A new q-analogue of Van Hamme's (H.2) supercongruence for primes $p \equiv 1 \pmod 4$

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Abstract. Long and Ramakrishna generalized the (H.2) supercongruence of Van Hamme to the modulus p^3 case. Wei and Wang gave two different q-analogues of this supercongruence for primes $p \equiv 1 \pmod 4$. The author and Zudilin ever presented a new q-analogue of Van Hamme's original (H.2) supercongruence for primes $p \equiv 1 \pmod 4$. In this paper, we further extend this q-congruence to the modulus $\Phi_n(q)^3$ case, where $\Phi_n(q)$ is the n-th cyclotomic polynomial in q. The main ingredients of our proof are the creative microscoping method, a q-analogue of Watson's ${}_3F_2$ summation, and the Chinese remainder theorem for polynomials.

Keywords: supercongruence; q-congruence; creative microscoping; q-analogue of Watson's ${}_3F_2$ summation.

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1. Introduction

Bauer's formula [1] from 1859,

$$\sum_{k=0}^{\infty}(-1)^k \frac{4k+1}{64^k}\binom{2k}{k}^3 = \frac{2}{\pi} \tag{1.1}$$

became famous after Ramanujan [20] in 1914 gave a long list of analogous series for the constant but with a rapid convergence. In 1994, Ekhad (Zeilberger's computer) and Zeilberger [2] presented a remarkable computer proof of it by applying the WZ (Wilf–Zeilberger) method of creative telescoping.

In 1997, Van Hamme [26] noticed that a number of Ramanujan's and Ramanujan-type evaluations possess neat p-adic analogues. For instance, the congruence

$$\sum_{k=0}^{(p-1)/2} (-1)^k \frac{4k+1}{64^k}\binom{2k}{k}^3 \equiv p(-1)^{(p-1)/2} \pmod{p^3} \tag{1.2}$$

(tagged (B.2) in [26]), corresponding to the equality (1.1), is valid for any odd prime p. The congruence (1.2) was first confirmed by Mortenson [19] by making use of a ${}_6F_5$ hypergeometric transformation. Later Zudilin [31] gave a new proof of (1.2) through the WZ method (in fact, he borrowed the very same "WZ certificate" for (1.1) in [2]). Note that (1.2) is also called a *supercongruence* indicating that it is true modulo a power of p greater than one.

Another famous entry on Van Hamme's list [26] is the following supercongruence (tagged (H.2) in [26]):

$$\sum_{k=0}^{(p-1)/2} \frac{1}{64^k}\binom{2k}{k}^3 \equiv \begin{cases} -\Gamma_p(\tfrac14)^4 \pmod{p^2} & \text{if } p\equiv 1 \pmod 4,\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4, \end{cases} \tag{1.3}$$

again for any odd prime p. Here $\Gamma_p(x)$ denotes the p-adic Gamma function. Van Hamme himself also proved (1.3) in [26]. Many different generalizations of (1.3) can be found in [9, 15–17, 21–23]. For example, Long and Ramakrishna [17, Theorem 3] obtained the following result:

$$\sum_{k=0}^{(p-1)/2} \frac{1}{64^k}\binom{2k}{k}^3 \equiv \begin{cases} -\Gamma_p(\tfrac14)^4 \pmod{p^3} & \text{if } p\equiv 1 \pmod 4,\\ -\dfrac{p^2}{16}\,\Gamma_p(\tfrac14)^4 \pmod{p^3} & \text{if } p\equiv 3 \pmod 4. \end{cases} \tag{1.4}$$

In 2019, the author and Zudilin [12, Theorem 2] established a q-analogue of (1.3) as follows: modulo $\Phi_n(q)^2$,

$$\sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{2k} \equiv \begin{cases} \dfrac{(q^2;q^4)_{(n-1)/4}^2}{(q^4;q^4)_{(n-1)/4}^2}q^{(n-1)/2} & \text{if } n\equiv 1 \pmod 4,\\ 0 & \text{if } n\equiv 3 \pmod 4 \end{cases} \tag{1.5}$$

