Further q-supercongruences from a transformation of Rahman

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Abstract. Employing a quadratic transformation formula of Rahman and the method of 'creative microscoping' (introduced by the author and Zudilin in 2019), we provide some new q-supercongruences for truncated basic hypergeometric series. In particular, we confirm two recent conjectures of Liu and Wang. We also propose some related conjectures on supercongruences and q-supercongruences.

Keywords: cyclotomic polynomials; q-supercongruences; Rahman's transformation; creative microscoping

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1. Introduction

In his first letter to Hardy on the 16th January 1913, Ramanujan mentioned the following formula:

$$\sum_{k=0}^{\infty}(8k+1)\frac{(\frac{1}{4})_k^4}{k!^4}=\frac{2\sqrt{2}}{\sqrt{\pi}\,\Gamma(\frac{3}{4})^2} \tag{1.1}$$

(see [1, p. 25, (2)]), where $(a)_n=a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol and $\Gamma(x)$ denotes the Gamma function. Recently, Chen and Chu [2] gave a q-analogue of (1.1) as follows:

$$\sum_{k=0}^{\infty}[8k+1]\frac{(q;q^4)_k^4}{(q^4;q^4)_k^4}q^{2k}=\frac{(q^5,q^3,q^3,q^3;q^4)_\infty}{(q^2,q^4,q^4,q^4;q^4)_\infty}. \tag{1.2}$$

They also obtained the following similar formula:

$$\sum_{k=0}^{\infty}[6k+1]\frac{(q;q^4)_k(q;q^2)_k^3}{(q^2;q^2)_k(q^4;q^4)_k^3}q^{k^2+k}=\frac{(q^5,q^3,q^3,q^3;q^4)_\infty}{(q^2,q^4,q^4,q^4;q^4)_\infty}. \tag{1.3}$$

Here and in what follows, $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ is the *q-shifted factorial*, and $[n]:=[n]_q=(1-q^n)/(1-q)$ denotes the *q-integer*. For simplicity, we also write $(a_1,a_2,\dots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n$ for $n\geqslant 0$ or $n=\infty$.

For any odd prime p, let $\Gamma_p(x)$ be the p-adic Gamma function [18]. In 2015, Swisher [21] proved the following p-adic analogue of (1.1):

$$\sum_{k=0}^{(p-1)/4}(8k+1)\frac{(\frac{1}{4})_k^4}{k!^4}\equiv p\frac{\Gamma_p(\frac{1}{2})\Gamma_p(\frac{1}{4})}{\Gamma_p(\frac{3}{4})} \pmod{p^3},\quad \text{if } p\equiv 1 \pmod{4}, \tag{1.4}$$

which was originally conjectured by Van Hamme [22, (G.2)]. Liu and Wang [16] showed that (1.4) can also be deduced from the following q-supercongruence:

$$\sum_{k=0}^{(n-1)/4}[8k+1]\frac{(q;q^4)_k^4}{(q^4;q^4)_k^4}q^{2k}\equiv \frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}[n]q^{(1-n)/4} \pmod{[n]\Phi_n(q)^2} \tag{1.5}$$

(for a more general form, see [8, Theorem 4.3]). On the other hand, Guo and Schlosser [7, Theorem 2 with $d=4$] proved that,

$$\sum_{k=0}^{n-1}[8k+1]\frac{(q;q^4)_k^4}{(q^4;q^4)_k^4}q^{2k}\equiv 0 \pmod{\Phi_n(q)^2}\quad \text{for } n\equiv 3 \pmod{4}.$$

Here, $\Phi_n(q)$ is the n-th *cyclotomic polynomial* in q, which may be given by

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(n,k)=1}}(q-\zeta^k),$$

where ζ is an n-th primitive root of unity. Moreover, two rational functions $A(q)$ and $B(q)$ in q are called congruent modulo a polynomial $P(q)$, denoted by $A(q)\equiv B(q) \pmod{P(q)}$, if $P(q)$ divides the numerator of the reduced form of $A(q)-B(q)$ in the polynomial ring $\mathbb{Z}[q]$.

