A FAMILY OF q-CONGRUENCES MODULO THE SQUARE OF A CYCLOTOMIC POLYNOMIAL

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Abstract. Using Watson's terminating ${}_8\phi_7$ transformation formula, we prove a family of q-congruences modulo the square of a cyclotomic polynomial, which were originally conjectured by the author and Zudilin [J. Math. Anal. Appl. 475 (2019), 1636–646]. As an application, we deduce two supercongruences modulo p^4 (p is an odd prime) and their q-analogues. This also partially confirms a special case of Swisher's (H.3) conjecture.

1. Introduction

In 1997, Van Hamme [19, (H.2)] proved the following supercongruence: for any prime $p \equiv 3 \pmod 4$,

$$\sum_{k=0}^{(p-1)/2} \frac{(\frac{1}{2})_k^3}{k!^3} \equiv 0 \pmod{p^2}, \tag{1.1}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ is the rising factorial. It is easy to see that (1.1) is also true when the sum is over k from 0 to $p-1$, since $(1/2)_k/k! \equiv 0 \pmod p$ for $(p-1)/2 < k \leqslant p-1$. Nowadays various generalizations of (1.1) can be found in [8, 10–14, 16, 17]. For example, Liu [12] proved that, for any prime $p \equiv 3 \pmod 4$ and positive integer m,

$$\sum_{k=0}^{mp-1} \frac{(\frac{1}{2})_k^3}{k!^3} \equiv 0 \pmod{p^2}. \tag{1.2}$$

The first purpose of this note is to prove the following q-analogue of (1.2), which was originally conjectured by the author and Zudilin [10, Conjecture 2].

Theorem 1.1. *Let m and n be positive integers with $n \equiv 3 \pmod 4$. Then*

$$\sum_{k=0}^{mn-1} \frac{(1+q^{4k+1})(q^2;q^4)_k^3}{(1+q)(q^4;q^4)_k^3} q^k \equiv 0 \pmod{\Phi_n(q)^2}, \tag{1.3}$$

$$\sum_{k=0}^{mn+(n-1)/2} \frac{(1+q^{4k+1})(q^2;q^4)_k^3}{(1+q)(q^4;q^4)_k^3} q^k \equiv 0 \pmod{\Phi_n(q)^2}. \tag{1.4}$$

1991 *Mathematics Subject Classification.* 33D15, 11A07, 11B65.

Key words and phrases. cyclotomic polynomial; q-congruence; supercongruence; Watson's transformation.

The author was partially supported by the National Natural Science Foundation of China (grant 11771175).

Here and in what follows, the *q-shifted factorial* is defined by $(a;q)_0 = 1$ and $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n \geqslant 1$, and the n-th *cyclotomic polynomial* $\Phi_n(q)$ is defined as

$$\Phi_n(q) = \prod_{\substack{1\leqslant k\leqslant n\\ \gcd(n,k)=1}} (q-\zeta^k),$$

where ζ is an n-th primitive root of unity. Moreover, the *q-integer* is given by $[n] = [n]_q = 1 + q + \cdots + q^{n-1}$.

The $m = 1$ case of (1.3) was first conjectured by the author and Zudilin [9, Conjecture 4.13] and has already been proved by themselves in a recent paper [11]. For some other recent progress on q-congruences, the reader may consult [2–8, 10, 15].

In 2016, Swisher [18, (H.3) with $r = 2$] conjectured that, for primes $p \equiv 3 \pmod 4$ and $p > 3$,

$$\sum_{k=0}^{(p^2-1)/2} \frac{(\frac12)_k^3}{k!^3} \equiv p^2 \pmod{p^5}, \tag{1.5}$$

The second purpose of this note is to prove the following q-congruences related to (1.5) modulo p^4.

Theorem 1.2. *Let $n \equiv 3 \pmod 4$ be a positive integer. Then, modulo $\Phi_n(q)^2\Phi_{n^2}(q)^2$, we have*

$$\sum_{k=0}^{(n^2-1)/2} \frac{(1+q^{4k+1})(q^2;q^4)_k^3}{(1+q)(q^4;q^4)_k^3} q^k \equiv \frac{[n^2]_{q^2}(q^3;q^4)_{(n^2-1)/2}}{(q^5;q^4)_{(n^2-1)/2}} q^{(1-n^2)/2}, \tag{1.6}$$

$$\sum_{k=0}^{n^2-1} \frac{(1+q^{4k+1})(q^2;q^4)_k^3}{(1+q)(q^4;q^4)_k^3} q^k \equiv \frac{[n^2]_{q^2}(q^3;q^4)_{(n^2-1)/2}}{(q^5;q^4)_{(n^2-1)/2}} q^{(1-n^2)/2}. \tag{1.7}$$

Let $n = p \equiv 3 \pmod 4$ be a prime and take $q \to 1$ in Theorem 1.2. Then $\Phi_p(1) = \Phi_{p^2}(1) = p$, and

$$\lim_{q\to 1} \frac{(q^3;q^4)_{(p^2-1)/2}}{(q^5;q^4)_{(p^2-1)/2}} = \prod_{k=1}^{(p^2-1)/2} \frac{4k-1}{4k+1} = \frac{(\frac34)_{(p^2-1)/2}}{(\frac54)_{(p^2-1)/2}}.$$

and we obtain the following conclusion.

