A generalization of a q-congruence of Liu and Wang

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Abstract. In 2017, He proved that, for primes $p \equiv 1 \pmod 4$,

$$\sum_{k=0}^{p-1}(6k+1)\frac{(\frac{1}{2})_k^3(\frac{1}{4})_k}{k!^4 4^k} \equiv (-1)^{(p+3)/4} p\Gamma_p(\tfrac{1}{2})\Gamma_p(\tfrac{1}{4})^2 \pmod{p^2},$$

where $(x)_n = \Gamma(x+n)/\Gamma(x)$ is the Pochhammer symbol and $\Gamma_p(x)$ is the p-adic Gamma function. Liu proved that the above congruence is true modulo p^3. Liu and Wang gave a q-analogue Liu's congruence. In this note, we give a further generalization of Liu and Wang's q-congruence.

Keywords: q-congruences; p-adic Gamma function; Rahman's transformation; creative microscoping

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1. Introduction

In 2017, He [5] proved the following congruence: for any prime $p \equiv 1 \pmod 4$,

$$\sum_{k=0}^{p-1}(6k+1)\frac{(\frac{1}{2})_k^3(\frac{1}{4})_k}{k!^4 4^k} \equiv (-1)^{(p+3)/4} p\Gamma_p(\tfrac{1}{2})\Gamma_p(\tfrac{1}{4})^2 \pmod{p^2}. \tag{1.1}$$

Here and in what follows, $(x)_n = \Gamma(x+n)/\Gamma(x)$ denotes the Pochhammer symbol also for n not being a non-negative integer, and $\Gamma_p(x)$ is the p-adic Gamma function (see [8]). Note that the sign $(-1)^{(p+3)/4}$ was lost in He's paper. Liu [6] further proved that (1.1) also holds modulo p^3.

Applying the 'creative microscoping' method introduced by the second author and Zudilin [4], Liu and Wang [7] gave a q-analogue of (1.1) modulo p^3: for positive integers $n \equiv 1 \pmod 4$, modulo $[n]\Phi_n(q)^2$,

$$\sum_{k=0}^{n-1}[6k+1]\frac{(q;q^2)_k^3(q;q^4)_k}{(q^2;q^2)_k(q^4;q^4)_k^3}q^{k^2+k} \equiv \frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}[n]q^{(1-n)/4}. \tag{1.2}$$

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Here we need to recall the standard q-notation. The *q-shifted factorial* is defined by

$$(a;q)_n = \begin{cases} (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if } n=1,2,\dots, \\ 1, & \text{if } n=0, \\ \dfrac{1}{(1-aq^{-1})(1-aq^{-2})\cdots(1-aq^{n})}, & \text{if } n=-1,-2,\dots \end{cases}$$

For simplicity, we will often use the condensed notation

$$(a_1,a_2,\dots,a_m;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n$$

for $n=0,\pm1,\pm2,\dots$, or $n=\infty$. The *q-integer* is defined as $[n]=(1-q^n)/(1-q)$, and $\Phi_n(q)$ represents the n-th *cyclotomic polynomial*. Namely,

$$\Phi_n(q) = \prod_{\substack{1\leqslant k\leqslant n \\ \gcd(k,n)=1}} (q-\zeta^k),$$

where ζ is an n-th primitive root of unity. For more results on q-congruences, see [2–4, 9–12].

In this note, we shall give a generalization of (1.2) modulo $\Phi_n(q)^3$ as follows.

