SOME q-SUPERCONGRUENCES FROM THE BAILEY TRANSFORMATION

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Abstract. Using the Bailey transformation formula together with the 'creative microscoping' method (recently introduced by the author and Zudilin), we give q-analogues of two supercongruences of He:

$$\sum_{k=0}^{(p-1)/2}(-1)^k\frac{(\frac{1}{2})_k^3}{k!^3}\equiv(-1)^{(p-1)/2}p\sum_{k=0}^{(p-1)/2}\frac{(\frac{1}{2})_k^3}{k!(\frac{3}{4})_k(\frac{5}{4})_k}\pmod{p^2},$$

$$\sum_{k=0}^{(p-1)/2}(-1)^k\frac{(\frac{1}{2})_k^2}{k!^2}\equiv(-1)^{(p-1)/2}p\sum_{k=0}^{(p-1)/2}\frac{(\frac{1}{2})_k}{k!(4k+1)}\pmod{p^2},$$

where p is an odd prime. One of our results implies that He's second supercongruence is still true modulo p^3 for $p\equiv 3 \pmod 4$. We also give two similar q-supercongruences.

1. Introduction

Summation and transformation formulas for hypergeometric series play an important role in the study of supercongruences. See, for example, [15, 17]. In particular, using the $q=1$ case of the Bailey transformation formula, He [15] proved the following supercongruences: for any odd prime p,

$$\sum_{k=0}^{(p-1)/2}(-1)^k\frac{(\frac{1}{2})_k^3}{k!^3}\equiv(-1)^{(p-1)/2}p\sum_{k=0}^{(p-1)/2}\frac{(\frac{1}{2})_k^3}{k!(\frac{3}{4})_k(\frac{5}{4})_k}\pmod{p^2},\tag{1.1}$$

$$\sum_{k=0}^{(p-1)/2}(-1)^k\frac{(\frac{1}{2})_k^2}{k!^2}\equiv(-1)^{(p-1)/2}p\sum_{k=0}^{(p-1)/2}\frac{(\frac{1}{2})_k}{k!(4k+1)}\pmod{p^2},\tag{1.2}$$

where $(a)_n=a(a+1)\cdots(a+n-1)$ is the rising factorial. Note that Sun [24] established the following result: for odd primes p,

$$\sum_{k=0}^{(p-1)/2}(-1)^k\frac{(\frac{1}{2})_k^3}{k!^3}\equiv\begin{cases}(-1)^{(p-1)/2}(4x^2-2p)\pmod{p^2}, & \text{if } p=x^2+2y^2,\\ 0, & \text{if } p\equiv 5,7\pmod 8.\end{cases}\tag{1.3}$$

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where $x, y \in \mathbb{Z}$, and Sun [23] has already proved that, for any prime $p \equiv 1 \pmod 4$,

$$\sum_{k=0}^{(p-1)/2} (-1)^k \frac{(\frac{1}{2})_k^2}{k!^2} \equiv (-1)^{(p-1)/4} \left(2x - \frac{p}{2x}\right),$$

where $p = x^2 + y^2$ with $x, y \in \mathbb{Z}$ and $x \equiv 1 \pmod 4$.

Recently, the author and Zudilin [12] gave a partial q-analogue of (1.3) as follows: for any positive integer $n \equiv 1 \pmod 4$, modulo $\Phi_n(q)$,

$$\begin{aligned}&\sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k^2 (q^2;q^4)_k}{(q^2;q^2)_k^2 (q^4;q^4)_k} (-q)^k \\ &\quad\equiv \begin{cases} \dfrac{(q;q^4)_k^2 (q^2;q^4)_{(n-1)/8}^2 (-q;q)_{(n-1)/2}}{(q^4;q^4)_{(n-1)/4}^2} q^{-(1-n)^2/4} & \text{if } n \equiv 1 \pmod 8, \\ 0 & \text{if } n \equiv 5 \pmod 8. \end{cases}\end{aligned}$$

Here and throughout the paper, $|q| < 1$, the *q-shifted factorial* is defined by $(a;q)_0 = 1$ and $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n \geqslant 1$, or $n = \infty$, and the n-th *cyclotomic polynomial* $\Phi_n(q)$ is given by

$$\Phi_n(q) = \prod_{\substack{1 \leqslant k \leqslant n \\ \gcd(k,n)=1}} (q - e^{2\pi i k/n}) \in \mathbb{Z}[q].$$

It is easy to see that $\Phi_n(q^2) = \Phi_n(q)\Phi_n(-q)$ for any odd integer n. Moreover, we define the *q-integer* by $[n] = [n]_q = (1-q^n)/(1-q)$. For some other recent progress on q-congruences, we refer the reader to [2–11, 13, 14, 16, 18, 19, 22, 25, 27, 28, 30].