Corollary 1.3. *Let $p \equiv 3 \pmod 4$ be a prime. Then*

$$\sum_{k=0}^{(p^2-1)/2} \frac{(\frac12)_k^3}{k!^3} \equiv p^2 \frac{(\frac34)_{(p^2-1)/2}}{(\frac54)_{(p^2-1)/2}} \pmod{p^4}, \tag{1.8}$$

$$\sum_{k=0}^{p^2-1} \frac{(\frac12)_k^3}{k!^3} \equiv p^2 \frac{(\frac34)_{(p^2-1)/2}}{(\frac54)_{(p^2-1)/2}} \pmod{p^4}. \tag{1.9}$$

Comparing (1.5) and (1.8), we would like to propose the following conjecture, which was recently confirmed by Wang and Pan [20].

Conjecture 1.4. *Let $p \equiv 3 \pmod 4$ be a prime and r a positive integer. Then*

$$\prod_{k=1}^{(p^{2r}-1)/2} \frac{4k-1}{4k+1} \equiv 1 \pmod{p^2}. \tag{1.10}$$

Note that the $r=1$ case is equivalent to say that (1.5) is true modulo p^4.

2. Proof of Theorem 1.1

We need to use Watson's terminating ${}_8\phi_7$ transformation formula (see [1, Appendix (III.18)] and [1, Section 2]):

$$\begin{aligned} &{}_8\phi_7\left[\begin{matrix} a, & qa^{\frac12}, & -qa^{\frac12}, & b, & c, & d, & e, & q^{-n} \\ & a^{\frac12}, & -a^{\frac12}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1} \end{matrix}; q, \frac{a^2q^{n+2}}{bcde}\right] \\ &\quad = \frac{(aq;q)_n(aq/de;q)_n}{(aq/d;q)_n(aq/e;q)_n}\, {}_4\phi_3\left[\begin{matrix} aq/bc, \ d, \ e, \ q^{-n} \\ aq/b, \ aq/c, \ deq^{-n}/a \end{matrix}; q, q\right], \end{aligned} \tag{2.1}$$

where the basic hypergeometric ${}_{r+1}\phi_r$ series with $r+1$ upper parameters $a_1, \dots, a_{r+1}$, r lower parameters $b_1, \dots, b_r$, base q and argument z is defined as

$${}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix}; q, z\right] := \sum_{k=0}^{\infty} \frac{(a_1;q)_k(a_2;q)_k \dots (a_{r+1};q)_k}{(q;q)_k(b_1;q)_k \cdots (b_r;q)_k} z^k.$$

The left-hand side of (1.4) with $m \geqslant 0$ can be written as the following terminating ${}_8\phi_7$ series:

$${}_8\phi_7\left[\begin{matrix} q^2, & q^5, & -q^5, & q^2, & q, & q^2, & q^{4+(4m+2)n}, & q^{2-(4m+2)n} \\ & q, & -q, & q^4, & q^5, & q^4, & q^{2-(4m+2)n}, & q^{4+(4m+2)n} \end{matrix}; q^4, q\right]. \tag{2.2}$$

By Watson's transformation formula (2.1) with $q \mapsto q^4$, $a = b = d = q^2$, $c = q$, $e = q^{4+(4m+2)n}$, and $n \mapsto mn + (n-1)/2$, we see that (2.2) is equal to

$$\frac{(q^6;q^4)_{mn+(n-1)/2}(q^{-(4m+2)n};q^4)_{mn+(n-1)/2}}{(q^4;q^4)_{mn+(n-1)/2}(q^{2-(4m+2)n};q^4)_{mn+(n-1)/2}}\, {}_4\phi_3\left[\begin{matrix} q^3, \ q^2, \ q^{4+(4m+2)n}, \ q^{2-(4m+2)n} \\ q^4, \quad q^5, \quad q^6 \end{matrix}; q^4, q^4\right]. \tag{2.3}$$

