FURTHER GENERALIZATIONS OF FOUR SUPERCONGRUENCES OF RODRIGUEZ-VILLEGAS

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Abstract. In 2003, Fernando Rodriguez-Villegas discovered four supercongruences modulo p^2 (p an odd prime) for truncated ${}_2F_1$ hypergeometric series related to Calabi–Yau manifolds of dimension $d=1$. These four supercongruences were confirmed by Mortenson using Gaussian hypergeometric series and the p-adic Gamma function. q-Analogues of Rodriguez-Villegas' supercongruences were later established by Guo–Zeng and Guo–Pan–Zhang. In this paper, employing the 'creative microscoping' method developed by Guo–Zudilin, we give further extensions of these four q-supercongruences, which can also be considered as q-analogues of four supercongruences obtained by Liu. For example, we prove that, for all positive odd integers m and n, modulo $\Phi_n(q)^2$,

$$\sum_{k=0}^{(mn-1)/2}\frac{(q;q^2)_k^2}{(q^2;q^2)_k^2}q^{2k}\equiv(-1)^{(n-1)/2}q^{(n^2-1)/4}\sum_{k=0}^{(m-1)/2}\frac{(q^{n^2};q^{2n^2})_k^2}{(q^{2n^2};q^{2n^2})_k^2}q^{2n^2k},$$

where $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ and $\Phi_n(q)$ is the n-th cyclotomic polynomial in q.

1. Introduction

Rodriguez-Villegas [26] observed some astonishing supercongruences between a truncated hypergeometric function associated to Calabi-Yau manifolds at a prime p and the number of its $\mathbb{F}_p$-points. In particular, he [26, (36)] mentioned four such supercongruences associated to elliptic curves:

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\equiv\left(\frac{-1}{p}\right)\pmod{p^2}\quad\text{for } p>2,\tag{1.1}$$

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{2k}\binom{2k}{k}}{27^k}\equiv\left(\frac{-3}{p}\right)\pmod{p^2}\quad\text{for } p>3,\tag{1.2}$$

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{64^k}\equiv\left(\frac{-2}{p}\right)\pmod{p^2}\quad\text{for } p>2,\tag{1.3}$$

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$$\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{432^k}\equiv\left(\frac{-1}{p}\right)\pmod{p^2}\quad\text{for } p>3, \tag{1.4}$$

where $(\frac{a}{b})$ denotes the Kronecker symbol. It should be pointed out that the sum in (1.1) can be truncated at $(p-1)/2$, since $\binom{2k}{k}\equiv 0 \pmod p$ for $(p-1)/2<k\leqslant p-1$. Rodriguez-Villegas' supercongruences (1.1)–(1.4) were first confirmed by Mortenson [21, 22] using the theory of Gaussian hypergeometric series and properties of the p-adic Gamma function (following a strategy devised by Ahlgren and Ono [1]). For an elementary proof of them, see [27]. Some q-analogues of (1.1)–(1.4) can be found in [7, 10, 12, 23]. For instance, Guo, Pan, and Zhang [10] proved that, for odd n,

$$\sum_{k=0}^{n-1}\frac{(q;q^2)_k^2}{(q^2;q^2)_k^2}q^{2k}\equiv\left(\frac{-1}{n}\right)q^{(n^2-1)/4}\pmod{\Phi_n(q)^2}, \tag{1.5}$$

$$\sum_{k=0}^{n-1}\frac{(q;q^3)_k(q^2;q^3)_k}{(q^3;q^3)_k^2}q^{3k}\equiv\left(\frac{-3}{n}\right)q^{(n^2-1)/3}\pmod{\Phi_n(q)^2}\quad\text{if } \gcd(n,3)=1, \tag{1.6}$$

$$\sum_{k=0}^{n-1}\frac{(q;q^4)_k(q^3;q^4)_k}{(q^4;q^4)_k^2}q^{4k}\equiv\left(\frac{-2}{n}\right)q^{3(n^2-1)/8}\pmod{\Phi_n(q)^2}, \tag{1.7}$$

$$\sum_{k=0}^{n-1}\frac{(q;q^6)_k(q^5;q^6)_k}{(q^6;q^6)_k^2}q^{6k}\equiv\left(\frac{-1}{n}\right)q^{5(n^2-1)/12}\pmod{\Phi_n(q)^2}\quad\text{if } \gcd(n,3)=1, \tag{1.8}$$

(the $n=p$ case of (1.5) was first given by Guo and Zeng [12]). Here we need to familiarize ourselves with the standard q-notation. The *q-integer* is defined by $[n]_q=1+q+\cdots+q^{n-1}$, the *q-shifted factorial* is defined as

$$(a;q)_n=\begin{cases}(1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if } n\geq 1,\\ 1, & \text{if } n=0,\end{cases}$$

and the *n-th cyclotomic polynomial* in q is given by

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}}(q-\zeta^k),$$

where ζ is an n-th primitive root of unity. More recent q-supercongruences can be found in [4, 8, 11, 13–15, 19, 20, 24, 25, 31, 32].

