5) is true modulo $\Phi_{n^r}(q)^2$.

Besides, we can conclude from (5.4) that, for all positive integers $m$ and $n$ with $n\equiv 1 \pmod 4$, modulo $\Phi_n(q)$,

$$\sum_{k=0}^{mn-1}\frac{(q;q^2)_k^2q^{2k}}{(q^2;q^2)_k(q^4;q^4)_k}\equiv\left(\frac{-2}{n}\right)q^{(n-1)(n+3)/8}\frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}\sum_{k=0}^{m-1}\frac{(\frac12)_k^2}{2^kk!^2},$$

and so, for $1\leqslant j\leqslant r-1$, modulo $\pmod{\Phi_{n^j}(q)}$,

$$\sum_{k=0}^{n^r-1}\frac{(q;q^2)_k^2q^{2k}}{(q^2;q^2)_k(q^4;q^4)_k}\equiv\left(\frac{-2}{n^j}\right)q^{(n^j-1)(n^j+3)/8}\frac{(q^2;q^4)_{(n^j-1)/4}}{(q^4;q^4)_{(n^j-1)/4}}\sum_{k=0}^{n^{r-j}-1}\frac{(\frac12)_k^2}{2^kk!^2},$$

$$\sum_{k=0}^{n^{r-1}-1}\frac{(q^n;q^{2n})_k^2q^{2nk}}{(q^{2n};q^{2n})_k(q^{4n};q^{4n})_k}\equiv\left(\frac{-2}{n^{j-1}}\right)q^{n(n^{j-1}-1)(n^{j-1}+3)/8}\frac{(q^{2n};q^{4n})_{(n^{j-1}-1)/4}}{(q^{4n};q^{4n})_{(n^{j-1}-1)/4}}\sum_{k=0}^{n^{r-j}-1}\frac{(\frac12)_k^2}{2^kk!^2}.$$

It follows that, for $1\leqslant j\leqslant r-1$, modulo $\Phi_{n^j}(q)$,

$$\begin{aligned}\sum_{k=0}^{n^r-1}&\frac{(q;q^2)_k^2}{(q^2;q^2)_k(q^4;q^4)_k}q^{2k}\\&\equiv\left(\frac{-2}{n}\right)q^{(n-1)(n^{2j-1}+3))/8}\frac{(q^2;q^4)_{(n^j-1)/4}(q^{4n};q^{4n})_{(n^{j-1}-1)/4}}{(q^4;q^4)_{(n^j-1)/4}(q^{2n};q^{4n})_{(n^{j-1}-1)/4}}\\&\quad\times\sum_{k=0}^{n^{r-1}-1}\frac{(q^n;q^{2n})_k^2}{(q^{2n};q^{2n})_k(q^{4n};q^{4n})_k}q^{2nk}.\end{aligned}$$

Like the proof of Lemma 2.3, we can prove that

$$\frac{(q^2;q^4)_{(n^r-1)/4}(q^{4n};q^{4n})_{(n^{r-1}-1)/4}}{(q^4;q^4)_{(n^r-1)/4}(q^{2n};q^{4n})_{(n^{r-1}-1)/4}}\equiv\frac{(q^2;q^4)_{(n^j-1)/4}(q^{4n};q^{4n})_{(n^{j-1}-1)/4}}{(q^4;q^4)_{(n^j-1)/4}(q^{2n};q^{4n})_{(n^{j-1}-1)/4}}\pmod{\Phi_{n^j}(q)}.$$

Using $q^{(n-1)(n^{2j-1}+3)/8}\equiv q^{(n-1)(n^{2r-1}+3)/8}\pmod{\Phi_{n^j}(q)}$, we complete the proof of (5.5) for $d=1$. The proof of the $d=2$ case is exactly the same. ☐

When $n$ is a prime and $q$ tends to 1 in (5.5), we arrive at the following result: for any prime $p\equiv 1\pmod 4$,

$$\sum_{k=0}^{(p^r-1)/d}\frac{(\frac12)_k^2}{2^kk!^2}\equiv\left(\frac{-2}{p}\right)\frac{(\frac12)_{(p^r-1)/4}(1)_{(p^{r-1}-1)/4}}{(1)_{(p^r-1)/4}(\frac12)_{(n^{r-1}-1)/4}}\sum_{k=0}^{(p^{r-1}-1)/d}\frac{(\frac12)_k^2}{2^kk!^2}\pmod{p^{r+1}},$$

where $d=1,2$. We point out that the $r=1$ case was first proved by Sun [31]. Moreover, the author and Zudilin [18] conjectured that the above supercongruence is true modulo $p^{2r}$.