Theorem 1.1. *Let $n>1$ be an integer with $n\equiv 1 \pmod 4$. Let s be a non-negative integer with $s\leqslant (n-1)/2$. Then, modulo $\Phi_n(q)^3$,*

$$\sum_{k=s}^{n-s-1} [6k+1]\frac{(q;q^2)_{k-2s}(q;q^2)_{k+2s}(q;q^2)_k(q;q^4)_k}{(q^2;q^2)_k(q^4;q^4)_{k-s}(q^4;q^4)_{k+s}(q^4;q^4)_k} q^{k^2+k}$$
$$\equiv \begin{cases} \dfrac{[6s+1](q;q^2)_{3s}(q;q^4)_s(q^{5+6s};q^4)_{(n-1-2s)/4}(q^{3-n};q^4)_{(n-1+2s)/4}q^{2s^2+s}}{(q^2;q^2)_s(q^3,q^4,q^4;q^4)_s(q^{4+2s};q^4)_{(n-1-2s)/4}(q^{4+4s-n};q^4)_{(n-1+2s)/4}}, & \text{if } s \text{ is even}, \\ -\dfrac{[6s+1](q;q^2)_{3s}(q;q^4)_s(q^{3+6s};q^4)_{(n+1-2s)/4}(q^{3-n};q^4)_{(n+1+2s)/4}q^{2s^2+s}}{(q^2;q^2)_s(q^3,q^4,q^4;q^4)_s(q^{2+2s};q^4)_{(n+1-2s)/4}(q^{4+4s-n};q^4)_{(n+1+2s)/4}}, & \text{otherwise}. \end{cases} \tag{1.3}$$

It is easy to see that, for $n\equiv 1 \pmod 4$,

$$\frac{(q^5;q^4)_{(n-1)/4}(q^{3-n};q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}(q^{4-n};q^4)_{(n-1)/4}} = \frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}[n]q^{(1-n)/4},$$

and so the q-congruence (1.3) reduces to the modulus $\Phi_n(q)^3$ case of (1.2). Moreover, for n prime, taking the limits as $q\to 1$ in (1.3), we get the congruence: for any prime $p\equiv 1$

(mod 4), and non-negative integer $s \leqslant (p-1)/2$, modulo p^3,

$$\sum_{k=s}^{p-s-1}(6k+1)\frac{(\frac{1}{2})_{k-2s}(\frac{1}{2})_{k+2s}(\frac{1}{2})_k(\frac{1}{4})_k}{(k-s)!(k+s)!k!^2 4^k}$$
$$\equiv \begin{cases} \dfrac{(6s+1)(\frac{1}{2})_{3s}(\frac{1}{4})_s(\frac{5+6s}{4})_{(p-1-2s)/4}(\frac{3-p}{4})_{(p-1+2s)/4}}{4^s(1)_s^3(\frac{3}{4})_s(1+\frac{s}{2})_{(p-1-2s)/4}(\frac{4+4s-p}{4})_{(p-1+2s)/4}}, & \text{if } s \text{ is even,} \\ -\dfrac{(6s+1)(\frac{1}{2})_{3s}(\frac{1}{4})_s(\frac{3+6s}{4})_{(p+1-2s)/4}(\frac{3-p}{4})_{(p+1+2s)/4}}{4^s(1)_s^3(\frac{3}{4})_s(\frac{1+s}{2})_{(p+1-2s)/4}(\frac{4+4s-p}{4})_{(p+1+2s)/4}}, & \text{otherwise.} \end{cases}$$

Summation and transformation formulas for basic hypergeometric series are very useful in the investigation of q-congruences. Here we would like to mention Gasper and Rahman's quadratic summation (see [1, (3.8.12)]), which may be written as

$$\sum_{k=0}^{\infty}\frac{1-aq^{3k}}{1-a}\frac{(a,b,q/b;q)_k(d,f,a^2q/df;q^2)_k}{(aq/d,aq/f,df/a;q)_k(q^2,aq^2/b,abq;q^2)_k}q^k$$
$$+\frac{(aq,f/a,b,q/b;q)_\infty(d,aq^2/df,fq^2/d,df^2q/a^2;q^2)_\infty}{(a/f,fq/a,aq/d,df/a;q)_\infty(aq^2/b,abq,fq/ab,bf/a;q^2)_\infty}$$
$$\times\, {}_3\phi_2\left[\begin{matrix} f,bf/a,fq/ab \\ fq^2/d,df^2q/a^2 \end{matrix};q^2,q^2\right]$$
$$=\frac{(aq,f/a;q)_\infty(aq^2/bd,abq/d,bdf/a,dfq/ab;q^2)_\infty}{(aq/d,df/a;q)_\infty(aq^2/b,abq,bf/a,fq/ab;q^2)_\infty}, \tag{1.4}$$

