A \(q\)-analogue of a hypergeometric congruence

Cheng-Yang Gu and Victor J. W. Guo*

School of Mathematical Sciences, Huaiyin Normal University, Huai’an 223300, Jiangsu, People’s Republic of China
525290408@qq.com, jwguo@math.ecnu.edu.cn

Abstract. We give a \(q\)-analogue of the following congruence: for any odd prime \(p\),
\[
\sum_{k=0}^{(p-1)/2} (-1)^k (6k + 1) \left(\frac{1}{2}\right)_k^{3} k!^{15} 8^k \sum_{j=1}^{k} \left(\frac{1}{(2j-1)^2} - \frac{1}{16j^2}\right) \equiv 0 \pmod{p},
\]
which was originally conjectured by Long and later proved by Swisher. This confirms a conjecture in [J. Math. Anal. Appl. 466 (2018), 749–761].

Keywords: basic hypergeometric series; \(q\)-congruence; congruence; cyclotomic polynomial.
AMS Subject Classifications: 33D15; Secondary 11A07, 11F33

1. Introduction

In 1914, Ramanujan [11] obtained a number of fast approximations of \(1/\pi\). Although the following is not in the list of [11], it gives such an example:
\[
\sum_{k=0}^{\infty} (-1)^k (6k + 1) \left(\frac{1}{2}\right)_k^{3} k!^{15} 8^k = \frac{2\sqrt{2}}{\pi},
\]
which is a special case of a \(4F_3\) summation formula of Gosper [1]. Here \((a)_n = a(a + 1) \cdots (a + n - 1)\) is the Pochhammer symbol. In 1997, Van Hamme [14] proposed 13 amazing \(p\)-adic analogues of Ramanujan-type formulas, such as
\[
\sum_{k=0}^{(p-1)/2} (-1)^k (6k + 1) \left(\frac{1}{2}\right)_k^{3} k!^{15} 8^k \equiv p \left(\frac{-2}{p}\right) \pmod{p^3},
\]
where \(p\) is an odd prime, and \((\cdot)_p\) denotes the Legendre symbol modulo \(p\). Van Hamme’s supercongruence (1.2) was first proved by Swisher [13]. We point out that the last supercongruence of Van Hamme was proved by Osburn and Zudilin [9] in 2016. In recent years, quite a few authors have studied \(q\)-analogues of supercongruences (see, for example, [2–5, 7, 12]).

*Corresponding author.
Swisher [13] also deduced the following interesting congruence from (1.2): for any odd prime $p$,
\[
\sum_{k=0}^{(p-1)/2} (-1)^k(6k+1)\left(\frac{1}{2}\right)_k^3 \sum_{j=1}^{k} \left( \frac{1}{(2j-1)^2} - \frac{1}{16j^2} \right) \equiv 0 \pmod{p},
\]
which was conjectured by Long [8]. In this note we shall prove the following $q$-analogue of (1.3), which was originally observed by the second author [2, Conjecture 4.4].

**Theorem 1.** Let $n$ be a positive odd integer. Then
\[
\sum_{k=0}^{(n-1)/2} (-1)^k[6k+1] \left(\frac{q; q^2} {q^4; q^4}\right)_k \sum_{j=1}^{k} \left( \frac{q^{2j-1}}{(2j-1)^2} - \frac{q^{4j}}{|4j|^2} \right) \equiv 0 \pmod{\Phi_n(q)}.
\]

Here and throughout the paper, we adopt the standard $q$-notation: $(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$ is the $q$-shifted factorial; $[n] = 1 + q + \cdots + q^{n-1}$ is the $q$-integer; and $\Phi_n(q)$ stands for the $n$-th cyclotomic polynomial in $q$, which may be defined as
\[
\Phi_n(q) = \prod_{1 \leq k \leq n, \gcd(n, k) = 1} (q - \zeta^k),
\]
where $\zeta$ is an $n$-th primitive root of unity.

## 2. Proof of Theorem 1

We first give the following lemma, which is a $q$-analogue of [8, Lemma 4.4].

**Lemma 2.** Let $n$ be a positive odd integer. Then
\[
\sum_{k=0}^{(n-1)/2} (-1)^k[6k+1] \left(\frac{q; q^2} {q^4; q^4}\right)_k \sum_{j=1}^{k} \left( \frac{q^{2j-1}}{(2j-1)^2} - \frac{q^{4j}}{|4j|^2} \right) = [n](-q)^{-(n-1)(n+5)/8}.
\]

**Proof.** The second author and Zudilin [6] gave a $q$-analogue of a formula for $1/\pi$ by using the following formula of Rahman [10, eq. (4.6)]:
\[
\sum_{k=0}^{\infty} (a; q)_k(1 - aq^{2k})(d; q)_k(q/d; q)_k(b; q^2)_k \frac{a^k q^{(k+1)/2}}{b^k} (q^2; q^2)_k(1 - a)(aq^2/d; q^2)_k(adq; q^2)_k(aaq/b; q)_k = \frac{(aq; q^2)_\infty (aq^2/b; q^2)_\infty (aq^2/bd; q^2)_\infty (aq^2/d; q^2)_\infty (adq; q^2)_\infty}{(aq/b; q^2)_\infty (aq^2/b; q^2)_\infty (aq^2/bd; q^2)_\infty (aq^2/d; q^2)_\infty (adq; q^2)_\infty}.
\]