The author [5] established the $q$-congruence

$$\sum_{k=0}^{n-1}\frac{q^k}{(-q;q)_k}\begin{bmatrix}2k\\k\end{bmatrix}\equiv(-1)^{(n-1)/2}q^{(n^2-1)/4}\pmod{\Phi_n(q)^2}, \tag{5.6}$$

which was conjectured by Tauraso [34] for $n$ being an odd prime. The author also conjectured that

$$\sum_{k=0}^{n-1}q^k\begin{bmatrix}2k\\k\end{bmatrix}\equiv\left(\frac{-3}{n}\right)q^{(n^2-1)/3}\pmod{\Phi_n(q)^2}, \tag{5.7}$$

which was recently confirmed by Liu and Petrov [21].

Here we shall prove the following Dwork-type $q$-generalizations of (5.6) and (5.7), confirming a conjecture of the author and Zudilin [18, Conjecture 4.8].

**Theorem 5.3.** *Let $n>1$ be an odd integer and let $r\geqslant 1$. Then, modulo $\Phi_{n^r}(q)^{2-d}\prod_{j=1}^{r}\Phi_{n^j}(q)$,*

$$\sum_{k=0}^{(n^r-1)/d}\frac{q^k}{(-q;q)_k}\begin{bmatrix}2k\\k\end{bmatrix}\equiv(-1)^{(n-1)/2}q^{(n-1)(1+n^{2r-1})/4}\sum_{k=0}^{(n^{r-1}-1)/d}\frac{q^{nk}}{(-q^n;q^n)_k}\begin{bmatrix}2k\\k\end{bmatrix}_{q^n}, \tag{5.8}$$

$$\sum_{k=0}^{(n^r-1)/d}q^k\begin{bmatrix}2k\\k\end{bmatrix}\equiv q^{(n-1)(1+n^{2r-1})/3}\left(\frac{-3}{n}\right)\sum_{k=0}^{(n^{r-1}-1)/d}q^{nk}\begin{bmatrix}2k\\k\end{bmatrix}_{q^n}, \tag{5.9}$$

*where $d=1,2$. When $d=1$, the second $q$-congruence is still true for even integers $n$.*

*Sketch of proof.* For $d=1$, the $q$-congruences (5.8) and (5.9) follow from (5.6) and (5.7) and the following generalizations of them: for all positive integers $m$ and $n$, modulo $\Phi_n(q)$,

$$\sum_{k=0}^{mn-1}\frac{q^k}{(-q;q)_k}\begin{bmatrix}2k\\k\end{bmatrix}\equiv(-1)^{(n-1)/2}q^{(n^2-1)/4}\sum_{k=0}^{m-1}\frac{1}{2^k}\binom{2k}{k}\quad(n\text{ is odd}), \tag{5.10}$$

$$\sum_{k=0}^{mn-1}q^k\begin{bmatrix}2k\\k\end{bmatrix}\equiv\left(\frac{-3}{n}\right)q^{(n^2-1)/3}\sum_{k=0}^{m-1}\binom{2k}{k}. \tag{5.11}$$

From (5.6) and (5.7), we immediately obtain

$$\sum_{k=0}^{(n-1)/2}\frac{q^k}{(-q;q)_k}\begin{bmatrix}2k\\k\end{bmatrix}_q\equiv(-1)^{(n-1)/2}q^{(n^2-1)/4}\pmod{\Phi_n(q)}, \tag{5.12}$$

$$\sum_{k=0}^{(n-1)/2}q^k\begin{bmatrix}2k\\k\end{bmatrix}\equiv\left(\frac{-3}{n}\right)q^{(n^2-1)/3}\pmod{\Phi_n(q)}. \tag{5.13}$$