where the *basic hypergeometric series* ${}_{r+1}\phi_r$ is defined by

$${}_{r+1}\phi_r\left[\begin{matrix} a_1,a_2,\dots,a_{r+1} \\ b_1,b_2,\dots,b_r \end{matrix};q,\ z\right]=\sum_{k=0}^{\infty}\frac{(a_1,a_2,\dots,a_{r+1};q)_k z^k}{(q,b_1,\dots,b_r;q)_k}.$$

We shall prove Theorem 1.1 by employing the method of 'creative microscoping' and Gasper and Rahman's summation (1.4).

2. Proof of Theorem 1.1

We require the following three lemmas.

Lemma 2.1. *Let $n>1$ be an odd integer, and let s be a non-negative integer with $s \leqslant (n-1)/2$. Then*

$$\sum_{k=s}^{n-s-1}\frac{1-q^{1+6k-n}}{1-q^{1-n}}\frac{(aq;q^2)_{k-2s}(q^{1-n};q^2)_{k+2s}(q/a;q^2)_k(q/b;q^4)_k}{(bq^2;q^2)_k(q^4;q^4)_{k-s}(q^{4-n}/a;q^4)_{k+s}(aq^{4-n};q^4)_k}(bq^{-n})^k q^{k^2+k}=0. \tag{2.1}$$

Proof. It is easy to see that the left-hand side of (2.1) can be written as

$$\begin{aligned}
&\sum_{k=0}^{n-2s-1}\frac{(1-q^{1+6k+6s-n})(aq;q^2)_{k-s}(q^{1-n};q^2)_{k+3s}(q/a;q^2)_{k+s}(q/b;q^4)_{k+s}}{(1-q^{1-n})(bq^2;q^2)_{k+s}(q^4;q^4)_k(q^{4-n}/a;q^4)_{k+2s}(aq^{4-n};q^4)_{k+s}}(bq^{-n})^{k+s}q^{(k+s)^2+k+s}\\
&=\frac{(aq;q^2)_{-s}(q^{1-n};q^2)_{3s}(q/a;q^2)_s(q/b;q^4)_s}{(bq^2;q^2)_s(q^{4-n}/a;q^4)_{2s}(aq^{4-n};q^4)_s}(bq^{-n})^s q^{s^2+s}\\
&\quad\times\sum_{k=0}^{n-2s-1}\frac{(1-q^{1+6k+6s-n})(aq^{1-2s},q^{1+6s-n},q^{1+2s}/a;q^2)_k(q^{1+4s}/b;q^4)_k}{(1-q^{1-n})(bq^{2+2s};q^2)_k(q^4,q^{4+8s-n}/a,aq^{4+4s-n};q^4)_k}(bq^{-n})^k q^{k^2+2sk+k}.
\end{aligned}\tag{2.2}$$

If $s\geqslant(n-1)/6$, then $(q^{1-n};q^2)_{3s}=0$ or $(1-q^{1+6k+6s-n})(q^{1+6s-n};q^2)_k=0$, and so the right-hand side of (2.2) vanishes. We now assume that $0\leqslant s<(n-1)/6$. Putting $d=q^{-2n}$ and then taking $n\to\infty$ in (1.4), we get

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{1-aq^{3k}}{1-a}\frac{(a,b,q/b;q)_k(f;q^2)_k}{(q^2,aq^2/b,abq;q^2)_k(aq/f;q)_k}\left(\frac{a}{f}\right)^k q^{\binom{k+1}{2}}\\
&\quad=\frac{(aq,aq^2,aq^2/bf,abq/f;q^2)_\infty}{(aq/f,aq^2/f,aq^2/b,abq;q^2)_\infty}.
\end{aligned}\tag{2.3}$$