It is not difficult to see that there are exactly $m+1$ factors of the form $1-q^{an}$ (a is an integer) among the $mn+(n-1)/2$ factors of $(q^6;q^4)_{mn+(n-1)/2}$. So are $(q^{-(4m+2)n};q^4)_{mn+(n-1)/2}$. But there are only m factors of the form $1-q^{an}$ (a is an integer) in each of $(q^4;q^4)_{mn+(n-1)/2}$ and $(q^{2-(4m+2)n};q^4)_{mn+(n-1)/2}$. Since $\Phi_n(q)$ is a factor of $1-q^N$ if and only if n divides N, we conclude that the fraction before the ${}_4\phi_3$ series is congruent to 0 modulo $\Phi_n(q)^2$. Moreover, for any integer x, let $f_n(x)$ be the least non-negative integer k such that $(q^x;q^4)_k \equiv 0$ modulo $\Phi_n(q)$. Since $n \equiv 3 \pmod 4$, we have $f_n(2) = (n+1)/2$, $f_n(3) = (n+1)/4$, $f_n(4) = n$, $f_n(5) = (3n-1)/4$, and $f_n(6) = (n-1)/2$. It follows that the denominator of the reduced form of the k-th summand

$$\frac{(q^3;q^4)_k(q^2;q^4)_k(q^{4+(4m+2)n};q^4)_k(q^{2-(4m+2)n};q^4)_k}{(q^4;q^4)_k^2(q^5;q^4)_k(q^6;q^4)_k} q^{4k}$$

in the ${}_4\phi_3$ summation is always relatively prime to $\Phi_n(q)$ for any non-negative integer k. This proves that (2.3) (i.e. (2.2)) is congruent to 0 modulo $\Phi_n(q)^2$, thus establishing (1.4) for $m \geqslant 0$.

It is easy to see that $(q^2;q^4)_k^3/(q^4;q^4)_k^3$ is congruent to 0 modulo $\Phi_n(q)^3$ for $mn+(n-1)/2 < k \leqslant (m+1)n-1$. Therefore, the q-congruence (1.3) with $m \mapsto m+1$ follows from (1.4).

3. Proof of Theorem 1.2

The author and Zudilin [11, Theorem 1.1] proved that, for any positive odd integer n,

$$\sum_{k=0}^{(n-1)/2} \frac{(1+q^{4k+1})\,(q^2;q^4)_k^3}{(1+q)\,(q^4;q^4)_k^3}\, q^k \equiv \frac{[n]_{q^2}(q^3;q^4)_{(n-1)/2}}{(q^5;q^4)_{(n-1)/2}}\, q^{(1-n)/2} \pmod{\Phi_n(q)^2}, \tag{3.1}$$

which is also true when the sum on the left-hand side of (3.1) is over k from 0 to $n-1$. Replacing n by n^2 in (3.1) and its equivalent form, we see that the q-congruences (1.6) and (1.7) hold modulo $\Phi_{n^2}(q)^2$.

It is easy to see that, for $n \equiv 3 \pmod 4$,

$$\frac{[n^2]_{q^2}(q^3;q^4)_{(n^2-1)/2}}{(q^5;q^4)_{(n^2-1)/2}} q^{(1-n^2)/2} \equiv 0 \pmod{\Phi_n(q)^2}$$

because $[n^2]_{q^2} = (1-q^{n^2})/(1-q^2)$ is divisible by $\Phi_n(q)$, and $(q^3;q^4)_{(n^2-1)/2}$ contains $(n+1)/2$ factors of the form $1-q^{an}$ (a is an integer), while $(q^5;q^4)_{(n^2-1)/2}$ only has $(n-1)/2$ such factors. Meanwhile, by Theorem 1.1, the left-hand sides of (1.6) and (1.7) are both congruent to 0 modulo $\Phi_n(q)^2$ since $(n^2-1)/2 = (n-1)n/2+(n-1)/2$. This proves that the q-congruences (1.6) and (1.7) also hold modulo $\Phi_n(q)^2$. Since the polynomials $\Phi_n(q)$ and $\Phi_{n^2}(q)$ are relatively prime, we finish the proof of the theorem.

4. Discussion

Swisher's (H.3) conjecture also indicates that, for positive integer r and primes $p \equiv 3 \pmod 4$ with $p > 3$, we have

$$\sum_{k=0}^{(p^{2r}-1)/2} \frac{(\frac{1}{2})_k^3}{k!^3} \equiv p^{2r} \pmod{p^{2r+3}}. \tag{4.1}$$

Motivated by (4.1), we shall give the following generalization of Theorem 1.2.