Note that we can truncate the sum in (1.5) at $(n-1)/2$, since $(q;q^2)_k/(q^2;q^2)_k\equiv 0 \pmod{\Phi_n(q)}$ for $(n-1)/2<k\leqslant n-1$. In this paper, we first give a generalization of this shortened form of (1.5).

Theorem 1.1. *Let m and n be positive odd integers. Then*

$$\sum_{k=0}^{(mn-1)/2}\frac{(q;q^2)_k^2}{(q^2;q^2)_k^2}q^{2k}\equiv\left(\frac{-1}{n}\right)q^{(n^2-1)/4}\sum_{k=0}^{(m-1)/2}\frac{(q^{n^2};q^{2n^2})_k^2}{(q^{2n^2};q^{2n^2})_k^2}q^{2n^2k}\pmod{\Phi_n(q)^2}. \tag{1.9}$$

Liu [17] proved the following generalizations of (1.1)–(1.4): for any prime $p>3$ and positive odd integer m,

$$\sum_{k=0}^{mp-1}\frac{\binom{2k}{k}^2}{16^k}\equiv\left(\frac{-1}{p}\right)\sum_{k=0}^{m-1}\frac{\binom{2k}{k}^2}{16^k}\pmod{p^2}, \tag{1.10}$$

$$\sum_{k=0}^{mp-1}\frac{\binom{3k}{k}\binom{2k}{k}}{27^k}\equiv\left(\frac{-3}{p}\right)\sum_{k=0}^{m-1}\frac{\binom{3k}{k}\binom{2k}{k}}{27^k}\pmod{p^2}, \tag{1.11}$$

$$\sum_{k=0}^{mp-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{64^k}\equiv\left(\frac{-2}{p}\right)\sum_{k=0}^{m-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{64^k}\pmod{p^2}, \tag{1.12}$$

$$\sum_{k=0}^{mp-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{432^k}\equiv\left(\frac{-1}{p}\right)\sum_{k=0}^{m-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{432^k}\pmod{p^2}. \tag{1.13}$$

Let $\langle x\rangle_n$ denote the least nonnegative residue of x modulo n. In this paper, we shall also give a common generalization of (1.5)–(1.8) and (1.10)–(1.13) by establishing the following q-supercongruence.

Theorem 1.2. *Let d,m,r be positive integers with $r<d$. Let $n>1$ be an odd integer satisfying $n\equiv\pm1\pmod{d}$. Then, modulo $\Phi_n(q)^2$,*

$$\sum_{k=0}^{mn-1}\frac{(q^r;q^d)_k(q^{d-r};q^d)_k}{(q^d;q^d)_k^2}q^{dk}$$
$$\equiv(-1)^{\langle -r/d\rangle_n}q^{r(d-r)(n^2-1)/(2d)}\sum_{k=0}^{m-1}\frac{(q^{rn^2};q^{n^2})_k(q^{(d-r)n^2};q^{dn^2})_k}{(q^{dn^2};q^{dn^2})_k^2}q^{dn^2k}. \tag{1.14}$$

It is easy to see that, for any positive odd n,

$$(-1)^{\langle -1/2\rangle_n}=\left(\frac{-1}{n}\right),\quad (-1)^{\langle -1/4\rangle_n}=\left(\frac{-2}{n}\right),$$

and for any positive n with $\gcd(n,6)=1$,

$$(-1)^{\langle -1/3\rangle_n}=\left(\frac{-3}{n}\right),\quad (-1)^{\langle -1/6\rangle_n}=\left(\frac{-1}{n}\right).$$

Thus, for $d=2,3,4,6$, letting $m=1$ in Theorem 1.2, we get the q-supercongruences (1.5)–(1.8). It is well known that $\Phi_p(1)=1$ for any prime p. Moreover, for $d=2,3,4,6$, and any prime $p>3$, we always have $p\equiv\pm1\pmod{d}$. Thus, for these d with $r=1$, letting n be a prime and then taking $q\to1$ in Theorem 1.2, we are led to the supercongruences (1.10)–(1.13).

The paper is arranged as follows. We prove Theorem 1.1 in the next section by adopting the method of 'creative microscoping', which was recently developed by Guo and Zudilin [13]. More precisely, we shall first give a generalization of Theorem 1.1 with an extra parameter a, and then educe Theorem 1.1 from this generalization by choosing $a=1$. Here we want to emphasize that our method of inserting the parameter a is a little different from that of the paper [14], where many

Dwork-type q-supercongruences are proved. In Section 3, we first give a parametric generalization of the aforementioned result of Guo, Pang, and Zhang [10], and then prove Theorem 1.2 using the 'creative microscoping' method again.

2. Proof of Theorem 1.1

We need the following generalization of (1.5), which was proved by the first author [7, Corollary 4.3].

