q-Analogues of two Ramanujan-type formulas for $1/\pi$

Victor J. W. Guo$^1$ and Ji-Cai Liu$^2$

$^1$School of Mathematical Sciences, Huaiyin Normal University, Huaian, Jiangsu 223300, People’s Republic of China

jwguo@hytc.edu.cn

$^2$Department of Mathematics, Wenzhou University, Wenzhou 325035, People’s Republic of China

jcliu2016@gmail.com

Abstract. We give $q$-analogues of the following two Ramanujan-type formulas for $1/\pi$:

\[
\sum_{k=0}^{\infty} (6k + 1) \frac{(\frac{1}{2})_{k}^3}{k!^3 4^k} = \frac{4}{\pi} \quad \text{and} \quad \sum_{k=0}^{\infty} (-1)^k (6k + 1) \frac{(\frac{1}{2})_{k}^3}{k!^3 8^k} = \frac{2\sqrt{2}}{\pi}.
\]

Our proof is based on two $q$-WZ pairs found by the first author in his earlier work.

Keywords: Ramanujan; $q$-WZ pair; supercongruences; cyclotomic polynomial

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1. Introduction

In 1997, van Hamme [8] conjectured 13 Ramanujan-type $\pi$ series including

\[
\sum_{k=0}^{\infty} (6k + 1) \frac{\left(\frac{1}{2}\right)_k^3}{k!^3 4^k} = \frac{4}{\pi},
\]

\[
\sum_{k=0}^{\infty} (-1)^k (6k + 1) \frac{\left(\frac{1}{2}\right)_k^3}{k!^3 8^k} = \frac{2\sqrt{2}}{\pi},
\]

have nice $p$-adic analogues (called Ramanujan-type supercongruences). Here we use the Pochhammer symbol $(a)_k = a(a+1)\cdots(a+k-1)$. All the 13 Ramanujan-type supercongruences have now been confirmed by different authors (see [5]). Note that Ekhad and Zeilberger [1] first applied the Wilf–Zeilberger method to prove a Ramanujan-type formula for $\pi$. Recently, the first author [3,4] has formulated $q$-analogues of the (J.2) and (L.2) supercongruences of van Hamme [8], and confirmed the following special cases: for any positive odd integer $n$,

\[
\sum_{k=0}^{n-1} q^{k^2} [6k + 1] \frac{(q; q^2)_k^2 (q^2; q^4)_k}{(q^4; q^4)_k^3} \equiv [n] \left( -q \right)^{\frac{1-n}{2}} \pmod{[n] \Phi_n(q)},
\]

*Corresponding author.
and, for any odd prime power $n$,
\[
\sum_{k=0}^{n-1} (-1)^k [6k + 1] \frac{(q; q^2)^3_k}{(q^4; q^4)^3_k} \equiv [n] (-q)^{-\frac{(n-1)(n+5)}{2}} \pmod{[n] \Phi_n(q)}.
\]

Here and in what follows, the $q$-shifted factorial is defined by $(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$ for $n \geq 1$ and $(a; q)_0 = 1$, the $q$-integer is defined as $[n] = 1 + a + \cdots + a^n$, and $\Phi_n(q)$ is the $n$-th cyclotomic polynomial.

The first purpose of this paper is to prove the following $q$-analogues of (1.1) and (1.2), which were originally conjectured by the first author [3, Conjecture 4.2] and [4, Conjecture 4.5], respectively.

**Theorem 1.1.** For any complex number $q$ with $|q| < 1$, we have
\[
\sum_{k=0}^{\infty} q^{k^2} [6k + 1] \frac{(q; q^2)_k(q^2; q^4)_k}{(q^4; q^4)_k} = \frac{(1 + q)(q^2; q^4)\infty(q^6; q^4)\infty}{(q^4; q^4)_2\infty},
\]
(1.3)

\[
\sum_{k=0}^{\infty} (-1)^k q^{3k^2} [6k + 1] \frac{(q; q^2)_k^3}{(q^4; q^4)_k^3} = \frac{(q^3; q^4)\infty(q^5; q^4)\infty}{(q^4; q^4)_2\infty}.
\]
(1.4)

To see (1.3) and (1.4) are indeed $q$-analogues of (1.1) and (1.2), just notice that the $q$-Gamma function $\Gamma_q(x)$ defined by
\[
\Gamma_q(x) = \frac{(q; q)\infty}{(q^x; q)\infty} (1 - q)^{1-x}, \quad 0 < q < 1
\]
(see [2, page 20]) has the property $\lim_{q \to 1} \Gamma_q(x) = \Gamma(x)$, and moreover $\Gamma(x)\Gamma(1 - x) = \pi / \sin(\pi x)$.