Recall that the *basic hypergeometric series* ${}_{r+1}\phi_r$ (see Gasper and Rahman's monograph [4]) is defined as

$${}_{r+1}\phi_r\left[\begin{matrix}a_1,a_2,\dots,a_{r+1}\\ b_1,b_2,\dots,b_r\end{matrix};q,\ z\right]=\sum_{k=0}^{\infty}\frac{(a_1,a_2,\dots,a_{r+1};q)_k z^k}{(q,b_1,\dots,b_r;q)_k}.$$

A quadratic transformation of Rahman [4, (3.8.13)] can be stated as follows:

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(1-aq^{3k})(a,d,aq/d;q^2)_k(b,c,aq/bc;q)_k}{(1-a)(aq/d,d,q;q)_k(aq^2/b,aq^2/c,bcq;q^2)_k}q^k\\ &=\frac{(aq^2,bq,cq,aq^2/bc;q^2)_\infty}{(q,aq^2/b,aq^2/c,bcq;q^2)_\infty}\,{}_3\phi_2\left[\begin{matrix}b,c,aq/bc\\ dq,aq^2/d\end{matrix};q^2,q^2\right],\end{aligned} \tag{1.6}$$

provided that d or aq/d is not of the form q^{-2n} (n is a non-negative integer).

The author and Zudilin [11] introduced a method, called 'creative microscoping', to prove q-supercongruences through inserting one or more additional parameters and considering asymptotics at roots of unity. More concretely, to prove a q-supercongruence modulo $\Phi_n(q)^3$, we first establish a parametric q-congruence modulo $\Phi_n(q)(1-aq^n)(a-q^n)$ and then take $a=1$ to accomplish the proof. The crucial virtue is that this parametric q-congruence can be established modulo $\Phi_n(q)$, $1-aq^n$, and $a-q^n$ individually, since these three moduli are pairwise coprime as polynomials in q. Meanwhile, each parametric q-congruence can usually be easily verified by employing summation and transformation formulas for basic hypergeometric series. For instance, the author and Zudilin settled two particular cases of [11, Conjecture 4.6] by using the 'creative microscoping' method and Rahman's quadratic transformation (1.6). Later the author and Schlosser [9, Theorem 6.1] completely confirmed this conjecture by applying a special case of (1.6).

Recently, using the 'creative microscoping' method together with Rahman's quadratic transformation (1.6) again, Liu and Wang [17] proved that, modulo $[n]\Phi_n(q)^2$,

$$\sum_{k=0}^{M}[6k+1]\frac{(q;q^4)_k\,(q;q^2)_k^3}{(q^2;q^2)_k\,(q^4;q^4)_k^3}q^{k^2+k}\equiv\begin{cases}\dfrac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}[n]q^{(1-n)/4}, & \text{if } n\equiv 1 \pmod 4,\\ 0, & \text{if } n\equiv 3 \pmod 4,\end{cases} \tag{1.7}$$

where $M=(n-1)/2$ or $n-1$. They also gave the following generalization of the second case of (1.7) modulo $\Phi_n(q)^2$, for any positive integer $d\geqslant 2$ and positive odd integer n with $n\equiv d+1 \pmod{2d}$,

$$\sum_{k=0}^{(n-1)/d}[3dk+1]\frac{(q,e,q^{1+d}/e;q^{2d})_k(q,q,q^{d-1};q^d)_k}{(q^d,e,q^{1+d}/e;q^d)_k(q^{2d},q^{2d},q^{d+2};q^{2d})_k}q^{dk}\equiv 0 \pmod{\Phi_n(q)^2}. \tag{1.8}$$

For some other recent work on q-supercongruences, see the literature [3, 5, 6, 10, 12–15, 19, 20, 23–25].

The first aim of this paper is to establish a stronger version of (1.8) for even d as follows.

Theorem 1.1. *Let $d\geqslant 2$ be an even integer and e an indeterminate. Let $n\equiv d+1 \pmod{2d}$ be a positive integer. Then*

$$\sum_{k=0}^{(dn+n-1)/(2d)}[3dk+1]\frac{(q,e,q^{1+d}/e;q^{2d})_k(q,q,q^{d-1};q^d)_k}{(q^d,e,q^{1+d}/e;q^d)_k(q^{2d},q^{2d},q^{d+2};q^{2d})_k}q^{dk}\equiv 0 \pmod{\Phi_n(q)^3}. \tag{1.9}$$

Note that the q-supercongruence (1.9) modulo $\Phi_n(q)^2$ also follows from (1.8), since the k-th summand on the left-hand side of (1.8) is congruent to 0 modulo $\Phi_n(q)^2$ for all $(n-1)/d<k\leqslant(dn+n-1)/(2d)$. However, the q-supercongruence (1.8) does not hold modulo $\Phi_n(q)^3$ in general.