Lemma 2.1. *Let n be a positive odd integer. Then*

$$\sum_{k=0}^{(n-1)/2} \frac{(aq;q^2)_k(a^{-1}q;q^2)_k q^{2k}}{(q^2;q^2)_k^2} \equiv \left(\frac{-1}{n}\right) q^{(n^2-1)/4} \pmod{(1-aq^n)(a-q^n)}. \tag{2.1}$$

We have the following parametric generalization of Theorem 1.1.

Theorem 2.2. *Let m and n be positive odd integers with $n>1$. Then, modulo*

$$\prod_{j=0}^{(m-1)/2} (1-aq^{(2j+1)n})(a-q^{(2j+1)n}), \tag{2.2}$$

we have

$$\begin{aligned}\sum_{k=0}^{(mn-1)/2} & \frac{(aq;q^2)_k(a^{-1}q;q^2)_k}{(q^2;q^2)_k^2} q^{2k} \\ &\equiv \left(\frac{-1}{n}\right) q^{(n^2-1)/4} \sum_{k=0}^{(m-1)/2} \frac{(a^nq^{n^2};q^{2n^2})_k(a^{-n}q^{n^2};q^{2n^2})_k}{(q^{2n^2};q^{2n^2})_k^2} q^{2n^2k}. \end{aligned}\tag{2.3}$$

Proof. It suffices to show that both sides of (2.3) are equal for $a=q^{-(2j+1)n}$ and $a=q^{(2j+1)n}$ with $j=0,1,\dots,(m-1)/2$, i.e.,

$$\begin{aligned}\sum_{k=0}^{(mn-1)/2} & \frac{(q^{1-(2j+1)n};q^2)_k(q^{1+(2j+1)n};q^2)_k}{(q^2;q^2)_k^2} q^{2k} \\ &= \left(\frac{-1}{n}\right) q^{(n^2-1)/4} \sum_{k=0}^{(m-1)/2} \frac{(q^{-2jn^2};q^{2n^2})_k(q^{(2j+2)n^2};q^{2n^2})_k}{(q^{2n^2};q^{2n^2})_k^2} q^{2n^2k}. \end{aligned}\tag{2.4}$$

Clearly, $(mn-1)/2 \geqslant ((2j+1)n-1)/2$ for $0\leqslant j\leqslant (m-1)/2$, and $(q^{1-(2j+1)n};q^2)_k=0$ for $k>((2j+1)n-1)/2$. By the $a=q^{-n}$ case of (2.1) (it becomes an identity), the left-hand side of (2.4) is equal to $\left(\frac{-1}{(2j+1)n}\right)q^{((2j+1)^2n^2-1)/4}$. Similarly, the right-hand side of (2.4) can be simplified as

$$\left(\frac{-1}{n}\right) q^{(n^2-1)/4} \left(\frac{-1}{2j+1}\right) q^{n^2((2j+1)^2-1)/4} = \left(\frac{-1}{(2j+1)n}\right) q^{((2j+1)^2n^2-1)/4},$$

where $\left(\frac{-1}{1}\right)$ is understood to be 1. This proves the identity (2.4), and so the q-congruence (2.3) holds. $\square$

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. It is well known that, for any positive integer N,

$$q^N - 1 = \prod_{d|N} \Phi_d(q).$$

Thus, the $a = 1$ case of (2.2) has the factor $\Phi_n(q)^{m+1}$. On the other hand, the least common denominator of the left-hand side of (2.3) is $(q^2; q^2)^2_{(mn-1)/2}$, and its factor related to $\Phi_n(q)$ is $\Phi_n(q)^{m-1}$. Likewise, the least common denominator of the right-hand side of (2.3) only has the factor $\Phi_n(q)^{m-1}$ related to $\Phi_n(q)$ too. Hence, letting $a = 1$ in (2.3), we conclude that the q-congruence (1.9) holds. $\square$

3. Proof of Theorem 1.2

We first establish a parametric generalization of the $n \equiv 1 \pmod 2$ case of [10, Corollary 3.1].

Lemma 3.1. *Let $d, n \geqslant 2$ with $\gcd(d, n) = 1$ and n odd. Let r be an integer. Then, modulo*

$$(1 - aq^{r+d\langle -r/d\rangle_n})(a - q^{d-r+d\langle (r-d)/d\rangle_n}),$$

we have

$$\sum_{k=0}^{n-1} \frac{(aq^r; q^d)_k (a^{-1}q^{d-r}; q^d)_k}{(q^d; q^d)_k^2} q^{dk} \equiv (-1)^s a^{(n-1)/2-s} q^{d\binom{s+1}{2}-(r+ds)(s-(n-1)/2)}, \tag{3.1}$$

where $s = \langle -r/d\rangle_n$.