Z.-W. Sun [7, (1.6)] proved that for prime $p \geq 5$,
\[
\sum_{k=0}^{p-1} \frac{2^k}{8^k} \equiv \left( \frac{2}{p} \right) + \left( \frac{-2}{p} \right) \frac{p^2}{4} E_{p-3} \pmod{p^3},
\]
(1.5)

where $\left( \frac{a}{p} \right)$ is the Legendre symbol modulo $p$ and $E_n$ is the $n$-th Euler number. The second aim of this paper is to show the following $q$-analogue of (1.5) modulo $p^2$.

**Theorem 1.2.** For any odd positive integer $n$, we have
\[
\sum_{k=0}^{n-1} q^{k^2} \frac{(q; q^2)_k}{(q^4; q^4)_k} \equiv (-q)^{\frac{1-n^2}{8}} \pmod{\Phi_n(q^2)}.
\]

We shall prove Theorem 1.1 in Section 1, and show Theorem 1.2 in Section 2.
2. Proof of Theorem 1.1

Proof of (1.3). We begin with the identity [3] (2.11):
\[
\sum_{k=0}^{n-1} q^{k^2}[6k + 1] (q; q^2)_k^2 (q^2; q^4)_k = \sum_{k=1}^{n} \frac{q^{(n-k)}(q^2; q^4)_k n(q; q^2)_{n-k}(q; q^2)_{n+k-1}}{(1 - q)(q^4; q^4)_k^2(q^4; q^4)_{n-k}(q^4; q^4)_{n-k}}.
\]
(2.1)

For the sake of completeness, we sketch the proof of [3, (2.11)] here. Let
\[
F(n, k) = \frac{q^{(n-k)^2}[6n - 2k + 1](q^2; q^4)_n(q^2; q^2)_{n-k}(q^2; q^2)_{n+k}}{(q^4; q^4)_n^2(q^4; q^4)_{n-k}(q^4; q^4)_k},
\]
\[
G(n, k) = \frac{q^{(n-k)^2}(q^2; q^4)_n(q^2; q^2)_{n-k}(q^2; q^2)_{n+k-1}}{(1 - q)(q^4; q^4)_n^2(q^4; q^4)_{n-1}(q^4; q^4)_{n-k}(q^4; q^4)_k},
\]
where \(1/(q^4; q^4)_m = 0\) for any negative integer \(m\). Then
\[
F(n, k - 1) - F(n, k) = G(n + 1, k) - G(n, k).
\]
(2.2)

Namely, the functions \(F(n, k)\) and \(G(n, k)\) form a \(q\)-WZ pair. Moreover, the identity (2.1) is equivalent to
\[
\sum_{n=0}^{m-1} F(n, 0) = \sum_{k=1}^{m} G(m, k),
\]
which follows from (2.2) by first summing over \(n = 0, 1, \ldots, m - 1\) and then summing over \(k\) from 1 to \(m - 1\).

Letting \(k \to n - k\) on the right-hand side of (2.1), we obtain
\[
\sum_{k=0}^{n-1} q^{k^2}[6k + 1] (q; q^2)_k^2 (q^2; q^4)_k = \sum_{k=1}^{n-1} \frac{q^{k^2}(q^2; q^4)_n(q^2; q^2)_{2n-k-1}}{(1 - q)(q^4; q^4)_k^2(q^4; q^4)_{n-k}(q^4; q^4)_{n-k}}.
\]
(2.3)

Furthermore, letting \(n \to \infty\) on both sides of (2.3), we are led to
\[
\sum_{k=0}^{\infty} q^{k^2}[6k + 1] (q; q^2)_k^2 (q^2; q^4)_k = \frac{(q; q^2)_\infty}{(q^4; q^4)_\infty^2} \sum_{k=0}^{\infty} \frac{q^{k^2}(q; q^2)_k}{(1 - q)(q^4; q^4)_k}.
\]
(2.4)

Replacing \(q\) by \(-q\) in Slater’s identity [6] (4), we have
\[
\sum_{k=0}^{\infty} q^{k^2} (q; q^2)_{k-1}^2 (q^2; q^4)_{k-1} = \frac{(q^2; q^4)_\infty^2}{(q; q^2)_\infty}.
\]
(2.5)

The proof of (1.3) then follows from (2.4) and (2.5). \(\Box\)
Proof of (1.4). We start with the identity \(4, (2.14)\)

\[
\sum_{k=0}^{n-1} (-1)^k [6k + 1] \frac{(q; q^2)_k^3}{(q^4; q^4)_k^3} = \sum_{k=1}^{n} \frac{(-1)^{n+k} (q; q^2)_{n+k-1}(q; q^2)_{n-k}^2}{(1 - q)(q^4; q^4)_{n-1}(q^4; q^4)_{n-k}},
\]

