q-Analogues of two Ramanujan-type formulas for $1/\pi$

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Abstract. We give q-analogues of the following two Ramanujan-type formulas for $1/\pi$:

$$\sum_{k=0}^{\infty}(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 4^k}=\frac{4}{\pi} \quad \text{and} \quad \sum_{k=0}^{\infty}(-1)^k(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 8^k}=\frac{2\sqrt{2}}{\pi}.$$

Our proof is based on two q-WZ pairs found by the first author in his earlier work.

Keywords: Ramanujan; q-WZ pair; supercongruences; cyclotomic polynomial

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1. Introduction

In 1997, van Hamme [8] conjectured 13 Ramanujan-type π series including

$$\sum_{k=0}^{\infty}(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 4^k}=\frac{4}{\pi}, \tag{1.1}$$

$$\sum_{k=0}^{\infty}(-1)^k(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 8^k}=\frac{2\sqrt{2}}{\pi}, \tag{1.2}$$

have nice p-adic analogues (called Ramnujan-type supercongruences). Here we use the Pochhammer symbol $(a)_k=a(a+1)\cdots(a+k-1)$. All the 13 Ramanujan-type supercongruences have now been confirmed by different authors (see [5]). Note that Ekhad and Zeilberger [1] first applied the Wilf–Zeilberger method to prove a Ramanujan-type formula for π. Recently, the first author [3,4] has formulated q-analogues of the (J.2) and (L.2) supercongruences of van Hamme [8], and confirmed the following special cases: for any positive odd integer n,

$$\sum_{k=0}^{n-1} q^{k^2}[6k+1]\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^4;q^4)_k^3} \equiv [n](-q)^{\frac{1-n}{2}} \pmod{[n]\Phi_n(q)},$$

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and, for any odd prime power n,

$$\sum_{k=0}^{n-1}(-1)^k[6k+1]\frac{(q;q^2)_k^3}{(q^4;q^4)_k^3} \equiv [n](-q)^{-\frac{(n-1)(n+5)}{8}} \pmod{[n]\Phi_n(q)}.$$

Here and in what follows, the *q-shifted factorial* is defined by $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n \geqslant 1$ and $(a;q)_0 = 1$, the *q-integer* is defined as $[n] = 1+q+\cdots+q^{n-1}$, and $\Phi_n(q)$ is the n-th *cyclotomic polynomial*.

The first purpose of this paper is to prove the following q-analogues of (1.1) and (1.2), which were originally conjectured by the first author [3, Conjecture 4.2] and [4, Conjecture 4.5], respectively.

Theorem 1.1. *For any complex number q with $|q| < 1$, we have*

$$\sum_{k=0}^{\infty} q^{k^2}[6k+1]\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^4;q^4)_k^3} = \frac{(1+q)(q^2;q^4)_\infty(q^6;q^4)_\infty}{(q^4;q^4)_\infty^2}, \tag{1.3}$$

$$\sum_{k=0}^{\infty}(-1)^k q^{3k^2}[6k+1]\frac{(q;q^2)_k^3}{(q^4;q^4)_k^3} = \frac{(q^3;q^4)_\infty(q^5;q^4)_\infty}{(q^4;q^4)_\infty^2}. \tag{1.4}$$

To see (1.3) and (1.4) are indeed q-analogues of (1.1) and (1.2), just notice that the q-Gamma function $\Gamma_q(x)$ defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad 0 < q < 1$$

(see [2, page 20]) has the property $\lim_{q\to 1^-}\Gamma_q(x) = \Gamma(x)$, and moreover $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$.

Z.-W. Sun [7, (1.6)] proved that for prime $p \geq 5$,

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}}{8^k} \equiv \left(\frac{2}{p}\right) + \left(\frac{-2}{p}\right)\frac{p^2}{4}E_{p-3} \pmod{p^3}, \tag{1.5}$$

where $\left(\frac{a}{p}\right)$ is the Legendre symbol modulo p and E_n is the n-th Euler number. The second aim of this paper is to show the following q-analogue of (1.5) modulo p^2.