Letting $e\to 0$ and $e=-q$ in (1.9), respectively, we obtain

$$\sum_{k=0}^{(dn+n-1)/(2d)}[3dk+1]\frac{(q;q^{2d})_k(q,q,q^{d-1};q^d)_k}{(q^d;q^d)_k(q^{2d},q^{2d},q^{d+2};q^{2d})_k}q^{d(k^2+k)/2}\equiv 0 \pmod{\Phi_n(q)^3},$$

$$\sum_{k=0}^{(dn+n-1)/(2d)}[3dk+1]\frac{(q,-q,-q^d;q^{2d})_k(q,q,q^{d-1};q^d)_k}{(q^d,-q,-q^d;q^d)_k(q^{2d},q^{2d},q^{d+2};q^{2d})_k}q^{dk}\equiv 0 \pmod{\Phi_n(q)^3}.$$

Letting $n=p$ be an odd prime and $q\to 1$ in each of the above q-supercongruences, we get the following result: for even $d\geqslant 2$ and $p\equiv d+1 \pmod{2d}$,

$$\sum_{k=0}^{(dp+p-1)/(2d)}(3dk+1)\frac{(\frac{1}{d})_k^2(\frac{d-1}{d})_k(\frac{1}{2d})_k}{k!^3 4^k(\frac{d+2}{2d})_k}\equiv 0 \pmod{p^3}. \tag{1.10}$$

For $d=4$, we have $\frac{d-1}{d}=\frac{d+2}{2d}=\frac{3}{4}$. It is easy to see $(\frac12)_k\equiv(\frac14)_k\equiv 0 \pmod p$ for $(3p-1)/4<k\leqslant p-1$. Thus, from the $d=4$ case of (1.10) we deduce the following supercongruence, which was conjectured by Liu and Wang [17, Conjecture 4].

Corollary 1.2. *Let $p\equiv 5 \pmod 8$ be a prime. Then*

$$\sum_{k=0}^{p-1}(12k+1)\frac{(\frac14)_k^2(\frac18)_k}{k!^3 4^k}\equiv 0 \pmod{p^3}.$$

The second aim of this paper is to establish the following q-supercongruence, which is a generalization of [17, Theorem 8] for the second case with d even.

Theorem 1.3. *Let $d\geqslant 2$ be an even integer and e an indeterminate. Let $n\equiv d+1 \pmod{2d}$ be a positive integer. Then, modulo $\Phi_n(q)^3$,*

$$\sum_{k=0}^{M}[3dk-1]\frac{(q^{-1},e,q^{d-1}/e;q^{2d})_k(q,q,q^{d-3};q^d)_k q^{dk}}{(q^d,e,q^{d-1}/e;q^d)_k(q^{2d-2},q^{2d-2},q^{d+2};q^{2d})_k}\equiv 0, \tag{1.11}$$

where $M=(n-1)/2$ if $d=2$, and $M=(dn-n+1)/(2d)$ otherwise.

The third aim of this paper is to prove the following q-supercongruence, which was conjectured by Liu and Wang [17, Conjecture 5].

Theorem 1.4. *Let n be a positive odd integer. Then, modulo $\Phi_n(q)^3$,*

$$\sum_{k=0}^{(n-1)/2}[6k-1]\frac{(q^{-1};q^4)_k(q^{-1};q^2)_k(q;q^2)_k^2}{(q^2;q^2)_k(q^4;q^4)_k(q^2;q^4)_k^2}q^{k^2+k+1}$$
$$\equiv\begin{cases}-\dfrac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}q^{(n-1)/4}, & \text{if } n\equiv 1 \pmod 4,\\[2mm] 0, & \text{if } n\equiv 3 \pmod 4.\end{cases} \tag{1.12}$$

Liu and Wang's original conjecture states that (1.12) holds modulo $[n]\Phi_n(q)^2$, which is in fact not true (the first counterexample is $n=15$).

We shall prove Theorems 1.1, 1.3, and 1.4 by using the method of 'creative microscoping' and Rahman's quadratic transformation (1.6) once more. At the end of this paper, we put forward several open problems on supercongruences and q-supercongruences.

2. Proof of Theorem 1.1

We first give a generalization of Theorem 1.1 with an extra parameter a. Note that this q-congruence modulo $(1-aq^n)(a-q^n)$ was actually indicated by Liu and Wang [17]. In order to make the paper self-contained, we give a complete proof here.