Likewise, we can prove the $d=2$ case of (5.8) and (5.9) by using (5.10)–(5.13). □

Sun [32, Conjecture 3 (ii),(iii)] proposed the following conjecture: for any prime $p$,

$$\sum_{k=0}^{p^r-1}\frac{1}{2^k}\binom{2k}{k}\equiv\left(\frac{-1}{p}\right)\sum_{k=0}^{p^{r-1}-1}\frac{1}{2^k}\binom{2k}{k}\pmod{p^{2r}}\quad(p>2),\tag{5.14}$$

$$\sum_{k=0}^{p^r-1}\binom{2k}{k}\equiv\left(\frac{-3}{p}\right)\sum_{k=0}^{p^{r-1}-1}\binom{2k}{k}\pmod{p^{2r}},\tag{5.15}$$

and these expectations were recently confirmed by Zhang and Pan [38]. It is easy to see that (5.8) and (5.9) are $q$-analogues of (5.14) and (5.15) modulo $p^{r+1}$. Complete $q$-analogues of (5.14) and (5.15) are still not known.

## 6. Open problems and concluding remarks

Liu [19] established the following generalization of (1.1): for any odd prime $p$ and positive integer $m$,

$$\sum_{k=0}^{mp-1}\frac{(\frac{1}{2})_k^3}{k!^3}\equiv\begin{cases}-\Gamma_p(\frac14)^4\sum_{k=0}^{m-1}\frac{(\frac12)_k^3}{k!^3}\pmod{p^2}, & \text{if } p\equiv 1\pmod 4,\\ 0\pmod{p^2}, & \text{if } p\equiv 3\pmod 4,\end{cases}\tag{6.1}$$

Recently, the author [10] gave a $q$-analogue of the second case of (6.1): for positive integers $m$ and $n$ with $n\equiv 3\pmod 4$,

$$\sum_{k=0}^{mn-1}\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{2k}\equiv 0\pmod{\Phi_n(q)^2},\tag{6.2}$$

which were previously conjectured by the author and Zudilin [16].

We find the following $q$-analogue of the first case of (6.1), which is also a refinement of Lemma 2.2.

**Conjecture 6.1.** *Let $m$ and $n$ be positive integers with $n\equiv 1\pmod 4$ and $n>1$. Then, modulo $\Phi_n(q)^2$,*

$$\sum_{k=0}^{mn-1}\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{2k}\equiv[n]\frac{(q^3;q^4)_{(n-1)/2}}{(q^5;q^4)_{(n-1)/2}}\sum_{k=0}^{m-1}\frac{(\frac12)_k^3}{k!^3}.\tag{6.3}$$

For $n\equiv 3\pmod 4$, one sees that $[n](q^3;q^4)_{(n-1)/2}$ is divisible by $\Phi_n(q)^2$ while $(q^5;q^4)_{(n-1)/2}$ is coprime with $\Phi_n(q)$, and so (6.3) reduces to (6.2) in this case. For any prime $p\equiv 1\pmod 4$, we have

$$p\frac{(\frac34)_{(p-1)/2}}{(\frac54)_{(p-1)/2}}\equiv-\Gamma_p(\tfrac14)^4\pmod{p^2}.$$

Letting $n=p$ be a prime and taking the limits as $q\to 1$ in (6.3), we are led to the first part of (6.1).