Theorem 1.2. *For any odd positive integer n, we have*

$$\sum_{k=0}^{\frac{n-1}{2}} q^{k^2}\frac{(q;q^2)_k}{(q^4;q^4)_k} \equiv (-q)^{\frac{1-n^2}{8}} \pmod{\Phi_n(q)^2}.$$

We shall prove Theorem 1.1 in Section 1, and show Theorem 1.2 in Section 2.

2. Proof of Theorem 1.1

Proof of (1.3). We begin with the identity [3, (2.11)]:

$$\sum_{k=0}^{n-1} q^{k^2}[6k+1]\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^4;q^4)_k^3} = \sum_{k=1}^{n} \frac{q^{(n-k)^2}(q^2;q^4)_n(q;q^2)_{n-k}(q;q^2)_{n+k-1}}{(1-q)(q^4;q^4)_{n-1}^2(q^4;q^4)_{n-k}(q^2;q^4)_k}. \tag{2.1}$$

For the sake of completeness, we sketch the proof of [3, (2.11)] here. Let

$$F(n,k) = \frac{q^{(n-k)^2}[6n-2k+1](q^2;q^4)_n(q;q^2)_{n-k}(q;q^2)_{n+k}}{(q^4;q^4)_n^2(q^4;q^4)_{n-k}(q^2;q^4)_k},$$

$$G(n,k) = \frac{q^{(n-k)^2}(q^2;q^4)_n(q;q^2)_{n-k}(q;q^2)_{n+k-1}}{(1-q)(q^4;q^4)_{n-1}^2(q^4;q^4)_{n-k}(q^2;q^4)_k},$$

where $1/(q^4;q^4)_m = 0$ for any negative integer m. Then

$$F(n,k-1) - F(n,k) = G(n+1,k) - G(n,k). \tag{2.2}$$

Namely, the functions $F(n,k)$ and $G(n,k)$ form a q-WZ pair. Moreover, the identity (2.1) is equivalent to

$$\sum_{n=0}^{m-1} F(n,0) = \sum_{k=1}^{m} G\left(m,k\right),$$

which follows from (2.2) by first summing over $n = 0, 1, \ldots, m-1$ and then summing over k from 1 to $m-1$.

Letting $k \to n-k$ on the right-hand side of (2.1), we obtain

$$\sum_{k=0}^{n-1} q^{k^2}[6k+1]\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^4;q^4)_k^3} = \sum_{k=0}^{n-1} \frac{q^{k^2}(q^2;q^4)_n(q;q^2)_k(q;q^2)_{2n-k-1}}{(1-q)(q^4;q^4)_{n-1}^2(q^4;q^4)_k(q^2;q^4)_{n-k}}. \tag{2.3}$$

Furthermore, letting $n \to \infty$ on both sides of (2.3), we are led to

$$\sum_{k=0}^{\infty} q^{k^2}[6k+1]\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^4;q^4)_k^3} = \frac{(q;q^2)_\infty}{(q^4;q^4)_\infty^2}\sum_{k=0}^{\infty} \frac{q^{k^2}(q;q^2)_k}{(1-q)(q^4;q^4)_k}. \tag{2.4}$$

Replacing q by $-q$ in Slater's identity [6, (4)], we have

$$\sum_{k=0}^{\infty} q^{k^2}\frac{(q;q^2)_k}{(q^4;q^4)_k} = \frac{(q^2;q^4)_\infty^2}{(q;q^2)_\infty}. \tag{2.5}$$

The proof of (1.3) then follows from (2.4) and (2.5). $\square$

Proof of (1.4). We start with the identity [4, (2.14)]

$$\sum_{k=0}^{n-1}(-1)^k[6k+1]\frac{(q;q^2)_k^3}{(q^4;q^4)_k^3}=\sum_{k=1}^{n}\frac{(-1)^{n+k}(q;q^2)_{n+k-1}(q;q^2)_{n-k}^2}{(1-q)(q^4;q^4)_{n-1}^2(q^4;q^4)_{n-k}}, \tag{2.6}$$

of which the proof is exactly the same as that of (2.1). The q-WZ pair this time is

$$F(n,k)=\frac{(-1)^{n+k}[6n-2k+1](q;q^2)_{n+k}(q;q^2)_{n-k}^2}{(q^4;q^4)_n^2(q^4;q^4)_{n-k}},$$
$$G(n,k)=\frac{(-1)^{n+k}(q;q^2)_{n+k-1}(q;q^2)_{n-k}^2}{(1-q)(q^4;q^4)_{n-1}^2(q^4;q^4)_{n-k}}.$$