The aim of this paper to deduce some q-congruences from the Bailey transformation formula (see Andrews [1, Equation (2.9)] or Liu [20, Proposition 11.4]):

$$\begin{aligned}&\sum_{k=0}^{\infty} \frac{(1+\alpha q^{2k})(\alpha^2;q^2)_k (q^2/a;q^2)_k (q^2/b;q^2)_k (-q;q)_k (\alpha q/\lambda;q)_k (-\alpha^2\lambda ab)^k q^{k^2-3k}}{(1+\alpha)(q^2;q^2)_k (\alpha^2 a;q^2)_k (\alpha^2 b;q^2)_k (\alpha;q)_k (-\lambda;q)_k} \\ &\quad= \frac{(\alpha^2 q^2;q^2)_\infty (\alpha^2 ab/q^2;q^2)_\infty}{(\alpha^2 a;q^2)_\infty (\alpha^2 b;q^2)_\infty} \sum_{k=0}^{\infty} \frac{(q^2/a;q^2)_k (q^2/b;q^2)_k (\lambda;q^2)_k (\lambda q;q^2)_k}{(q^2;q^2)_k (\alpha q;q^2)_k (\alpha q^2;q^2)_k (\lambda^2;q^2)_k} \left(\frac{\alpha^2 ab}{q^2}\right)^k, \end{aligned} \tag{1.4}$$

where $|\alpha^2 ab/q^2| < 1$. Our first result is a q-analogue of (1.1).

Theorem 1.1. *Let n be a positive odd integer. Then*

$$\begin{aligned}&\sum_{k=0}^{(n-1)/2} (-1)^k \frac{(1+q^{4k+1})(q^2;q^4)_k^3}{(1+q)(q^4;q^4)_k^3} q^{2k^2+2k} \\ &\quad\equiv (-1)^{(n-1)/2} q^{(n-1)^2/2} [n]_{q^2} \sum_{k=0}^{(n-1)/2} \frac{(q^2;q^4)_k^3 q^{2k}}{(q^4;q^4)_k (q^3;q^4)_k (q^5;q^4)_k} \pmod{\Phi_n(q)^2 \Phi_n(-q)^3}.\end{aligned} \tag{1.5}$$

It is easy to see that when $n=p$ and $q\to 1$, the q-supercongruence (1.5) reduces to the supercongruence (1.1). Furthermore, if we let $n=p$ and $q\to -1$ in (1.5), then we get the following result:

$$\sum_{k=0}^{(p-1)/2}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^3}{k!^3}\equiv p(-1)^{(p-1)/2}\pmod{p^3}. \tag{1.6}$$

Note that the above supercongruence was observed by Van Hamme [26] in 1997 (tagged (B.2) in Van Hamme's list). The supercongruence (1.5) was first proved by Mortenson [21] using a ${}_6F_5$ hypergeometric transformation, and was later reproved by Zudilin [29] via the WZ (Wilf–Zeilberger) method. It should be pointed out that other different q-analogues of (1.6) can be found in [4, 5, 13].

Our second result is the following q-analogue of (1.2).

Theorem 1.2. *Let n be a positive odd integer. Then*

$$\begin{aligned}&\sum_{k=0}^{(n-1)/2}(-1)^k\frac{(1+q^{4k+1})(q^2;q^4)_k^2}{(1+q)(q^4;q^4)_k^2}q^{2k^2+k}\\&\quad\equiv(-1)^{(n-1)/2}q^{(n-1)^2/2}[n]_{q^2}\sum_{k=0}^{(n-1/2}\frac{(q^2;q^4)_kq^{2k}}{[4k+1](q^4;q^4)_k}\\&\qquad\begin{cases}\pmod{\Phi_n(q)^2\Phi_n(-q)^3} & \text{if } n\equiv 1 \pmod 4,\\ \pmod{\Phi_n(q)^3\Phi_n(-q)^3} & \text{if } n\equiv 3 \pmod 4.\end{cases}\end{aligned} \tag{1.7}$$

Likewise, the supercongruence (1.2) follows from (1.7) by taking $n=p$ and $q\to 1$. In fact, the q-supercongruence (1.7) implies the following more stronger result: for any prime $p\equiv 3 \pmod 4$, the supercongruence (1.2) is true modulo p^3. However, for the $n=p$ and $q\to -1$ case, the q-supercongruence (1.7) gives a less interesting supercongruence modulo p^3 (we omit it here), since both sides have closed forms in this case.