Proof. Recall that the q-Chu–Vandermonde summation formula [3, Appendix (II.6)] can be stated as follows:

$$\sum_{k=0}^{n} \frac{(q^{-n}; q)_k (b; q)_k}{(q; q)_k (c; q)_k} q^k = \frac{(c/b; q)_n}{(c; q)_n} b^n.$$

Thus, for $a = q^{-r-d\langle -r/d\rangle_n} = q^{-r-ds}$, the left-hand side of (3.1) is equal to

$$\begin{aligned}
\sum_{k=0}^{n-1} \frac{(q^{-ds}; q^d)_k (q^{d+ds}; q^d)_k}{(q^d; q^d)_k^2} q^{dk} &= \frac{(q^{-ds}; q^d)_s}{(q^d; q^d)_s} q^{(d+ds)s} \\
&= (-1)^s q^{-d\binom{s+1}{2}+(d+ds)s} \\
&= (-1)^s q^{-(r+ds)((n-1)/2-s)+d\binom{s+1}{2}-(r+ds)(s-(n-1)/2)},
\end{aligned}$$

which is just the right-hand side of (3.1). This proves that the congruence (3.1) holds modulo $(1 - aq^{r+d\langle -r/d\rangle_n})$.

Let $t = \langle -(d-r)/d\rangle_n$. It is not difficult to see that $s + t = n - 1$. Similarly, for $a = q^{d-r+d\langle (r-d)/d\rangle_n} = q^{(d-r)+dt}$, the left-hand side of (3.1) is equal to

$$\begin{aligned}
\sum_{k=0}^{n-1} \frac{(q^{d+dt}; q^d)_k (q^{-dt}; q^d)_k}{(q^d; q^d)_k^2} q^{dk} &= (-1)^t q^{d\binom{t+1}{2}} \\
&= (-1)^s q^{(d-r+dt)((n-1)/2-s)+d\binom{s+1}{2}-(r+ds)(s-(n-1)/2)},
\end{aligned}$$

which is again the right-hand side of (3.1). This proves that the congruence (3.1) holds modulo $(a-q^{d-r+d\langle (r-d)/d\rangle_n})$. Since $(1-aq^{r+d\langle -r/d\rangle_n})$ and $(a-q^{d-r+d\langle (r-d)/d\rangle_n})$ are relatively prime polynomials, we complete the proof. $\square$

We now give a parametric generalization of Theorem 1.2 for $n\equiv 1 \pmod d$.

Theorem 3.2. *Let d,m,r be positive integers with $r<d$. Let $n>1$ be an odd integer satisfying $n\equiv 1 \pmod d$. Then, modulo*

$$\prod_{j=0}^{m-1}(1-aq^{(dj+r)n})(a-q^{(dj+d-r)n}), \tag{3.2}$$

we have

$$\sum_{k=0}^{mn-1}\frac{(aq^r;q^d)_k(a^{-1}q^{d-r};q^d)_k}{(q^d;q^d)_k^2}q^{dk}\equiv (-1)^{\langle -r/d\rangle_n}a^{(n-1)/2-\langle -r/d\rangle_n}q^{r(d-r)(n^2-1)/(2d)} \times\sum_{k=0}^{m-1}\frac{(a^nq^{rn^2};q^{dn^2})_k(a^{-n}q^{(d-r)n^2};q^{dn^2})_k}{(q^{dn^2};q^{dn^2})_k^2}q^{dn^2k}. \tag{3.3}$$

Proof. Similarly as before, we need to show that both sides of (3.3) are equal for $a=q^{-(dj+r)n}$ and $a=q^{(dj+d-r)n}$ $(j=0,1,2,\dots,m-1)$. We first prove the $a=q^{-(dj+r)n}$ case. Namely,

$$\begin{aligned}\sum_{k=0}^{mn-1}&\frac{(q^{r-(dj+r)n};q^d)_k(q^{d-r+(dj+r)n};q^d)_k}{(q^d;q^d)_k^2}q^{dk}\\ &=(-1)^{\langle -r/d\rangle_n}q^{-(dj+r)n((n-1)/2-\langle -r/d\rangle_n)+r(d-r)(n^2-1)/(2d)}\\ &\quad\times\sum_{k=0}^{m-1}\frac{(q^{-djn^2};q^{dn^2})_k(q^{dn^2(j+1)};q^{dn^2})_k}{(q^{dn^2};q^{dn^2})_k^2}q^{dn^2k}.\end{aligned} \tag{3.4}$$