(2.6)
of which the proof is exactly the same as that of (2.1). The \(q\)-WZ pair this time is

\[
F(n, k) = \frac{(-1)^{n+k}[6n - 2k + 1](q; q^2)_{n+k}(q; q^2)_{n-k}^2}{(q^4; q^4)_n^2(q^4; q^4)_{n-k}},
\]

\[
G(n, k) = \frac{(-1)^{n+k}(q; q^2)_{n+k-1}(q; q^2)_{n-k}^2}{(1 - q)(q^4; q^4)_{n-1}(q^4; q^4)_{n-k}}.
\]

Replacing \(q\) by \(q^{-1}\) in (2.6) and noticing that \((q^{-1}; q^{-2})_k^2 = (-1)^k q^{k^2}(q; q^2)_k\) and \((q^{-4}; q^{-4})_k = (-1)^k q^{4^{k+1}}(q; q^4)_k\), we obtain

\[
\sum_{k=0}^{n-1} (-1)^k q^{3k^2} [6k + 1] \frac{(q; q^2)_k^3}{(q^4; q^4)_k^3} = \sum_{k=1}^{n} \frac{(-1)^{n+k} q^{(3n+k)(n-k)}(q; q^2)_{n+k-1}(q; q^2)_{n-k}^2}{(1 - q)(q^4; q^4)_{n-1}(q^4; q^4)_{n-k}}
\]

\[
= \sum_{k=0}^{n-1} \frac{(-1)^k q^{4n-k} k (q; q^2)_{2n-k-1}(q; q^2)_{k}^2}{(1 - q)(q^4; q^4)_{n-1}(q^4; q^4)_{k}},
\]

(2.7)
where the second equality follows from reversing the summation order. Finally, letting \(n \to \infty\) on both sides of (2.7), we get

\[
\sum_{k=0}^{\infty} (-1)^k q^{3k^2} [6k + 1] \frac{(q; q^2)_k^3}{(q^4; q^4)_k^3} = \frac{(q; q^2)_{\infty}}{(1 - q)(q^4; q^4)_{\infty}^2},
\]

since all the summands on the right-hand side of (2.7) except for the first one \((k = 0)\) vanish as \(n \to \infty\). The proof of (1.4) then follows from the obvious fact \((q; q^2)_{\infty}/(1 - q) = (q^3; q^4)_{\infty}(q^5; q^4)_{\infty}\). \(\square\)

3. Proof of Theorem 1.2

Since

\[
(1 - q^{n-2j+1})(1 - q^{n+2j-1}) + (1 - q^{2j-1})^2 q^{n-2j+1} = (1 - q^n)^2
\]

and \(1 - q^n \equiv 0 \pmod{\Phi_n(q)}\), we have

\[
(1 - q^{n-2j+1})(1 - q^{n+2j-1}) \equiv -(1 - q^{2j-1})^2 q^{n-2j+1} \pmod{\Phi_n(q^2)}.
\]
Therefore,
\[
(-1)^k q^{nk-k^2} (q^{1-n}; q^2)_k (q^{n+1}; q^2)_k = \prod_{j=1}^{k} (1 - q^{n-2j+1})(1 - q^{n+2j-1})
\]
\[
\equiv (-1)^k \prod_{j=1}^{k} (1 - q^{2j-1})^2 q^{n-2j+1}
\]
\[
= (-1)^k q^{nk-k^2} (q; q^2)_k^2 \pmod{\Phi_n(q^2)}.
\]

It follows that
\[
\sum_{k=0}^{n-1} q^{k^2} (q; q^2)_k (q^4; q^4)_k \equiv \sum_{k=0}^{n-1} q^{k^2} (q^{1-n}; q^2)_k (q^{n+1}; q^2)_k (q^4; q^4)_k = (-q)^{\frac{1-n^2}{2}} \pmod{\Phi_n(q^2)}. \tag{3.1}
\]

The last identity in (3.1) just follows from a terminating $q$-analogue of Whipple’s $3F_2$ sum [2, Appendix (II.19), page 355]:
\[
\phi_3 \left[ q^{-n}, q^{n+1}, c, -c \left/ e, c^2 q/e, -q \right; q, q \right] = \frac{(eq^{-n}, eq^{n+1}, c^2 q^{1-n}/e, c^2 q^{n+2}/e; q^2)_\infty q^{n(n+1)}}{(e, c^2 q; q^2)_\infty q^{n(n+1)/2}}
\]

with $n \to \frac{n-1}{2}$, $q \to q^2$, $c \to \infty$ and $e \to q$.

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**References**