The paper is organized as follows. We shall use the method of 'creative microscoping', recently introduced by the author and Zudilin [11], to prove Theorems 1.1 and 1.2 in Sections 2 and 3, respectively. Then we shall give two similar results in Section 4.

2. Proof of Theorem 1.1

We first establish the following parametric generalization of Theorem 1.1.

Theorem 2.1. *Let $n>1$ be an odd integer and a an indeterminate. Then, modulo $\Phi_n(-q)(1-aq^{2n})(a-q^{2n})$,*

$$\sum_{k=0}^{(n-1)/2}(-1)^k\frac{(1+q^{4k+1})(aq^2;q^4)_k(q^2/a;q^4)_k(q^2;q^4)_k}{(1+q)(aq^4;q^4)_k(q^4/a;q^4)_k(q^4;q^4)_k}q^{2k^2+2k}$$

$$\equiv(-1)^{(n-1)/2}q^{(n-1)^2/2}[n]_{q^2}\sum_{k=0}^{(n-1)/2}\frac{(aq^2;q^4)_k(q^2/a;q^4)_k(q^2;q^4)_k}{(q^4;q^4)_k(q^3;q^4)_k(q^5;q^4)_k}q^{2k}. \tag{2.1}$$

Proof. For $a=q^{-2n}$ or $a=q^{2n}$, the left-hand side of (2.1) is equal to

$$\sum_{k=0}^{(n-1)/2}(-1)^k\frac{(1+q^{4k+1})(q^{2-2n};q^4)_k(q^{2+2n};q^4)_k(q^2;q^4)_k}{(1+q)(q^{4-2n};q^4)_k(q^{4+2n};q^4)_k(q^4;q^4)_k}q^{2k^2+2k}$$

$$=\frac{(q^6;q^4)_\infty(q^2;q^6)_\infty}{(q^{4+2n};q^4)_\infty(q^{4-2n};q^4)_\infty}\sum_{k=0}^{(n-1/2}\frac{(q^{2-2n};q^4)_k(q^{2+2n};q^4)_k(q^2;q^4)_k}{(q^4;q^4)_k(q^3;q^4)_k(q^5;q^4)_k}q^{2k}.$$

by the Bailey transformation (1.4) with the parameter substitutions $q\mapsto q^2$, $\alpha=q$, $a=q^{2+2n}$, $b=q^{2-2n}$, and $\lambda=q^2$. It is not difficult to see that

$$\frac{(q^6;q^4)_\infty(q^2;q^4)_\infty}{(q^{4+2n};q^4)_\infty(q^{4-2n};q^4)_\infty}=\frac{(q^2;q^4)_{(n+1)/2}}{(q^{4-2n};q^4)_{(n+1)/2}}=[n]_{q^2}(-1)^{(n-1)/2}q^{(n-1)^2/2}. \tag{2.2}$$

This proves that the congruence (2.1) holds modulo $1-aq^{2n}$ and $a-q^{2n}$.

Moreover, by [10, Lemma 3.1], for $0\leqslant k\leqslant(n-1)/2$, we have

$$\frac{(aq;q^2)_{(n-1)/2-k}}{(q^2/a;q^2)_{(n-1)/2-k}}\equiv(-a)^{(n-1)/2-2k}\frac{(aq;q^2)_k}{(q^2/a;q^2)_k}q^{(n-1)^2/4+k}\pmod{\Phi_n(q)}. \tag{2.3}$$