Letting $q \to q^2$ and $b \to \infty$ in (2.2), then taking $a = q$ and $d = aq$ we are led to
\[
\sum_{k=0}^{\infty} (-1)^k[6k+1] \left(\frac{aq; q^2} {aq^4; q^4}\right)_k(q/q^2)_k(q^2)_k(q^3k^2) \frac{a^k q^{(k+1)/2}}{b^k} (aq^4; q^4)_k(q^4/a; q^4)_k(q^4)_k = \frac{(q^2; q^4)_\infty (q^5; q^4)_\infty}{(aq^4; q^4)_\infty (q^4/a; q^4)_\infty}.
\]

2
which for \( a = q^n \) gives

\[
\sum_{k=0}^{(n-1)/2} (-1)^k [6k + 1] \frac{(q; q^2)_k(q^{1+n}; q^2)_k(q^{1-n}; q^2)_k}{(q^4; q^4)_k(q^{4+n}; q^4)_k(q^{4-n}; q^4)_k} q^{3k^2} = (-1)^{(n-1)(n+5)/8} [n] q^{(n-1)(n-3)/8}.
\]

Replacing \( q \) by \( q^{-1} \), we obtain (2.1).

**Proof of Theorem 1.** Using (2.2), the second author and Zudilin (see [7, Theorem 4.4]) with \( a \to 1 \) proved that

\[
\sum_{k=0}^{(n-1)/2} (-1)^k [6k + 1] \frac{(q; q^2)_k^3}{(q^4; q^4)_k^3} \equiv [n] \left( \frac{-q}{q^2} \right)^{(n-1)(n+5)/8} \pmod{n \Phi_n(q)^2},
\]

which was first conjectured in [2, Conjecture 1.1]. Consider the difference of the left-hand side and the right-hand side of (2.3). By (2.1), we get

\[
\sum_{k=0}^{(n-1)/2} (-1)^k [6k + 1] \frac{(q; q^2)_k^3}{(q^4; q^4)_k^3} - [n] \left( \frac{-q}{q^2} \right)^{(n-1)(n+5)/8}
\]

\[
= \sum_{k=0}^{(n-1)/2} (-1)^k [6k + 1] \frac{(q; q^2)_k^3}{(q^4; q^4)_k^3} - \frac{(q^{1+n}; q^2)_k(q^{1-n}; q^2)_k}{(q^4; q^4)_k(q^{4+n}; q^4)_k(q^{4-n}; q^4)_k}
\]

\[
= \sum_{k=0}^{(n-1)/2} (-1)^k [6k + 1] \frac{(q; q^2)_k^3}{(q^4; q^4)_k^3} \times \frac{(q^4; q^4)_k(q^{1+n}; q^4)_k(q^{1-n}; q^2)_k}{(q^4; q^4)_k(q^{4+n}; q^4)_k(q^{4-n}; q^4)_k}
\]

Noticing that

\[
(1 - q^{a+n+d})(1 - q^{a-n+d}) = (1 - q^{a+d})^2 - (1 - q^n)^2 q^{a+dn-n}
\]

and \( 1 - q^n \equiv 0 \pmod{\Phi_n(q)} \), we have

\[
(q^{4+n}; q^4)_k(q^{4-n}; q^4)_k = \prod_{j=1}^{k} (1 - q^{n+4j})(1 - q^{-n+4j})
\]

\[
= \prod_{j=1}^{k} \left( (1 - q^{4j})^2 - (1 - q^n)^2 q^{4j-n} \right)
\]

\[
\equiv (q^4; q^4)_k^2 - (q^4; q^4)_k^2 \frac{(1 - q^n)^2}{(1 - q^{4j})^2} q^{4j-n} \pmod{\Phi_n(q^4)},
\]

3
since the remaining terms are multiples of \((1 - q^n)^4\). Similarly, there holds

\[
(q^{1+n}; q^2)_k(q^{1-n}; q^2)_k \equiv (q^2; q^2)_k^2 - (q^2; q^2)_k^2 \sum_{j=1}^{k} \frac{(1 - q^n)^2}{(1 - q^{2j-1})^2} q^{2j-n-1} \pmod{\Phi_n(q)^4}.
\]

It follows that

\[
(q; q^2)_k^2(q^{4+n}; q^4)_k(q^{4-n}; q^4)_k - (q^4; q^4)_k^2(q^{1+n}; q^2)_k(q^{1-n}; q^2)_k \equiv (q; q^2)_k^2(q^4; q^4)_k^2 \sum_{j=1}^{k} \left( \frac{q^{2j-n-1}}{[2j-1]^2} - \frac{q^{4j-n}}{[4j]^2} \right) \pmod{\Phi_n(q)^4}.
\]

Thus as a conclusion of (2.3), we obtain

\[
\sum_{k=0}^{n/2} (-1)^k[6k + 1](q; q^2)_k^2(q^{4+n}; q^4)_k(q^{4-n}; q^4)_k \sum_{j=1}^{k} \left( \frac{q^{2j-n-1}}{[2j-1]^2} - \frac{q^{4j-n}}{[4j]^2} \right) \equiv 0 \pmod{\Phi_n(q)},
\]

which is equivalent to the desired congruence (1.4) by observing that \(q^n \equiv 1 \pmod{\Phi_n(q)}\). \(\Box\)

**Acknowledgment.** The second author was partially supported by the National Natural Science Foundation of China (grant 11771175).

**References**