Theorem 4.1. *Let n and r be positive integers with $n \equiv 3 \pmod 4$. Then, modulo $\Phi_{n^{2r}}(q)^2 \prod_{j=1}^{r} \Phi_{n^{2j-1}}(q)^2$, we have*

$$\sum_{k=0}^{(n^{2r}-1)/2} \frac{(1+q^{4k+1})(q^2;q^4)_k^3}{(1+q)(q^4;q^4)_k^3} q^k \equiv \frac{[n^{2r}]_{q^2}(q^3;q^4)_{(n^{2r}-1)/2}}{(q^5;q^4)_{(n^{2r}-1)/2}} q^{(1-n^{2r})/2}, \tag{4.2}$$

$$\sum_{k=0}^{n^{2r}-1} \frac{(1+q^{4k+1})(q^2;q^4)_k^3}{(1+q)(q^4;q^4)_k^3} q^k \equiv \frac{[n^{2r}]_{q^2}(q^3;q^4)_{(n^{2r}-1)/2}}{(q^5;q^4)_{(n^{2r}-1)/2}} q^{(1-n^{2r})/2}. \tag{4.3}$$

Proof. Replacing n by n^{2r} in (3.1) and its equivalent form, we see that (4.2) and (4.3) are true modulo $\Phi_{n^{2r}}(q)^2$. Similarly as before, we can show that

$$\frac{[n^{2r}]_{q^2}(q^3;q^4)_{(n^{2r}-1)/2}}{(q^5;q^4)_{(n^{2r}-1)/2}} q^{(1-n^{2r})/2} \equiv 0 \pmod{\prod_{j=1}^{r} \Phi_{n^{2j-1}}(q)^2}.$$

Further, by Theorem 1.1, we can easily deduce that the left-hand sides of (4.2) and (4.3) are also congruent to 0 modulo $\prod_{j=1}^{r} \Phi_{n^{2j-1}}(q)^2$. □

Letting $n = p \equiv 3 \pmod 4$ be a prime and taking $q \to 1$ in Theorem 4.1, we are led to the following result.

Corollary 4.2. *Let $p \equiv 3 \pmod 4$ be a prime and let $r \geqslant 1$. Then*

$$\sum_{k=0}^{(p^{2r}-1)/2} \frac{(\frac{1}{2})_k^3}{k!^3} \equiv p^{2r} \frac{(\frac{3}{4})_{(p^{2r}-1)/2}}{(\frac{5}{4})_{(p^{2r}-1)/2}} \pmod{p^{2r+2}}, \tag{4.4}$$

$$\sum_{k=0}^{p^{2r}-1} \frac{(\frac{1}{2})_k^3}{k!^3} \equiv p^{2r} \frac{(\frac{3}{4})_{(p^{2r}-1)/2}}{(\frac{5}{4})_{(p^{2r}-1)/2}} \pmod{p^{2r+2}}. \tag{4.5}$$

In light of (1.10), the supercongruence (4.4) implies that (4.1) holds modulo p^{2r+2} for any odd prime p.

It is known that q-analogues of supercongruences are usually not unique. See, for example, [2]. The author and Zudilin [10, Conjecture 1] also gave another q-analogue of (1.2), which still remains open.

Conjecture 4.3 (Guo and Zudilin)**.** *Let m and n be positive integers with $n \equiv 3 \pmod 4$. Then*

$$\sum_{k=0}^{mn-1} \frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k} q^{2k} \equiv 0 \pmod{\Phi_n(q)^2}, \tag{4.6}$$

$$\sum_{k=0}^{mn+(n-1)/2} \frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k} q^{2k} \equiv 0 \pmod{\Phi_n(q)^2}.$$

The author and Zudilin [10, Theorem 2] themselves have proved (4.6) for the $m = 1$ case. Motivated by Conjecture 4.3, we would like to give the following new conjectural q-analogues of (1.8) and (1.9).

Conjecture 4.4. *Let $n \equiv 3 \pmod 4$ be a positive integer. Then, modulo $\Phi_n(q)^2\Phi_{n^2}(q)^2$, we have*

$$\sum_{k=0}^{(n^2-1)/2} \frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k} q^{2k} \equiv \frac{[n^2](q^3;q^4)_{(n^2-1)/2}}{(q^5;q^4)_{(n^2-1)/2}},$$

$$\sum_{k=0}^{n^2-1} \frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k} q^{2k} \equiv \frac{[n^2](q^3;q^4)_{(n^2-1)/2}}{(q^5;q^4)_{(n^2-1)/2}}.$$

There are similar such new q-analogues of (4.4) and (4.5). We omit them here and leave space for the reader's imagination.

Acknowledgment. The author is grateful to the two anonymous referees for their careful readings of this paper.

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