From this q-congruence we can easily deduce that, for $0\leqslant k\leqslant(n-1)/2$, the k-th and $((n-1)/2-k)$-th terms on the left-hand side of (2.1) modulo $\Phi_n(-q)$ cancel each other, i.e.,

$$(-1)^{(n-1)/2-k}\frac{(1+q^{2n-4k-1})(aq^2;q^4)_{(n-1)/2-k}(q^2/a;q^4)_{(n-1)/2-k}(q^2;q^4)_{(n-1)/2-k}}{(1+q)(aq^4;q^4)_{(n-1)/2-k}(q^4/a;q^4)_{(n-1)/2-k}(q^4;q^4)_{(n-1)/2-k}}$$

$$\times q^{2k((n-1)/2-k)^2+2((n-1)/2-k)}$$

$$\equiv-(-1)^k\frac{(1+q^{4k+1})(aq^2;q^4)_k(q^2/a;q^4)_k(q^2;q^4)_k}{(1+q)(aq^4;q^4)_k(q^4/a;q^4)_k(q^4;q^4)_k}q^{2k^2+2k}\pmod{\Phi_n(-q)}.$$

Notice that the above congruence is also true for $k=(n-1)/2-k$. In other words, if the $(n-1)/4$ term exists ($n\equiv1\pmod 4$), then it must be congruent to 0 modulo $\Phi_n(-q)$. Thus, we conclude that the left-hand side of (2.1) is congruent to 0 modulo $\Phi_n(-q)$. It is clear that the right-hand side of (2.1) is congruent to 0 modulo $\Phi_n(-q)$. and so the congruence (2.1) holds modulo $\Phi_n(-q)$. Since $1-aq^{2n}$, $a-q^{2n}$ and $\Phi_n(-q)$ are pairwise relatively prime polynomials, we complete the proof of (2.1). □

Proof of Theorem 1.1. It is easy to see that the denominators of the left-hand side of (2.1) are relatively prime to $\Phi_n(q^2)$ as $a \to 1$, and the denominators of the reduced form of the right-hand side of (2.1) are also relatively prime to $\Phi_n(q^2)$ because of the factor $[n]_{q^2}$ before the summation. On the other hand, the limit of $(1-aq^{2n})(a-q^{2n})$ as $a \to 1$ has the factor $\Phi_n(q^2)$. Therefore, letting $a \to 1$ in (2.1), we obtain the q-congruence (1.5). □

3. Proof of Theorem 1.2

Similarly as before, we first establish the following parametric generalization of Theorem 1.2.

Theorem 3.1. *Let $n > 1$ be an odd integer and a an indeterminate. Then*

$$\begin{aligned}
&\sum_{k=0}^{(n-1)/2} (-1)^k \frac{(1+q^{4k+1})(aq^2;q^4)_k(q^2/a;q^4)_k}{(1+q)(aq^4;q^4)_k(q^4/a;q^4)_k} q^{2k^2+k} \\
&\quad \equiv (-1)^{(n-1)/2} q^{(n-1)^2/2} [n]_{q^2} \sum_{k=0}^{(n-1)/2} \frac{(aq^2;q^4)_k(q^2/a;q^4)_k q^{2k}}{[4k+1](q^2;q^4)_k(q^4;q^4)_k} \\
&\quad \begin{cases} \pmod{\Phi_n(-q)(1-aq^{2n})(a-q^{2n})} & \text{if } n \equiv 1 \pmod 4, \\ \pmod{\Phi_n(q^2)(1-aq^{2n})(a-q^{2n})} & \text{if } n \equiv 3 \pmod 4. \end{cases}
\end{aligned} \tag{3.1}$$

Proof. For $a = q^{-2n}$ or $a = q^{2n}$, the left-hand side of (3.1) is equal to

$$\begin{aligned}
&\sum_{k=0}^{(n-1)/2} (-1)^k \frac{(1+q^{4k+1})(q^{2-2n};q^4)_k(q^{2+2n};q^4)_k}{(1+q)(q^{4-2n};q^4)_k(q^{4+2n};q^4)_k} q^{2k^2+k} \\
&\quad = \frac{(q^6;q^4)_\infty(q^2;q^6)_\infty}{(q^{4+2n};q^4)_\infty(q^{4-2n};q^4)_\infty} \sum_{k=0}^{(n-1/2} \frac{(q^{2-2n};q^4)_k(q^{2+2n};q^4)_k(q;q^4)_k}{(q^4;q^4)_k(q^5;q^4)_k(q^2;q^4)_k} q^{2k}.
\end{aligned}$$

by the Bailey transformation (1.4) with $q \mapsto q^2$, $\alpha = q$, $a = q^{2+2n}$, $b = q^{2-2n}$, and $\lambda = q$. By (2.2) and $(q;q^4)_k/(q^5;q^4)_k = 1/[4k+1]$, we see that (3.1) holds modulo $1-aq^{2n}$ and $a-q^{2n}$.