It is clear that $mn-1\geqslant nj+r(n-1)/d$ and $mn-1\geqslant nj+(d-r)(n-1)/d$ for $j=0,1,2,\dots,m-1$. Since $n\equiv 1 \pmod d$, we have $\langle -r/d\rangle_n=r(n-1)/d$ and $\langle (r-d)/d\rangle_n=(d-r)(n-1)/d$. By Lemma 3.1, the left-hand side of (3.4) is equal to

$$(-1)^{nj+r(n-1)/d}q^{d\binom{jn+r(n-1)/d+1}{2}}.$$

Similarly, the right-hand side of (3.4) is equal to

$$\begin{aligned}&(-1)^{j+r(n-1)/d}q^{-(dj+r)n((n-1)/2-r(n-1)/d)+r(d-r)(n^2-1)/(2d)+dn^2(j^2+j)/2}\\ &=(-1)^{j+r(n-1)/d}q^{d\binom{jn+r(n-1)/d+1}{2}},\end{aligned}$$

thus establishing (3.4). In the same way, we can prove that both sides of (3.4) are equal to $(-1)^{j+r(n-1)/d}q^{d\binom{jn+n-r(n-1)/d}{2}}$ for $a=q^{(dj+d-r)n}$ $(j=0,1,2,\dots,m-1)$. This proves the q-congruence (3.3). $\square$

Likewise, we have the following parametric generalization of Theorem 1.2 for $n\equiv -1 \pmod d$.

Theorem 3.3. *Let d, m, r be positive integers with $r < d$. Let $n > 1$ be an odd integer satisfying $n \equiv -1 \pmod{d}$. Then, modulo*

$$\prod_{j=0}^{m-1}(1-aq^{(dj+d-r)n})(a-q^{(dj+r)n}),$$

we have

$$\sum_{k=0}^{mn-1}\frac{(aq^r;q^d)_k(a^{-1}q^{d-r};q^d)_k}{(q^d;q^d)_k^2}q^{dk} \equiv (-1)^{\langle -r/d\rangle_n}a^{(n-1)/2-\langle -r/d\rangle_n}q^{r(d-r)(n^2-1)/(2d)} \times \sum_{k=0}^{m-1}\frac{(a^{-n}q^{rn^2};q^{dn^2})_k(a^nq^{(d-r)n^2};q^{dn^2})_k}{(q^{dn^2};q^{dn^2})_k^2}q^{dn^2k}. \tag{3.5}$$

Proof. For $a = q^{(dj+r)n}$ with $0 \leqslant j \leqslant m-1$, by Lemma 3.1, the left-hand side of (3.5) is equal to

$$\sum_{k=0}^{mn-1}\frac{(q^{r+(dj+r)n};q^d)_k(q^{d-r-(dj+r)n};q^d)_k q^{dk}}{(q^d;q^d)_k^2} = (-1)^{j-1+r(n+1)/d}q^{d\binom{jn+r(n+1)/d}{2}},$$

and the right-hand side of (3.5) is equal to

$$\begin{aligned}&(-1)^{\langle -r/d\rangle_n+j}q^{(dj+r)n((n-1)/2-\langle -r/d\rangle_n)+r(d-r)(n^2-1)/(2d)+dn^2(j^2+j)/2}\\ &\quad = (-1)^{1-(n+1)r/d+j}q^{d\binom{jn+r(n+1)/d}{2}},\end{aligned}$$

where we have used the fact that $\langle -r/d\rangle_n = n - r(n+1)/d$ since $n \equiv -1 \pmod{d}$. Hence, the q-congruence (3.5) is true modulo $\prod_{j=0}^{m-1}(a-q^{(dj+r)n})$.

Similarly, for $a = q^{-(dj+d-r)n}$ with $0 \leqslant j \leqslant m-1$, the left-hand side of (3.5) is equal to

$$\sum_{k=0}^{mn-1}\frac{(q^{r-(dj+d-r)n};q^d)_k(q^{(d-r)+(dj+d-r)n};q^d)_k q^{dk}}{(q^d;q^d)_k^2} = (-1)^{j-1-r(n+1)/d}q^{d\binom{jn+n-r(n+1)/d+1}{2}},$$

which is the same as the right-hand side of (3.5). This proves that (3.5) is also true modulo $\prod_{j=0}^{m-1}(1-aq^{(dj+d-r)n})$. □

Proof of Theorem 1.2. We first consider the $n \equiv 1 \pmod{d}$ case. It is clear that the $a = 1$ case of (3.2) has the factor $\Phi_n(q)^{2m}$. On the other hand, the least common denominator of the left-hand side of (3.3) only has the factor $\Phi_n(q)^{2m-2}$ related to $\Phi_n(q)$, and so is the left-hand side of (3.3). Thus, taking $a = 1$ in (3.3), we deduce that the q-congruence (1.14) is true modulo $\Phi_n(q)^2$ for $n \equiv 1 \pmod{d}$.

Similarly, we can prove the $d = -1$ case of (1.14) by taking $a = 1$ in (3.5). □

4. Two more such q-supercongruences

Sun [29, (1.7) and (1.8)] proved that, for any odd prime p and positive integer r,

$$\sum_{k=0}^{(p^r-1)/2} \frac{1}{8^k}\binom{2k}{k} \equiv \left(\frac{2}{p^r}\right) \pmod{p^2}, \tag{4.1}$$

$$\sum_{k=0}^{(p^r-1)/2} \frac{1}{16^k}\binom{2k}{k} \equiv \left(\frac{3}{p^r}\right) \pmod{p^2}. \tag{4.2}$$

The first author and Liu [9] gave the following q-analogue of (4.1): for any positive odd integer n,

$$\sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k q^{k^2}}{(q^4;q^4)_k} \equiv (-q)^{(1-n^2)/8} \pmod{\Phi_n(q)^2}. \tag{4.3}$$

Here we give a generalization of (4.3) similar to Theorem 1.1.