Theorem 2.1. *Let $d\geqslant 2$ be an even integer and a, e indeterminates. Let $n\equiv d+1 \pmod{2d}$ be a positive integer. Then, modulo $\Phi_n(q)(1-aq^n)(a-q^n)$,*

$$\sum_{k=0}^{(dn+n-1)/(2d)}[3dk+1]\frac{(q,e,q^{1+d}/e;q^{2d})_k(aq,q/a,q^{d-1};q^d)_k}{(q^d,e,q^{1+d}/e;q^d)_k(aq^{2d},q^{2d}/a,q^{d+2};q^{2d})_k}q^{dk}\equiv 0. \tag{2.1}$$

Proof. Put $q\mapsto q^d$, $a=q^{1-dn-n}$, $b=aq$, $c=q/a$ and $d=e$ in Rahman's transformation (1.6). Then, for $n\equiv d+1 \pmod{2d}$, we have

$$\begin{aligned}
&\sum_{k=0}^{(dn+n-1)/(2d)}\frac{(1-q^{3dk+1-dn-n})(q^{1-dn-n},e,q^{1+d-dn-n}/e;q^{2d})_k(aq,q/a,q^{d-1-dn-n};q^d)_k}{(1-q^{1-dn-n})(q^d,e,q^{1+d-dn-n}/e;q^d)_k(aq^{2d-dn-n},q^{2d-dn-n}/a,q^{d+2};q^{2d})_k}q^{dk}\\
&=\frac{(q^{2d+1-dn-n},q^{2d-1-dn-n},aq^{d+1},q^{d+1}/a;q^{2d})_\infty}{(q^d,q^{d+2},aq^{2d-dn-n},q^{2d-dn-n}/a;q^{2d})_\infty}\sum_{k=0}^{\infty}\frac{(aq,q/a,q^{d-1-dn-n};q^{2d})_k}{(q^{2d},eq^d,q^{2d+1-dn-n}/e;q^{2d})_k}q^{2dk}\\
&=0,
\end{aligned}\tag{2.2}$$

where we have used $(q^{2d+1-dn-n};q^{2d})_\infty=0$ for $n\equiv d+1 \pmod{2d}$. It is easy to see that, for $0\leqslant k\leqslant (dn+n-1)/(2d)$, the polynomial

$$(1-q^{1-dn-n})(q^d,e,q^{1+d-dn-n}/e;q^d)_k(aq^{2d-dn-n},q^{2d-dn-n}/a,q^{d+2};q^{2d})_k$$

is relatively prime to $\Phi_n(q)$. Since $q^n\equiv 1 \pmod{\Phi_n(q)}$, from (2.2) we deduce that the q-congruence (2.1) is true modulo $\Phi_n(q)$.

On the other hand, letting $q\mapsto q^d, a=q, b=q^{1-n}$, $c=q^{1+n}$ and $d=e$ in (1.6), we get

$$\begin{aligned}
&\sum_{k=0}^{(n-1)/d}[3dk+1]\frac{(q,e,q^{1+d}/e;q^{2d})_k(q^{1-n},q^{1+n},q^{d-1};q^d)_k}{(q^d,e,q^{1+d}/e;q^d)_k(q^{2d-n},q^{2d+n},q^{d+2};q^{2d})_k}q^{dk}\\
&=\frac{(q^{2d+1},q^{d+1-n},q^{d+1+n},q^{2d-1};q^{2d})_\infty}{(q^d,q^{2d+n},q^{2d-n},q^{d+2};q^{2d})_\infty}\sum_{k=0}^{\infty}\frac{(q^{1-n},q^{1+n},q^{d-1};q^{2d})_k}{(q^{2d},eq^d,q^{2d+1}/e;q^{2d})_k}q^{2dk}\\
&=0,
\end{aligned}$$

since $(q^{d+1-n};q^{2d})_\infty=0$ for $n\equiv d+1 \pmod{2d}$. Noticing that $(dn-1)/(2d)>(n-1)/d$, we conclude that the left-hand side of (2.1) is equal to 0 for $a=q^{-n}$ and $a=q^n$. Namely, the q-congruence (2.1) is true modulo $1-aq^n$ and $a-q^n$. Since $\Phi_n(q)$, $1-aq^n$, and $a-q^n$ are pairwise relatively prime polynomials in q, we complete the proof. ☐

Proof of Theorem 1.1. Since $n\equiv d+1 \pmod{2d}$, we have $\gcd(2d,n)=1$. Hence, $(q^{2d};q^{2d})_k$ is relatively prime to $\Phi_n(q)$ for any $0\leqslant k\leqslant n-1$. It is clear that $(dn+n-1)/(2d)\leqslant n-1$. Moreover, the polynomial $1-q^n$ contains the factor $\Phi_n(q)$. The proof of (1.9) then follows from (2.1) by taking $a=1$. ☐

3. Proof of Theorem 1.3

Like before, we first establish the following generalization of Theorem 1.3 with an additional parameter a.