From (2.3) we can deduce that, for $0 \leqslant k \leqslant (n-1)/2$, the k-th term plus the $((n-1)/2-k)$-th terms on the left-hand side of (3.1) is congruent to 0 modulo $\Phi_n(-q)$ if $n \equiv 1 \pmod 4$ and modulo $\Phi_n(q^2)$ if $n \equiv 3 \pmod 4$. This means that the left-hand side of (3.1) is congruent to 0 under the same modulus, and so the congruence (3.1) is true modulo $\Phi_n(-q)$ if $n \equiv 1 \pmod 4$ and modulo $\Phi_n(q^2)$ if $n \equiv 3 \pmod 4$. The proof of (3.1) then follows from the fact that $1-aq^{2n}$, $a-q^{2n}$ and $\Phi_n(q^2)$ are pairwise relatively prime polynomials. □

Proof of Theorem 1.2. The denominators of (the reduced forms) on both sides of (3.1) are relatively prime to $\Phi_n(q^2)$ as $a \to 1$ and the limit of $(1-aq^{2n})(a-q^{2n})$ contains the factor $\Phi_n(q^2)$ as $a \to 1$. We immediately get (1.5) by taking $a \to 1$ in (3.1). □

4. Two similar q-supercongruences

In this section, we give two results similar to Theorems 1.1 and 1.2.

Theorem 4.1. *Let n be a positive odd integer. Then*

$$\sum_{k=0}^{(n+1)/2}(-1)^k\frac{(1+q^{4k-1})(q^{-2};q^4)_k^3}{(1+q^{-1})(q^4;q^4)_k^3}q^{2k^2+6k}$$
$$\equiv(-1)^{(n-1)/2}q^{(n-1)^2/2}[n]_{q^2}\sum_{k=0}^{(n+1)/2}\frac{(q^{-2};q^4)_k^2(q^2;q^4)_kq^{6k}}{(q^4;q^4)_k(q^3;q^4)_k(q;q^4)_k}\pmod{\Phi_n(q)^2\Phi_n(-q)^3}. \tag{4.1}$$

Proof. This time we need to establish the following q-congruence with an extra parameter a: modulo $\Phi_n(-q)(1-aq^{2n})(a-q^{2n})$,

$$\sum_{k=0}^{(n+1)/2}(-1)^k\frac{(1+q^{4k-1})(aq^{-2};q^4)_k(q^{-2}/a;q^4)_k(q^{-2};q^4)_k}{(1+q^{-1})(aq^4;q^4)_k(q^4/a;q^4)_k(q^4;q^4)_k}q^{2k^2+6k}$$
$$\equiv(-1)^{(n-1)/2}q^{(n-1)^2/2}[n]_{q^2}\sum_{k=0}^{(n+1)/2}\frac{(aq^{-2};q^4)_k(q^{-2}/a;q^4)_k(q^2;q^4)_k}{(q^4;q^4)_k(q^3;q^4)_k(q;q^4)_k}q^{6k}.. \tag{4.2}$$

The q-congruence (4.2) modulo $(1-aq^{2n})(a-q^{2n})$ follows from the Bailey transformation (1.4) with $q\mapsto q^2$, $\alpha=q^{-1}$, $a=q^{6+2n}$, $b=q^{6-2n}$, and $\lambda=q^2$. On the other hand, using the following q-congruence (see [10, (5.4)])

$$\frac{(aq^{-1};q^2)_{(n+1)/2-k}}{(q^2/a;q^2)_{(n+1)/2-k}}=(-a)^{(n+1)/2-2k}\frac{(aq^{-1};q^2)_k}{(q^2/a;q^2)_k}q^{(n-1)^2/4+3k-1}\pmod{\Phi_n(q)} \tag{4.3}$$

with $q\mapsto q^2$, we can show that (4.2) also holds modulo $\Phi_n(-q)$.