It is not difficult to verify that, for any positive odd integer $n$,

$$\frac{(\frac{1}{2})_k^3}{k!^3} \equiv \frac{(q^n;q^{2n})_k^2(q^{2n};q^{4n})_k}{(q^{2n};q^{2n})_k^2(q^{4n};q^{4n})_k} q^{2nk} \pmod{\Phi_n(q)^2}. \tag{6.4}$$

Thus, the $q$-congruence can also be written as follows: modulo $\Phi_n(q)^2$,

$$\sum_{k=0}^{mn-1} \frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k} q^{2k} \equiv [n]\frac{(q^3;q^4)_{(n-1)/2}}{(q^5;q^4)_{(n-1)/2}} \sum_{k=0}^{m-1} \frac{(q^n;q^{2n})_k^2(q^{2n};q^{4n})_k}{(q^{2n};q^{2n})_k^2(q^{4n};q^{4n})_k} q^{2nk}.$$

Recently, the author [6] gave another $q$-analogues of the second case of (6.1): for positive integers $m$ and $n$ with $n \equiv 3 \pmod 4$,

$$\sum_{k=0}^{mn-1} \frac{(1+q^{4k+1})(q^2;q^4)_k^3}{(1+q)(q^4;q^4)_k^3} q^k \equiv 0 \pmod{\Phi_n(q)^2}, \tag{6.5}$$

which were previously conjectured by the author and Zudilin [16].

We have the following different $q$-analogue of the first case of (6.1), which is also a refinement of (4.2).

**Conjecture 6.2.** *Let $m$ and $n$ be positive integers with $n \equiv 1 \pmod 4$ and $n > 1$. Then, modulo $\Phi_n(q)^2$,*

$$\sum_{k=0}^{mn-1} \frac{(1+q^{4k+1})(q^2;q^4)_k^3}{(1+q)(q^4;q^4)_k^3} q^k \equiv \frac{[n]_{q^2}(q^3;q^4)_{(n-1)/2}}{(q^5;q^4)_{(n-1)/2}} q^{(1-n)/2} \sum_{k=0}^{m-1} \frac{(\frac{1}{2})_k^3}{k!^3}. \tag{6.6}$$

Likewise, the $q$-congruence (6.6) has the following equivalent form: modulo $\Phi_n(q)^2$,

$$\sum_{k=0}^{mn-1} \frac{(1+q^{4k+1})(q^2;q^4)_k^3}{(1+q)(q^4;q^4)_k^3} q^k \equiv \frac{[n]_{q^2}(q^3;q^4)_{(n-1)/2}}{(q^5;q^4)_{(n-1)/2}} q^{(1-n)/2} \sum_{k=0}^{m-1} \frac{(1+q^{n(4k+1)})(q^{2n};q^{4n})_k^3}{(1+q^n)(q^{4n};q^{4n})_k^3} q^{nk}.$$

We believe that Lemma 2.3 can be strengthened as follows.

**Conjecture 6.3.** *The $q$-congruence (2.3) holds modulo $\Phi_{n^s}(q)^2$.*

By the proof of Lemma 2.3, we know that the above conjecture can be easily derived from the following conjectural $q$-congruence with $q \to q^2$.

**Conjecture 6.4.** *Let $n \equiv 1 \pmod 4$ be an integer greater than 1 and let $r, s$ be positive integers with $r > s$. Then, modulo $\Phi_{n^s}(q)^2$,*

$$\frac{(q;q^2)_{(n^r-1)/4}(q^{2n};q^{2n})_{(n^{r-1}-1)/4}}{(q^2;q^2)_{(n^r-1)/4}(q^n;q^{2n})_{(n^{r-1}-1)/4}} \equiv \frac{(q;q^2)_{(n^s-1)/4}(q^{2n};q^{2n})_{(n^{s-1}-1)/4}}{(q^2;q^2)_{(n^s-1)/4}(q^n;q^{2n})_{(n^{s-1}-1)/4}}.$$

We point out that if Conjectures 6.1 and 6.3 are true, then we can prove that (1.7) holds modulo $\prod_{j=1}^r \Phi_{n^j}(q)^2$, which was conjectured by the author [10, Conjecture 6.3]. Similarly, if Conjectures 6.2 and 6.3 are confirmed, then we can conclude that (1.11) holds modulo $\prod_{j=1}^r \Phi_{n^j}(q)^2$, as already conjectured by the author and Zudilin [18, Conjecture 4.3]. Both [10, Conjecture 6.3] and [18, Conjecture 4.3] might be the best ways to prove the truth of (1.9) and (1.10) modulo $p^{2r}$.