Then, performing the parameter substitutions $q\mapsto q^2$, $a\mapsto q^{1+6s-n}$, $b\mapsto aq^{1-2s}$, and $f\mapsto q^{1+4s}/b$ in the above identity yields that

$$\sum_{k=0}^{(n-1)/6-s}\frac{(1-q^{1+6k+6s-n})(aq^{1-2s},q^{1+6s-n},q^{1+2s}/a;q^2)_k(q^{1+2s}/b;q^4)_k}{(1-q^{1+6s-n})(bq^{2+2s};q^2)_k(q^4,q^{4+8s-n}/a,aq^{4+4s-n};q^4)_k}(bq^{2s-n})^k q^{k^2+k}=0,$$

where we have used the fact that $(q^{1+6s-n};q^2)_k=0$ for $k>(n-1)/6-s$. Thus, the right-hand side of (2.2) vanishes. □

Lemma 2.2. *Let $n>1$ be an integer with $n\equiv 1 \pmod 4$, and let s be a non-negative integer with $s\leqslant(n-1)/2$. Then, modulo $a-q^n$,*

$$\begin{aligned}
&\sum_{k=s}^{n-s-1}[6k+1]\frac{(aq;q^2)_{k-2s}(q;q^2)_{k+2s}(q/a;q^2)_k(q/b;q^4)_k}{(bq^2;q^2)_k(q^4;q^4)_{k-s}(q^4/a;q^4)_{k+s}(aq^4;q^4)_k}b^k q^{k^2+k}\\
&=[6s+1]\frac{(aq;q^2)_{-s}(q;q^2)_{3s}(q/a;q^2)_s(q/b;q^4)_s}{(bq^2;q^2)_s(q^4/a;q^4)_{2s}(aq^4;q^4)_s}b^s q^{s^2+s}\\
&\quad\times\begin{cases}\dfrac{(q^{5+6s},bq^{3+4s-n};q^4)_{(n-1-2s)/4}}{(bq^{4+2s},q^{4+8s-n};q^4)_{(n-1-2s)/4}}, & \text{if } s \text{ is even},\\[2ex]
\dfrac{(q^{3+6s},bq^{3+4s-n};q^4)_{(n+1-2s)/4}}{(bq^{2+2s},q^{4+8s-n};q^4)_{(n+1-2s)/4}}, & \text{otherwise}.\end{cases}
\end{aligned}\tag{2.4}$$

Proof. For $a = q^n$, the left-hand side of (2.4) can be written as

$$\sum_{k=s}^{n-s-1} [6k+1] \frac{(q^{1+n};q^2)_{k-2s}(q;q^2)_{k+2s}(q^{1-n};q^2)_k(q/b;q^4)_k}{(bq^2;q^2)_k(q^4;q^4)_{k-s}(q^{4-n};q^4)_{k+s}(q^{4+n};q^4)_k} b^k q^{k^2+k}$$
$$= \frac{(q^{1+n};q^2)_{-s}(q;q^2)_{3s}(q^{1-n};q^2)_s(q/b;q^4)_s}{(bq^2;q^2)_s(q^{4-n};q^4)_{2s}(q^{4+n};q^4)_s} b^s q^{s^2+s}$$
$$\times \sum_{k=0}^{n-2s-1} [6k+6s+1] \frac{(q^{1-2s+n}, q^{1+6s}, q^{1+2s-n};q^2)_k (q^{1+4s}/b;q^4)_k}{(bq^{2+2s};q^2)_k(q^4, q^{4+8s-n}, q^{4+4s+n};q^4)_k} b^k q^{k^2+2sk+k}. \quad (2.5)$$

Putting $d = q^{-2n}$ and then taking $n \to \infty$ in (1.4), we get

$$\sum_{k=0}^{\infty} \frac{1-aq^{3k}}{1-a} \frac{(a,b,q/b;q)_k (f;q^2)_k}{(q^2, aq^2/b, abq;q^2)_k (aq/f;q)_k} \left(\frac{a}{f}\right)^k q^{\binom{k+1}{2}}$$
$$= \frac{(aq, aq^2, aq^2/bf, abq/f;q^2)_\infty}{(aq/f, aq^2/f, aq^2/b, abq;q^2)_\infty}. \quad (2.6)$$