Theorem 4.1. *Let m and n be positive odd integers. Then*

$$\sum_{k=0}^{(mn-1)/2} \frac{(q;q^2)_k q^{k^2}}{(q^4;q^4)_k} \equiv (-q)^{(1-n^2)/8} \sum_{k=0}^{(m-1)/2} \frac{(q^{n^2};q^{2n^2})_k q^{n^2k^2}}{(q^{4n^2};q^{4n^2})_k} \pmod{\Phi_n(q)^2}. \tag{4.4}$$

Proof. We need the following parametric version of (4.3) (see [5, (5.1)]):

$$\sum_{k=0}^{(n-1)/2} \frac{(aq;q^2)_k (a^{-1}q;q^2)_k q^{k^2}}{(q;q^2)_k (q^4;q^4)_k} \equiv (-q)^{(1-n^2)/8} \pmod{(1-aq^n)(a-q^n)}. \tag{4.5}$$

As before, we first give a parametric generalization of (4.4): modulo $\prod_{j=0}^{(m-1)/2}(1-aq^{(2j+1)n})(a-q^{(2j+1)n})$,

$$\begin{aligned}\sum_{k=0}^{(mn-1)/2} &\frac{(aq;q^2)_k (a^{-1}q;q^2)_k q^{k^2}}{(q;q^2)_k (q^4;q^4)_k} \\ &\equiv (-q)^{(1-n^2)/8} \sum_{k=0}^{(m-1)/2} \frac{(a^n q^{n^2};q^{2n^2})_k (a^{-n}q^{n^2};q^{2n^2})_k q^{n^2k^2}}{(q^{n^2};q^{2n^2})_k (q^{4n^2};q^{4n^2})_k}.\end{aligned} \tag{4.6}$$

For $a=q^{-(2j+1)n}$ or $q^{(2j+1)n}$ with $j=0,1,\dots,(m-1)/2$, the left-hand side of (4.6) is equal to

$$\sum_{k=0}^{(mn-1)/2} \frac{(q^{1-(2j+1)n};q^2)_k (q^{1+(2j+1)n};q^2)_k}{(q;q^2)_k (q^4;q^4)_k^2} q^{k^2}. \tag{4.7}$$

Note that the above sum is in fact truncated at $((2j+1)n-1)/2$. By the $a=q^{-n}$ case of (4.5) (an identity), the sum (4.7) is equal to $(-q)^{(1-(2j+1)^2n^2)/8}$. Similarly, for the same a, the right-hand side of (4.6) may be written as

$$\begin{aligned}&(-q)^{(1-n^2)/8}\sum_{k=0}^{(m-1)/2}\frac{(q^{-2jn^2};q^{2n^2})_k(q^{(2j+2)n^2};q^{2n^2})_k}{(q^{n^2};q^{2n^2})_k(q^{4n^2};q^{4n^2})_k}q^{n^2k^2}\\&\quad=(-q)^{(1-n^2)/8}(-q^{n^2})^{(1-(2j+1)^2)/8}=(-q)^{(1-(2j+1)^2n^2)/8}.\end{aligned}$$

This proves that both sides of (4.6) are equal for $a=q^{-(2j+1)n}$ and $a=q^{(2j+1)n}$ with $j=0,1,\dots,(m-1)/2$. Namely, the q-congruence (4.6) holds.

In what follows, we deduce the q-supercongruence (4.4) from (4.6). The $a=1$ case of the modulus in the congruence (4.6) has the factor $\Phi_n(q)^{m+1}$. Moreover, the least common denominators of both sides of (4.6) merely have the factor $\Phi_n(q)^{m-1}$ related to $\Phi_n(q)$. Hence, letting $a=1$ in (4.6), we conclude that (4.4) holds. □

Gu and the second author [5] also established the following q-analogue of (4.2):

$$\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_kq^{2k}}{(-q;q^2)_k(q^4;q^4)_k}\equiv\left(\frac{3}{n}\right)q^{(n^2-1)/12}\pmod{\Phi_n(q)^2},\tag{4.8}$$

Here we shall give a generalization of (4.8) as well.