Theorem 3.1. *Let $d \geqslant 2$ be an even integer and a, e indeterminates. Let $n \equiv d+1 \pmod{2d}$ be a positive integer. Then, modulo $\Phi_n(q)(1-aq^n)(a-q^n)$,*

$$\sum_{k=0}^{M}[3dk-1]\frac{(q^{-1},e,q^{d-1}/e;q^{2d})_k(aq,q/a,q^{d-3};q^d)_k q^{dk}}{(q^d,e,q^{d-1}/e;q^d)_k(aq^{2d-2},q^{2d-2}/a,q^{d+2};q^{2d})_k}\equiv 0, \tag{3.1}$$

where $M=(n-1)/2$ if $d=2$, and $M=(dn-n+1)/(2d)$ otherwise.

Proof. Set $q\mapsto q^d$, $a=q^{-1-dn+n}$, $b=aq$, $c=q/a$ and $d=e$ in (1.6). Then, for $n\equiv d+1 \pmod{2d}$, we have

$$\begin{aligned}
&\sum_{k=0}^{(dn-n+1)/(2d)}\frac{(1-q^{3dk-1-dn+n})(q^{-1-dn+n},e,q^{d-1-dn+n}/e;q^{2d})_k(aq,q/a,q^{d-3-dn+n};q^d)_k}{(1-q^{-1-dn+n})(q^d,e,q^{d-1-dn+n}/e;q^d)_k(aq^{2d-2-dn+n},q^{2d-2-dn+n}/a,q^{d+2};q^{2d})_k}q^{dk}\\
&=\frac{(q^{2d-1-dn+n},q^{2d-3-dn+n},aq^{d+1},q^{d+1}/a;q^{2d})_\infty}{(q^d,q^{d+2},aq^{2d-dn+n},q^{2d-dn+n}/a;q^{2d})_\infty}\sum_{k=0}^{\infty}\frac{(aq,q/a,q^{d-3-dn+n};q^{2d})_k}{(q^{2d},eq^d,q^{2d-1-dn+n}/e;q^{2d})_k}q^{2dk}\\
&=0,
\end{aligned}\tag{3.2}$$

where we have utilized $(q^{2d-1-dn+n};q^{2d})_\infty=0$ for $n\equiv d+1 \pmod{2d}$. Moreover, it is not difficult to see that $(q^{2d-2};q^{2d})_k$ is relatively prime to $\Phi_n(q)$ for $0\leqslant k\leqslant M$ (in fact, this is true for $0\leqslant k\leqslant (d-1)(n-1)/d$). Noticing $q^n\equiv 1 \pmod{\Phi_n(q)}$ again, the modulus $\Phi_n(q)$ case of the q-congruence (3.1) follows from (3.2) immediately (for $d=2$, we need to use the fact that $(q^{-1};q^4)_k\equiv 0 \pmod{\Phi_n(q)}$ for $(n+1)/4<k\leqslant (n-1)/2$).

On the other hand, letting $q\mapsto q^d, a=q^{-1}, b=q^{1-n}$, $c=q^{1+n}$ and $d=e$ in (1.6), we obtain

$$\begin{aligned}
&\sum_{k=0}^{(n-1)/d}[3dk-1]\frac{(q^{-1},e,q^{d-1}/e;q^{2d})_k(q^{1-n},q^{1+n},q^{d-3};q^d)_k}{(q^d,e,q^{d-1}/e;q^d)_k(q^{2d-2-n},q^{2d-2+n},q^{d+2};q^{2d})_k}q^{dk}\\
&\quad=-\frac{(q^{2d-1},q^{2d-3},q^{d+1-n},q^{d+1+n};q^{2d})_\infty}{q(q^d,q^{d+2},q^{2d-2-n},q^{2d-2+n};q^{2d})_\infty}\sum_{k=0}^{(n-1)/(2d)}\frac{(q^{1-n},q^{1+n},q^{d-3};q^{2d})_k}{(q^{2d},eq^d,q^{2d-1}/e;q^{2d})_k}q^{2dk}\\
&\quad=0,
\end{aligned}$$

as was first given by Liu and Wang [17]. In view of $(dn-n-1)/(2d)<(n-1)/2$ for $d=2$, and $(dn-n-1)/(2d)>(n-1)/d$ for $d\geqslant 4$, one sees that the left-hand side of (3.1) is equal to 0 for $a=q^{-n}$ and $a=q^n$. Thus, the q-congruence (3.1) holds modulo $1-aq^n$ and $a-q^n$. Since the polynomials $\Phi_n(q)$, $1-aq^n$, and $a-q^n$ are relatively prime to one another, we accomplish the proof. □