Making the parameter substitutions $q \mapsto q^2$, $a \mapsto q^{1+6s}$, $b \mapsto q^{1-2s+n}$, and $f \mapsto q^{1+4s}/b$ in (2.6), we arrive at

$$\sum_{k=0}^{n-2s-1} \frac{[6k+6s+1](q^{1-2s+n}, q^{1+6s}, q^{1+2s-n};q^2)_k (q^{1+4s}/b;q^4)_k}{[6s+1](bq^{2+2s};q^2)_k (q^4, q^{4+8s-n}, q^{4+4s+n};q^4)_k} b^k q^{k^2+2sk+k}$$
$$= \frac{(q^{3+6s}, q^{5+6s}, bq^{3+4s-n}, bq^{3+n};q^4)_\infty}{(bq^{2+2s}, bq^{4+2s}, q^{4+8s-n}, q^{4+4s+n};q^4)_\infty}$$
$$= \begin{cases} \dfrac{(q^{5+6s}, bq^{3+4s-n};q^4)_{(n-1-2s)/4}}{(bq^{4+2s}, q^{4+8s-n};q^4)_{(n-1-2s)/4}}, & \text{if } s \text{ is even,} \\[2ex] \dfrac{(q^{3+6s}, bq^{3+4s-n};q^4)_{(n+1-2s)/4}}{(bq^{2+2s}, q^{4+8s-n};q^4)_{(n+1-2s)/4}}, & \text{otherwise,} \end{cases}$$

where we have utilized $(q^{1+2s-n};q^2)_k = 0$ for $k > (n-1)/2 - s$. Substituting the above identity into (2.5), we conclude that both sides of (2.4) are equal for $a = q^n$. That is, the q-congruence (2.4) holds. □

Lemma 2.3. *Let $n > 1$ be an integer with $n \equiv 1 \pmod 4$, and let s be a non-negative integer with $s \leqslant (n-1)/2$. Then, modulo $b - q^n$,*

$$\sum_{k=s}^{n-s-1} [6k+1] \frac{(aq;q^2)_{k-2s}(q;q^2)_{k+2s}(q/a;q^2)_k(q/b;q^4)_k}{(bq^2;q^2)_k(q^4;q^4)_{k-s}(q^4/a;q^4)_{k+s}(aq^4;q^4)_k} b^k q^{k^2+k}$$
$$= [6s+1] \frac{(aq;q^2)_{-s}(q;q^2)_{3s}(q/a;q^2)_s(q/b;q^4)_s}{(bq^2;q^2)_s(q^4/a;q^4)_{2s}(aq^4;q^4)_s} b^s q^{s^2+s} \frac{(q^{3+6s}, q^{5+6s};q^4)_{(n-1)/4-s}}{(aq^{4+4s}, q^{4+8s}/a;q^4)_{(n-1)/4-s}}. \quad (2.7)$$

Proof. For $b=q^n$, the left-hand side of (2.4) is equal to

$$
\frac{(aq;q^2)_{-s}(q;q^2)_{3s}(q/a;q^2)_s(q^{1-n};q^4)_s}{(q^{2+n};q^2)_s(q^4/a;q^4)_{2s}(aq^4;q^4)_s}q^{s^2+ns+s}
\times\sum_{k=0}^{n-2s-1}[6k+6s+1]\frac{(aq^{1-2s},q^{1+6s},q^{1+2s}/a;q^2)_k(q^{1+4s-n};q^4)_k}{(q^{2+2s+n};q^2)_k(q^4,q^{4+8s}/a,aq^{4+4s};q^4)_k}q^{k^2+2sk+nk+k} \tag{2.8}
$$