Replacing q by q^{-1} in (2.6) and noticing that $(q^{-1};q^{-2})_k^2=(-1)^kq^{k^2}(q;q^2)_k$ and $(q^{-4};q^{-4})_k=(-1)^kq^{4\binom{k+1}{2}}(q^4;q^4)_k$, we obtain

$$\begin{aligned}\sum_{k=0}^{n-1}(-1)^kq^{3k^2}[6k+1]\frac{(q;q^2)_k^3}{(q^4;q^4)_k^3}&=\sum_{k=1}^{n}\frac{(-1)^{n+k}q^{(3n+k)(n-k)}(q;q^2)_{n+k-1}(q;q^2)_{n-k}^2}{(1-q)(q^4;q^4)_{n-1}^2(q^4;q^4)_{n-k}}\\&=\sum_{k=0}^{n-1}\frac{(-1)^kq^{(4n-k)k}(q;q^2)_{2n-k-1}(q;q^2)_k^2}{(1-q)(q^4;q^4)_{n-1}^2(q^4;q^4)_k},\end{aligned} \tag{2.7}$$

where the second equality follows from reversing the summation order. Finally, letting $n\to\infty$ on both sides of (2.7), we get

$$\sum_{k=0}^{\infty}(-1)^kq^{3k^2}[6k+1]\frac{(q;q^2)_k^3}{(q^4;q^4)_k^3}=\frac{(q;q^2)_\infty}{(1-q)(q^4;q^4)_\infty^2},$$

since all the summands on the right-hand side of (2.7) except for the first one ($k=0$) vanish as $n\to\infty$. The proof of (1.4) then follows from the obvious fact $(q;q^2)_\infty/(1-q)=(q^3;q^4)_\infty(q^5;q^4)_\infty$. □

3. Proof of Theorem 1.2

Since

$$(1-q^{n-2j+1})(1-q^{n+2j-1})+(1-q^{2j-1})^2q^{n-2j+1}=(1-q^n)^2$$

and $1-q^n\equiv 0 \pmod{\Phi_n(q)}$, we have

$$(1-q^{n-2j+1})(1-q^{n+2j-1})\equiv-(1-q^{2j-1})^2q^{n-2j+1}\pmod{\Phi_n(q)^2}.$$

Therefore,

$$
\begin{aligned}
(-1)^k q^{nk-k^2}(q^{1-n};q^2)_k(q^{n+1};q^2)_k &= \prod_{j=1}^{k}(1-q^{n-2j+1})(1-q^{n+2j-1}) \\
&\equiv (-1)^k \prod_{j=1}^{k}(1-q^{2j-1})^2 q^{n-2j+1} \\
&= (-1)^k q^{nk-k^2}(q;q^2)_k^2 \pmod{\Phi_n(q)^2}.
\end{aligned}
$$

It follows that

$$
\sum_{k=0}^{\frac{n-1}{2}} q^{k^2}\frac{(q;q^2)_k}{(q^4;q^4)_k} \equiv \sum_{k=0}^{\frac{n-1}{2}} q^{k^2}\frac{(q^{1-n};q^2)_k(q^{n+1};q^2)_k}{(q;q^2)_k(q^4;q^4)_k} = (-q)^{\frac{1-n^2}{8}} \pmod{\Phi_n(q)^2}. \tag{3.1}
$$

The last identity in (3.1) just follows from a terminating q-analogue of Whipple's ${}_3F_2$ sum [2, Appendix (II.19), page 355]:

$$
{}_4\phi_3\left[\begin{matrix} q^{-n},\ q^{n+1},\ c,\ -c \\ e,\ c^2q/e,\ -q \end{matrix};q,\ q\right] = \frac{(eq^{-n}, eq^{n+1}, c^2q^{1-n}/e, c^2q^{n+2}/e;q^2)_\infty}{(e, c^2q;q)_\infty} q^{\frac{n(n+1)}{2}}
$$

with $n \to \frac{n-1}{2}$, $q \to q^2$, $c \to \infty$ and $e \to q$.

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