Finally, letting $a\to1$ in (4.2), we obtain (4.1). □

When $n=p$ is an odd prime and $q\to1$, the q-supercongruence (4.1) reduces to the following supercongruence:

$$\sum_{k=0}^{(p+1)/2}(-1)^k\frac{(-\frac12)_k^3}{k!^3}\equiv(-1)^{(p-1)/2}p\sum_{k=0}^{(p+1)/2}\frac{(-\frac12)_k^2(\frac12)_k}{k!(\frac14)_k(\frac34)_k}\pmod{p^2}.$$

Besides, letting $n=p$ be an odd prime and $q\to-1$ in (1.5), we get

$$\sum_{k=0}^{(p+1)/2}(-1)^k(4k-1)\frac{(-\frac12)_k^3}{k!^3}\equiv p(-1)^{(p+1)/2}\pmod{p^3}. \tag{4.4}$$

Note that a different q-analogue of (4.4) was already given in [11, Theorem 4.9] with $r=-1$, $d=2$ and $a=1$ (see also [10, Section 5]).

Theorem 4.2. *Let n be a positive odd integer. Then*

$$\sum_{k=0}^{(n+1)/2}(-1)^k\frac{(1+q^{4k-1})(q^{-2};q^4)_k^2}{(1+q^{-1})(q^4;q^4)_k^2}q^{2k^2+3k}$$
$$\equiv(-1)^{(n-1)/2}q^{(n-1)^2/2}[n]_{q^2}\sum_{k=0}^{(n+1)/2}\frac{(q^{-2};q^4)_k(q^{-1};q^4)_k}{(q^4;q^4)_k(q^3;q^4)_k}q^{6k}$$
$$\begin{cases}\pmod{\Phi_n(q)^3\Phi_n(-q)^3} & \text{if } n\equiv 1 \pmod 4,\\ \pmod{\Phi_n(q)^2\Phi_n(-q)^3} & \text{if } n\equiv 3 \pmod 4.\end{cases} \tag{4.5}$$

Proof. We first establish the following parametric generalization of (4.5):

$$\sum_{k=0}^{(n+1)/2}(-1)^k\frac{(1+q^{4k-1})(aq^{-2};q^4)_k(q^{-2}/a;q^4)_k}{(1+q^{-1})(aq^4;q^4)_k(q^4/a;q^4)_k}q^{2k^2+3k}$$
$$\equiv(-1)^{(n-1)/2}q^{(n-1)^2/2}[n]_{q^2}\sum_{k=0}^{(n+1)/2}\frac{(aq^{-2};q^4)_k(q^{-2}/a;q^4)_k(q^{-1};q^4)_k}{(q^4;q^4)_k(q^3;q^4)_k(q^{-2};q^4)_k}q^{6k}$$
$$\begin{cases}\pmod{\Phi_n(q^2)(1-aq^{2n})(a-q^{2n})} & \text{if } n\equiv 1 \pmod 4,\\ \pmod{\Phi_n(-q)(1-aq^{2n})(a-q^{2n})} & \text{if } n\equiv 3 \pmod 4.\end{cases} \tag{4.6}$$

This q-congruence modulo $(1-aq^{2n})(a-q^{2n})$ follows from the Bailey transformation (1.4) with $q\mapsto q^2$, $\alpha=q^{-1}$, $a=q^{6+2n}$, $b=q^{6-2n}$, and $\lambda=q^{-1}$. On the other hand, using (4.3) with $q\mapsto q^2$, we can show that (4.2) also holds modulo $\Phi_n(q^2)$ if $n\equiv 1 \pmod 4$ and modulo $\Phi_n(-q)$ if $n\equiv 3 \pmod 4$.

Finally, letting $a\to 1$ in (4.6), we are led to (4.5). □

We can also deduce some supercongruences from Theorem 4.2 as before. In the case $n=p$ is an odd prime and $q\to 1$, the q-supercongruence (4.5) reduces to

$$\sum_{k=0}^{(p+1)/2}(-1)^k\frac{(-\frac{1}{2})_k^2}{k!^2}$$
$$\equiv(-1)^{(p+1)/2}p\sum_{k=0}^{(p+1)/2}\frac{(-\frac{1}{2})_k}{k!(4k-1)}\begin{cases}\pmod{p^3} & \text{if } p\equiv 1 \pmod 4,\\ \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Meanwhile, letting $n=p$ be an odd prime and $q\to -1$ in (1.5), we can also obtain another supercongruence modulo p^3. But this supercongruence is not so interesting because both sides are summable.

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