If $s>(n-1)/4$, then $(q^{1-n};q^4)_s=0$ and so both sides of (2.7) are equal to 0. We now assume that $0\leqslant s\leqslant (n-1)/4$. Letting $q\mapsto q^2$, $a\mapsto q^{1+6s}$, $b\mapsto aq^{1-2s}$, and $f\mapsto q^{1+4s-n}$ in (2.6), and noticing $(q^{1+4s-n};q^4)_k=0$ for $k>(n-1)/4-s$ and $n-2s-1\geqslant (n-1)/4-s$. (2.8) may be written as

$$
[6s+1]\frac{(aq;q^2)_{-s}(q;q^2)_{3s}(q/a;q^2)_s(q^{1-n};q^4)_s}{(q^{2+n};q^2)_s(q^4/a;q^4)_{2s}(aq^4;q^4)_s}q^{s^2+ns+s}
\times\frac{(q^{3+6s},q^{5+6s},q^{3+4s+n}/a,aq^{3+n};q^4)_\infty}{(q^{2+2s+n},q^{4+2s+n},q^{4+8s}/a,aq^{4+4s};q^4)_\infty},
$$

which is just the $b=q^n$ case of the right-hand side of (2.7). This completes the proof. □

Proof of Theorem 1.1. Since $q^n\equiv 1 \pmod{\Phi_n(q)}$, we deduce from (2.1) that

$$
\sum_{k=s}^{n-s-1}[6k+1]\frac{(aq;q^2)_{k-2s}(q;q^2)_{k+2s}(q/a;q^2)_k(q/b;q^4)_k}{(bq^2;q^2)_k(q^4;q^4)_{k-s}(q^4/a;q^4)_{k+s}(aq^4;q^4)_k}b^kq^{k^2+k}\equiv 0 \pmod{\Phi_n(q)}. \tag{2.9}
$$

It is easily seen that the right-hand sides of (2.4) and (2.7) are both congruent to 0 modulo $\Phi_n(q)$. Thus, applying the following congruences:

$$
\frac{b-q^n}{b-a}\equiv 1 \pmod{a-q^n},
$$
$$
\frac{a-q^n}{a-b}\equiv 1 \pmod{b-q^n},
$$

and the Chinese remainder theorem for coprime polynomials, we derive from (2.4), (2.7), and (2.9) that, for even s with $0\leqslant s\leqslant (n-1)/2$, modulo $\Phi_n(q)(a-q^n)(b-q^n)$,

$$
\sum_{k=s}^{n-s-1}[6k+1]\frac{(aq;q^2)_{k-2s}(q;q^2)_{k+2s}(q/a;q^2)_k(q/b;q^4)_k}{(bq^2;q^2)_k(q^4;q^4)_{k-s}(q^4/a;q^4)_{k+s}(aq^4;q^4)_k}b^kq^{k^2+k}
\equiv[6s+1]\frac{(aq;q^2)_{-s}(q;q^2)_{3s}(q/a;q^2)_s(q/b;q^4)_s}{(bq^2;q^2)_s(q^4/a;q^4)_{2s}(aq^4;q^4)_s}b^sq^{s^2+s}
\times\left\{\frac{(q^{5+6s},bq^{3+4s-n};q^4)_{(n-1-2s)/4}}{(bq^{4+2s},q^{4+8s-n};q^4)_{(n-1-2s)/4}}\frac{b-q^n}{b-a}+\frac{(q^{3+6s},q^{5+6s};q^4)_{(n-1)/4-s}}{(aq^{4+4s},q^{4+8s}/a;q^4)_{(n-1)/4-s}}\frac{a-q^n}{a-b}\right\}. \tag{2.10}
$$

It is not difficult to see that

$$\frac{(bq^{3+4s-n};q^4)_{(n-1-2s)/4}}{(q^{4+8s-n};q^4)_{(n-1-2s)/4}} \equiv \frac{(bq^3/a;q^4)_{(n-1+2s)/4}(q^{4+4s}/a;q^4)_s}{(q^{4+4s}/a;q^4)_{(n-1+2s)/4}(q^3/a;q^4)_s} \pmod{a-q^n},$$