Theorem 4.2. *Let m and n be positive odd integers. Then, modulo $\Phi_n(q)^2$,*

$$\sum_{k=0}^{(mn-1)/2}\frac{(q;q^2)_kq^{2k}}{(-q;q^2)_k(q^4;q^4)_k}\equiv\left(\frac{3}{n}\right)q^{(n^2-1)/12}\sum_{k=0}^{(m-1)/2}\frac{(q^{n^2};q^{2n^2})_kq^{2n^2k}}{(-q^{n^2};q^{2n^2})_k(q^{4n^2};q^{4n^2})_k}.\tag{4.9}$$

Sketch of proof. We need the following extension of (4.8) (see [5, (5.3)]):

$$\sum_{k=0}^{(n-1)/2}\frac{(aq;q^2)_k(a^{-1}q;q^2)_kq^{2k}}{(q^2;q^4)_k(q^4;q^4)_k}\equiv\left(\frac{3}{n}\right)q^{(n^2-1)/12}\pmod{(1-aq^n)(a-q^n)}.\tag{4.10}$$

Along the same lines as before, we can deduce the following congruence from (4.10): modulo $\prod_{j=0}^{(m-1)/2}(1-aq^{(2j+1)n})(a-q^{(2j+1)n})$,

$$\begin{aligned}&\sum_{k=0}^{(mn-1)/2}\frac{(aq;q^2)_k(a^{-1}q;q^2)_kq^{2k}}{(q^2;q^4)_k(q^4;q^4)_k}\\&\equiv\left(\frac{3}{n}\right)q^{(n^2-1)/12}\sum_{k=0}^{(m-1)/2}\frac{(a^nq^{n^2};q^{2n^2})_k(a^{-n}q^{n^2};q^{2n^2})_kq^{2n^2k}}{(q^{2n^2};q^{4n^2})_k(q^{4n^2};q^{4n^2})_k}.\end{aligned}\tag{4.11}$$

Finally, letting $a=1$ in (4.11), we conclude that (4.9) holds. □

5. Concluding remarks and open problems

Recall that the *q-binomial coefficient* is defined as

$$\begin{bmatrix}m\\n\end{bmatrix}_q=\begin{cases}\dfrac{(q;q)_m}{(q;q)_n(q;q)_{m-n}}, & \text{if } 0\leqslant n\leqslant m,\\ 0, & \text{otherwise.}\end{cases}$$

In 2019, the first author [6] proved the q-supercongruence: for any positive odd integer n,

$$\sum_{k=0}^{n-1} \frac{q^k}{(-q;q)_k} {2k \brack k}_q \equiv \left(\frac{-1}{n}\right) q^{(n^2-1)/4} \pmod{\Phi_n(q)^2}, \tag{5.1}$$

which was conjectured earlier by Tauraso [30] for n being a prime.

We conjecture that (5.1) has the following generalization.

Conjecture 5.1. *Let m and n be positive integers with n odd. Then*

$$\sum_{k=0}^{mn-1} \frac{q^k}{(-q;q)_k} {2k \brack k}_q \equiv \left(\frac{-1}{n}\right) q^{(n^2-1)/4} \sum_{k=0}^{m-1} \frac{q^k}{(-q^{n^2};q^{n^2})_k} {2k \brack k}_{q^{n^2}} \pmod{\Phi_n(q)^2}, \tag{5.2}$$

The first author [6] also conjectured that

$$\sum_{k=0}^{n-1} q^k {2k \brack k}_q \equiv \left(\frac{-3}{n}\right) q^{(n^2-1)/3} \pmod{\Phi_n(q)^2}, \tag{5.3}$$

which was confirmed by Liu and Petrov [20]. On the other hand, Apagodu and Zeilberger [2] conjectured that, for any prime $p>3$ and positive integer m,

$$\sum_{k=0}^{mp-1} \binom{2k}{k} \equiv \left(\frac{-3}{n}\right) \sum_{k=0}^{m-1} \binom{2k}{k} \pmod{p^2}, \tag{5.4}$$

which was later proved by Liu [16].

We believe the following common generalization of (5.3) and (5.4) should be true.

Conjecture 5.2. *Let m and n be positive integers. Then*

$$\sum_{k=0}^{mn-1} q^k {2k \brack k}_q \equiv \left(\frac{-3}{n}\right) q^{(n^2-1)/3} \sum_{k=0}^{m-1} q^{n^2 k} {2k \brack k}_{q^{n^2}} \pmod{\Phi_n(q)^2}. \tag{5.5}$$

The difficulty in proving (5.2) and (5.5) is that we do not know any parametric generalizations of (5.1) and (5.4). Both of (5.1) and (5.3) are proved by very special techniques (using curious q-identities). Note that the other two q-supercongruences in [6, Conjecture 5.1] do not have such generalizations like (5.5). We hope that an interested reader can make progress on Conjectures 5.1 and 5.2.

References

[1] S. Ahlgren and K. Ono, A Gaussian hypergeometric series evaluation and Apéry number congruences, J. Rine Angew. Math. 518 (2000), 187–212.

[2] M. Apagodu and D. Zeilberger, Using the "freshman's dream" to prove combinatorial congruences, Amer. Math. Monthly 124 (2017), 597–608.

[3] G. Gasper and M. Rahman, Basic Hypergeometric Series, second edition, Encyclopedia of Mathematics and Its Applications 96, Cambridge University Press, Cambridge, 2004.