Proof of Theorem 1.3. In the proof of Theorem 3.1, we have mentioned that $(q^{2d-2};q^{2d})_k$ is relatively prime to $\Phi_n(q)$ for $0\leqslant k\leqslant (dn-n+1)/(2d)$. The proof of (1.11) then follows from the $a=1$ case of (3.1). $\square$

4. Proof of Theorem 1.4

We require the following result, which was first given in [5, Lemma 2.1]. For the reader's convenience, we include a short proof here.

Lemma 4.1. *Let n be a positive odd integer and a an indeterminate. Then*

$$(aq,q/a;q^2)_{(n-1)/2}\equiv(-1)^{(n-1)/2}\frac{(1-a^n)q^{(1-n^2)/4}}{(1-a)a^{(n-1)/2}}\pmod{\Phi_n(q)}.\tag{4.1}$$

Proof. It is easy to see that

$$\begin{aligned}(q/a;q^2)_{(n-1)/2}&=(1-q/a)(1-q^3/a)\cdots(1-q^{n-2}/a)\\&\equiv(1-q^{1-n}/a)(1-q^{3-n}/a)\cdots(1-q^{-2}/a)\\&=(-1)^{(n-1)/2}(aq^2;q^2)_{(n-1)/2}\frac{q^{(1-n^2)/4}}{a^{(n-1)/2}}\pmod{\Phi_n(q)}.\end{aligned}$$

Hence, the left-hand side of (4.1) is congruent to

$$(-1)^{(n-1)/2}(aq;q)_{n-1}\frac{q^{(1-n^2)/4}}{a^{(n-1)/2}}.$$

For any n-th primitive root of unity ζ, we have

$$(a\zeta;\zeta)_{n-1}=\frac{(a;\zeta)_n}{1-a}=\frac{1-a^n}{1-a},$$

and so $(aq;q)_{n-1}$ is congruent to $(1-a^n)/(1-a)$ modulo $\Phi_n(q)$. This completes the proof. $\square$

We have the following parametric generalization of Theorem 1.4 for $n\equiv 1\pmod 4$.

Theorem 4.2. *Let $n\equiv 1\pmod 4$ be a positive integer and a an indeterminate. Then, modulo $\Phi_n(q)(1-aq^n)(a-q^n)$,*

$$\sum_{k=0}^{(n-1)/2}[6k-1]\frac{(q^{-1};q^4)_k(q^{-1},aq,q/a;q^2)_k}{(q^2;q^2)_k(q^4,aq^2,q^2/a;q^4)_k}q^{k^2+k+1}\equiv\frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}q^{(n-1)/4}.\tag{4.2}$$

Proof. Letting $d \to 0$ in (1.6), we have

$$\sum_{k=0}^{\infty} \frac{(1-aq^{3k})(a;q^2)_k(b,c,aq/bc;q)_k}{(1-a)(q;q)_k(aq^2/b,aq^2/c,bcq;q^2)_k} q^{(k^2+k)/2} = \frac{(aq^2,bq,cq,aq^2/bc;q^2)_\infty}{(q,aq^2/b,aq^2/c,bcq;q^2)_\infty}. \tag{4.3}$$

We then take $q \mapsto q^2$, $a = q^{-1-n}$, $b = aq$, $c = q/a$ in the above formula to obtain

$$\begin{aligned}
&\sum_{k=0}^{(n+1)/2} \frac{(1-q^{6k-1-n})(q^{-1-n};q^4)_k(aq,q/a,q^{-1-n};q^2)_k}{(1-q^{-1-n})(q^2;q^2)_k(aq^{2-n},q^{2-n}/a,q^4;q^4)_k} q^{k^2+k} \\
&= \frac{(q^{3-n},aq^3,q^3/a,q^{1-n};q^4)_\infty}{(q^2,q^{2-n}/a,aq^{2-n},q^4;q^4)_\infty} \\
&= 0.
\end{aligned} \tag{4.4}$$

Since $q^n \equiv 1 \pmod{\Phi_n(q)}$, we conclude from the above equality that

$$\sum_{k=0}^{(n+1)/2} [6k-1] \frac{(q^{-1};q^4)_k(q^{-1},aq,q/a;q^2)_k}{(q^2;q^2)_k(q^4,aq^2,q^2/a;q^4)_k} q^{k^2+k+1} \equiv 0 \pmod{\Phi_n(q)}.$$