and, modulo $b-q^n$,

$$\begin{aligned}
&\frac{(q^{3+6s},q^{5+6s};q^4)_{(n-1)/4-s}}{(aq^{4+4s},q^{4+8s}/a;q^4)_{(n-1)/4-s}} \\
&\quad\equiv \frac{(q^3;q^4)_{(n-1+2s)/4}(q^{5+6s};q^4)_{(n-1-2s)/4}(q^{4+2s}/a;q^4)_{3s/2}(abq^3;q^4)_{3s/2}}{(q^3;q^4)_{3s/2}(bq^{4+2s};q^4)_{s/2}(q^{4+2s}/a;q^4)_{(n-1-2s)/4}(bq^3/a;q^4)_s(aq^{4+4s};q^4)_{(n-1+2s)/4}}.
\end{aligned}$$

Thus, the q-congruence (2.10) is equivalent to the following one: modulo $\Phi_n(q)(a-q^n)(b-q^n)$,

$$\begin{aligned}
&\sum_{k=s}^{n-s-1}[6k+1]\frac{(aq;q^2)_{k-2s}(q;q^2)_{k+2s}(q/a;q^2)_k(q/b;q^4)_k}{(bq^2;q^2)_k(q^4;q^4)_{k-s}(q^4/a;q^4)_{k+s}(aq^4;q^4)_k}b^kq^{k^2+k} \\
&\quad\equiv [6s+1]\frac{(aq;q^2)_{-s}(q;q^2)_{3s}(q/a;q^2)_s(q/b;q^4)_s}{(bq^2;q^2)_s(q^4/a;q^4)_{2s}(aq^4;q^4)_s}b^sq^{s^2+s} \\
&\quad\times\Bigg\{\frac{(q^{5+6s};q^4)_{(n-1-2s)/4}(bq^3/a;q^4)_{(n-1+2s)/4}(q^{4+4s}/a;q^4)_s}{(bq^{4+2s};q^4)_{(n-1-2s)/4}(q^{4+4s}/a;q^4)_{(n-1+2s)/4}(q^3/a;q^4)_s}\frac{b-q^n}{b-a} \\
&\quad+\frac{(q^3;q^4)_{(n-1+2s)/4}(q^{5+6s};q^4)_{(n-1-2s)/4}(q^{4+2s}/a;q^4)_{3s/2}(abq^3;q^4)_{3s/2}\frac{a-q^n}{a-b}}{(q^3;q^4)_{3s/2}(bq^{4+2s};q^4)_{s/2}(q^{4+2s}/a;q^4)_{(n-1-2s)/4}(bq^3/a;q^4)_s(aq^{4+4s};q^4)_{(n-1+2s)/4}}\Bigg\}
\end{aligned}$$

(Note that both sides are congruent to 0 modulo $\Phi_n(q)$). By first letting $b=1$ and then taking the limits as $a\to 1$ in the above q-congruence, we are led to the following q-supercongrunece: for even s with $0\leqslant s\leqslant (n-1)/2$,

$$\begin{aligned}
&\sum_{k=s}^{n-s-1}[6k+1]\frac{(q;q^2)_{k-2s}(q;q^2)_{k+2s}(q;q^2)_k(q;q^4)_k}{(q^2;q^2)_k(q^4;q^4)_{k-s}(q^4;q^4)_{k+s}(q^4;q^4)_k}q^{k^2+k} \\
&\quad\equiv [6s+1]\frac{(q;q^2)_{-s}(q;q^2)_{3s}(q;q^2)_s(q;q^4)_s}{(q^2;q^2)_s(q^4;q^4)_{2s}(q^4;q^4)_s}q^{s^2+s} \\
&\quad\times\frac{(q^{5+6s};q^4)_{(n-1-2s)/4}(q^3;q^4)_{(n-1+2s)/4}(q^{4+4s};q^4)_s}{(q^{4+2s};q^4)_{(n-1-2s)/4}(q^{4+4s};q^4)_{(n-1+2s)/4}(q^3;q^4)_s} \pmod{\Phi_n(q)^3},
\end{aligned}$$

which is just the the first part of (1.3) after simplifications.

In the same way, we can prove the second part of (1.3). □

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