[4] J.-P. Fang and V.J.W. Guo, Two q-congruences involving double basic hypergeometric sums, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. RACSAM 116 (2022), Art. 25.

[5] C.-Y. Gu and V.J.W. Guo, q-Analogues of two supercongruences of Z.-W. Sun, Czechoslovak Math. J. 70 (2020), 757–765.
[6] V.J.W. Guo, Proof of a q-congruence conjectured by Tauraso, Int. J. Number Theory 15 (2019), 37–41.
[7] V.J.W. Guo, Some q-congruences with parameters, Acta Arith. 190 (2019), 381–393.
[8] V.J.W. Guo, A new extension of the (A.2) supercongruence of Van Hamme, Results Math. 77 (2022), Art. 96.
[9] V.J.W. Guo and J.-C. Liu, q-Analogues of two Ramanujan-type formulas for $1/\pi$, J. Difference Equ. Appl. 24 (2018), 1368–1373.
[10] V.J.W. Guo, H. Pan and Y. Zhang, The Rodriguez-Villegas type congruences for truncated q-hypergeometric functions, J. Number Theory 174 (2017), 358–368.
[11] V.J.W. Guo and S.-D. Wang, Factors of certain sums involving central q-binomial coefficients, Rev. R. Acad. Cienc. Exactas Fś. Nat., Ser. A Mat. RACSAM 116 (2022), Art. 46.
[12] V.J.W. Guo and J. Zeng, Some q-analogues of supercongruences of Rodriguez-Villegas, J. Number Theory 145 (2014), 301–316.
[13] V.J.W. Guo and W. Zudilin, A q-microscope for supercongruences, Adv. Math. 346 (2019), 329–358.
[14] V.J.W. Guo and W. Zudilin, Dwork-type supercongruences through a creative q-microscope, J. Combin. Theory, Ser. A 178 (2021), Art. 105362.
[15] L. Li and S.-D. Wang, Proof of a q-supercongruence conjectured by Guo and Schlosser, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. RACSAM 114 (2020), Art. 190.
[16] J.-C. Liu, On two conjectural supercongruences of Apagodu and Zeilberger, J. Difference Equ. Appl. 22 (2016), 1791–1799.
[17] J.-C. Liu, Congruences for truncated hypergeometric series ${}_2F_1$, Bull. Aust. Math. Soc. 96 (2017), 14–23.
[18] J-C. Liu, On a congruence involving q-Catalan numbers, C. R. Math. Acad. Sci. Paris 358 (2020), 211–215.
[19] J.-C. Liu and X.-T. Jiang, On the divisibility of sums of even powers of q-binomial coefficients, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM 116 (2022), Art. 76.
[20] J.-C. Liu and F. Petrov, Congruences on sums of q-binomial coefficients, Adv. Appl. Math. 116 (2020), Art. 102003.
[21] E. Mortenson, A supercongruence conjecture of Rodriguez-Villegas for a certain truncated hypergeometric function, J. Number Theory 99 (2003), 139–147.
[22] E. Mortenson, Supercongruences between truncated ${}_2F_1$ hypergeometric functions and their Gaussian analogs, Trans. Amer. Math. Soc. 355 (2003), 987–1007.
[23] H.-X. Ni and H. Pan, On a conjectured q-congruence of Guo and Zeng, Int. J. Number Theory 14 (2018), 1699–1707.
[24] H.-X. Ni and H. Pan, Some symmetric q-congruences modulo the square of a cyclotomic polynomial, J. Math. Anal. Appl. 481 (2020), Art. 123372.
[25] H.-X. Ni and L.-Y. Wang, Two q-supercongruences from Watson's transformation, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM 116 (2022), Art. 30.
[26] F. Rodriguez-Villegas, Hypergeometric families of Calabi–Yau manifolds, in: Calabi–Yau Varieties and Mirror Symmetry (Toronto, ON, 2001), Fields Inst. Commun., 38, Amer. Math. Soc., Providence, RI, 2003, pp. 223–231.
[27] Z.-H. Sun, Generalized Legendre polynomials and related supercongruences, J. Number Theory 143 (2014), 293–319.
[28] Z.-W. Sun, Super congruences and Euler numbers, Sci. China Math. 54 (2011), 2509–2535.
[29] Z.-W. Sun, Fibonacci numbers modulo cubes of primes, Taiwanese J. Math. 17 (2013), 1523–1543.
[30] R. Tauraso, Some q-analogs of congruences for central binomial sums, Colloq. Math. 133 (2013), 133–143.

[31] C. Wei, Some q-supercongruences modulo the fourth power of a cyclotomic polynomial, J. Combin. Theory, Ser. A 182 (2021), Art. 105469.

[32] C. Xu and X. Wang, Proofs of Guo and Schlosser's two conjectures, Period. Math. Hungar. 85 (2022), 474–480.

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