Namely,

$$\begin{aligned}
&\sum_{k=0}^{(n-1)/2} [6k-1] \frac{(q^{-1};q^4)_k(q^{-1},aq,q/a;q^2)_k}{(q^2;q^2)_k(q^4,aq^2,q^2/a;q^4)_k} q^{k^2+k+1} \\
&\equiv -[2] \frac{(q^{-1};q^4)_{(n+1)/2}(q^{-1},aq,q/a;q^2)_{(n+1)/2}}{(q^2;q^2)_{(n+1)/2}(q^4,aq^2,q^2/a;q^4)_{(n+1)/2}} q^{(n+1)(n+3)/4+1} \pmod{\Phi_n(q)}.
\end{aligned} \tag{4.5}$$

By Lemma 4.1, we have

$$\begin{aligned}
\frac{(aq,q/a;q^2)_{(n+1)/2}}{(aq^2,q^2/a;q^4)_{(n+1)/2}} &= \frac{(aq,q/a;q^2)_{(n-1)/2}(1-aq^n)(1-q^n/a)}{(aq^2,q^2/a;q^4)_{(n-1)/2}(1-aq^{2n})(1-q^{2n}/a)} \\
&= q^{(n^2-1)/4} \pmod{\Phi_n(q)}.
\end{aligned}$$

Moreover, modulo $\Phi_n(q)$,

$$\begin{aligned}
\frac{(q^{-1};q^2)_{(n+1)/2}}{(q^2;q^2)_{(n+1)/2}} &= \frac{(1-q^{-1})(1-q)\cdots(1-q^{n-2})}{(1-q^2)(1-q^4)\cdots(1-q^{n+1})} \equiv (-1)^{(n+1)/2} q^{-(n+1)(n+3)/4}, \\
-[2]\frac{(q^{-1};q^4)_{(n+1)/2}}{(q^4;q^4)_{(n+1)/2}} q &\equiv \frac{(q^3;q^4)_{(n-1)/2}}{(q^4;q^4)_{(n-1)/2}} = \frac{(q^3;q^4)_{(n-1)/4}(q^{n+2};q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}(q^{n+3};q^4)_{(n-1)/4}} \\
&\equiv \frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}.
\end{aligned}$$

Employing the above three q-congruences, we see that the right-hand side of (4.5) reduces to

$$\frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}} q^{(n-1)/4} \pmod{\Phi_n(q)}.$$

This proves that (4.2) is true modulo $\Phi_n(q)$.

The modulus $(1-aq^n)(a-q^n)$ case of (4.2) was already given by Liu and Wang [17, (4.2) with $e\to 0$], and this can be easily checked by putting $q\mapsto q^2$, $a=q^{-1}$, $b=q^{1-n}$, $c=q^{1+n}$ in (4.3). Since the polynomials $\Phi_n(q)$ and $(1-aq^n)(a-q^n)$ are relatively prime, we finish the proof of the theorem. □

Proof of Theorem 1.4. Letting $a=1$ in (4.2), we arrive at (1.12) for $n\equiv 1 \pmod 4$. On the other hand, letting $d=2$ and $e\to 0$ in (1.11), we are led to (1.12) for $n\equiv 3 \pmod 4$. □

5. Concluding remarks and open problems

Numerical calculation suggests that we can replace the upper bound of the sum in (1.10) by $p-1$. Namely, the following variation of (1.10) should be true.

Conjecture 5.1. *Let $d\geqslant 2$ be an even integer and let $p\equiv d+1 \pmod{2d}$ be a prime. Then*

$$\sum_{k=0}^{p-1}(3dk+1)\frac{(\frac{1}{d})_k^2(\frac{d-1}{d})_k(\frac{1}{2d})_k}{k!^3 4^k(\frac{d+2}{2d})_k}\equiv 0 \pmod{p^3}.$$

Furthermore, it seems that (1.9) is also true modulo $\Phi_n(q)^3$ for $N=n-1$, which we state as the following conjecture (which is also a generalization of Conjecture 5.1).