Although we are unable to prove some interesting special cases of [18, Conjetures 4.1, 4.2, 4.5, 4.6], we shall give the following simplified versions of them.

**Conjecture 6.5.** *Let $m$ and $n$ be positive integers with $n \equiv 1 \pmod 4$. Then, modulo $\Phi_n(q)^3$,*

$$\sum_{k=0}^{mn-1}(-1)^k[4k+1]\frac{(q;q^2)_k^4(q^2;q^4)_k}{(q^2;q^2)_k^4(q^4;q^4)_k}q^k \equiv [n]\frac{(q^2;q^4)_{(n-1)/4}^2}{(q^4;q^4)_{(n-1)/4}^2}\sum_{k=0}^{m-1}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^5}{k!^5}. \tag{6.7}$$

The $m=1$ case of (6.7) was given by the author [7]. Conjecture 4.1 in [18] can be deduced from Conjectures 6.4 and 6.5 in this section. It is worth mentioning that (6.7) is equivalent to the following $q$-congruence: modulo $\Phi_n(q)^3$,

$$\begin{aligned}&\sum_{k=0}^{mn-1}(-1)^k[4k+1]\frac{(q;q^2)_k^4(q^2;q^4)_k}{(q^2;q^2)_k^4(q^4;q^4)_k}q^k\\ &\quad\equiv [n]\frac{(q^2;q^4)_{(n-1)/4}^2}{(q^4;q^4)_{(n-1)/4}^2}\sum_{k=0}^{m-1}(-1)^k[4k+1]_{q^n}\frac{(q^n;q^{2n})_k^4(q^{2n};q^{4n})_k}{(q^{2n};q^{2n})_k^4(q^{4n};q^{4n})_k}q^{nk}\end{aligned}$$

for the same reason as (6.4).

**Conjecture 6.6.** *Let $m$ and $n$ be positive integers with $n$ odd. Then, modulo $\Phi_n(q)^3$,*

$$\sum_{k=0}^{mn-1}(-1)^k[4k+1]\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3}q^k \equiv \frac{[n]_{q^2}(-q^3;q^4)_{(n-1)/2}}{(-q^5;q^4)_{(n-1)/2}}(-q)^{(1-n)/2}\sum_{k=0}^{m-1}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^3}{k!^3}. \tag{6.8}$$

The $m=1$ case of (6.8) was proved by the author and Zudilin [15]. Conjecture 4.2 in [18] is a consequence of Conjecture 6.6.

**Conjecture 6.7.** *Let $m$ and $n$ be positive integers with $n$ odd. Then, modulo $\Phi_n(q)^2$,*

$$\sum_{k=0}^{mn-1}(-1)^k[3k+1]\frac{(q;q^2)_k^3}{(q;q)_k^3} \equiv (-1)^{(n-1)/2}q^{(n-1)^2/4}[n]\sum_{k=0}^{m-1}(-1)^k(3k+1)\frac{8^k(\frac{1}{2})_k^3}{k!^3}. \tag{6.9}$$

We point out that the $m=1$ case of (6.9) is also true modulo $\Phi_n(q)^3$, which was established by the author [8]. Moreover, Conjecture 4.4 in [18] can be derived from Conjecture 6.7.

**Conjecture 6.8.** *Let $m$ and $n$ be positive integers with $n$ odd. Then, modulo $\Phi_n(q)^2$,*

$$\sum_{k=0}^{mn-1}(-1)^k[4k+1]\frac{(q;q^2)_k^3}{(q^2;q^2)_k^3}q^{k^2} \equiv (-1)^{(n-1)/2}q^{(n-1)^2/4}[n]\sum_{k=0}^{m-1}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^3}{k!^3}. \tag{6.10}$$

It should be mentioned that the $m=1$ case of (6.10) also holds modulo $\Phi_n(q)^3$ (see [15]). Besides, Conjecture 4.5 in [18] follows from Conjecture 6.8. Finally, the $q$-congruences (6.8)–(6.10) have equivalent forms as before. However, we will not formulate them specifically here.

**Data availability.** All data generated during this study are included in the published article.

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