Conjecture 5.2. *Let $d\geqslant 2$ be an even integer and e an indeterminate. Let $n\equiv d+1 \pmod{2d}$ be a positive integer. Then*

$$\sum_{k=0}^{n-1}[3dk+1]\frac{(q,e,q^{1+d}/e;q^{2d})_k(q,q,q^{d-1};q^d)_k}{(q^d,e,q^{1+d}/e;q^d)_k(q^{2d},q^{2d},q^{d+2};q^{2d})_k}q^{dk}\equiv 0 \pmod{\Phi_n(q)^3}. \tag{5.1}$$

It should be pointed out that the $d=2$ case of (5.1) was already proved by Liu and Wang themselves [17, Theorem 1]. For $d=4$, since $(q^{d-1};q^d)_k/(q^{d+2};q^{2d})_k=1/(-q^3;q^4)_k$, one can easily see that each k-th summand on the left-hand side of (1.9) is congruent to 0 modulo $\Phi_n(q)^3$ for $(3n-1)/4<k\leqslant n-1$. Therefore, By Theorem 1.1, the q-supercongruence (5.1) is also true for $d=4$. However, the same argument does not work for $d\geqslant 6$.

We find that Theorem 1.3 for $d=4$ can be further strengthened as follows.

Conjecture 5.3. *Let $n\equiv 5 \pmod 8$ be a positive integer and e an indeterminate. Then*

$$\sum_{k=0}^{(3n+1)/8}[12k-1]\frac{(q^{-1},e,q^3/e;q^8)_k(q;q^4)_k^3q^{dk}}{(q^4,e,q^3/e;q^4)_k(q^6;q^8)_k^3}\equiv 0 \pmod{\Phi_n(q)^4}.$$

In particular, for any prime $p\equiv 5 \pmod 8$,

$$\sum_{k=0}^{(3p+1)/8}(12k-1)\frac{(-\frac{1}{8})_k(\frac{1}{4})_k^3}{k!4^k(\frac{3}{4})_k^3}\equiv 0 \pmod{p^4}.$$

Recently, the author and Zudilin [12] have extended many classical q-supercongruences to the so-called Dwork-type q-supercongruences through a creative q-microscope. They also proposed several difficult conjectures on Dwork-type q-supercongruences. Here we would like to propose such extensions of (1.7) and (1.12) for $n\equiv 1 \pmod 4$. We notice that a similar conjecture related to (1.5) was already made by Liu and Wang [16].

Conjecture 5.4. *Let $n\equiv 1 \pmod 4$ be a positive integer and let $r\geqslant 1$. Then, modulo $[n^r]\prod_{j=1}^r\Phi_{n^j}(q)^2$,*

$$\begin{aligned}
&\sum_{k=0}^{(n^r-1)/d}[6k+1]\frac{(q;q^4)_k(q;q^2)_k^3}{(q^2;q^2)_k(q^4;q^4)_k^3}q^{k^2+k}\\
&\quad\equiv \frac{(q^2;q^4)_{(n^r-1)/4}(q^{4n};q^{4n})_{(n^{r-1}-1)/4}}{(q^4;q^4)_{(n^r-1)/4}(q^{2n};q^{4n})_{(n^{r-1}-1)/4}}[n]q^{(1-n)/4}\\
&\quad\quad\times\sum_{k=0}^{(n^{r-1}-1)/d}[6k+1]\frac{(q^n;q^{4n})_k(q^n;q^{2n})_k^3}{(q^{2n};q^{2n})_k(q^{4n};q^{4n})_k^3}q^{(k^2+k)n},
\end{aligned}$$

where $d=1,2$.

Conjecture 5.5. *Let $n\equiv 1 \pmod 4$ be a positive integer and let $r\geqslant 1$. Then, modulo $\Phi_{n^r}(q)\prod_{j=1}^r\Phi_{n^j}(q)^2$,*

$$\begin{aligned}
&\sum_{k=0}^{(n^r-1)/2}[6k-1]\frac{(q^{-1};q^4)_k(q^{-1};q^2)_k(q;q^2)_k^2}{(q^2;q^2)_k(q^4;q^4)_k(q^2;q^4)_k^2}q^{k^2+k+1}\\
&\quad\equiv \frac{(q^2;q^4)_{(n^r-1)/4}(q^{4n};q^{4n})_{(n^{r-1}-1)/4}}{(q^4;q^4)_{(n^r-1)/4}(q^{2n};q^{4n})_{(n^{r-1}-1)/4}}q^{(n-1)/4}\\
&\quad\quad\times\sum_{k=0}^{(n^{r-1}-1)/2}[6k-1]\frac{(q^{-n};q^{4n})_k(q^{-n};q^{2n})_k(q^n;q^{2n})_k^2}{(q^{2n};q^{2n})_k(q^{4n};q^{4n})_k(q^{2n};q^{4n})_k^2}q^{(k^2+k+1)n}.
\end{aligned}$$

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