(for a special case of (1.5), see [9, Corollary 1.2]). Here we need to familiarize ourselves with the standard q-notation. Assume that $|q|<1$. The *q-shifted factorial* is defined as $(a;q)_0=1$ and $(a;q)_n=(1-a)(1-aq)\dots(1-aq^{n-1})$ for $n\geqslant 1$ or $n=\infty$. For convenience, we will now and then adopt the abbreviated notation $(a_1,\dots,a_m;q)_n=(a_1;q)_n\dots(a_m;q)_n$ for $n\geqslant 0$ or $n=\infty$. The n-th *cyclotomic polynomial* $\Phi_n(q)$ is defined by

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(n,k)=1}}(q-\zeta^k),$$

where ζ denotes an n-th primitive root of unity. Moreover, let $[n]=[n]_q=(1-q^n)/(1-q)$ be the *q-integer*.

For two rational functions $A(q)$ and $B(q)$, and a polynomial $P(q)\in\mathbb{Z}[q]$, we say that the q-congruence $A(q)\equiv B(q) \pmod{P(q)}$ holds if the numerator of $A(q)-B(q)$ is divisible by $P(q)$ in the polynomial ring $\mathbb{Z}[q]$.

For any prime $p\equiv 1 \pmod 4$, we have

$$\binom{-1/2}{(p-1)/4} \equiv -\frac{\Gamma_p(\frac14)^2}{\Gamma_p(\frac12)} \pmod{p^2}$$

(see [25, Theorem 3]) and $\Gamma_p(\frac12)^2=-1$. Taking the limits as $q\to 1$ in (1.5) for $n=p$ we immediately get (1.3).

One matter of (1.3) (not emphasized in [26]) is its connection with the coefficients

$$a(p) = \begin{cases} 2(a^2 - b^2) & \text{if } p = a^2 + b^2,\ a \text{ odd}, \\ 0 & \text{if } p \equiv 3 \pmod 4, \end{cases} \tag{1.6}$$

of CM modular form $q\prod_{j=1}^{\infty}(1-q^{4j})^6$ of weight 3. In other words, there holds

$$a(p) \equiv -\Gamma_p(\tfrac{1}{4})^4 \pmod{p^2} \quad \text{for primes } p \equiv 1 \pmod 4.$$

In fact, this is the main motivation of [12].

For some other recent progress on q-analogues of congruences, we refer the reader to [4,6,8,10,11,14,28–30]. In particular, the author and Zudilin [11] devised a new method of creative microscoping to build q-analogues of many classical supercongruences and also proposed some conjectures on q-congruences. Employing this method, the author [6] gave a further refinement of (1.5) for $n \equiv 3 \pmod 4$:

$$\sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k} q^{2k} \equiv [n]\frac{(q^3;q^4)_{(n-1)/2}}{(q^5;q^4)_{(n-1)/2}} \pmod{\Phi_n(q)^3}. \tag{1.7}$$

At the same time, Wei [28] gave a refinement of (1.5) for $n \equiv 1 \pmod 4$:

$$\begin{aligned} &\sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k} q^{2k} \\ &\quad\equiv \frac{(q^2;q^4)_{(n-1)/4}^2}{(q^4;q^4)_{(n-1)/4}^2} q^{(n-1)/2} \left(1 + 2[n]^2 \sum_{k=1}^{(n-1)/4} \frac{q^{4k-2}}{[4k-2]^2}\right) \pmod{\Phi_n(q)^3}, \end{aligned} \tag{1.8}$$

and another extension of (1.5) for $n \equiv 1 \pmod 4$ can be found in [27].

On the other hand, employing the q-Dixon sum, the author and Zudilin [13] proved that, for any positive odd integer n,

$$\begin{aligned} &\sum_{k=0}^{(n-1)/2} \frac{(1+q^{4k+1})(q^2;q^4)_k^3}{(1+q)\,(q^4;q^4)_k^3} q^k \\ &\quad\equiv \frac{[n]_{q^2}(q^3;q^4)_{(n-1)/2}}{(q^5;q^4)_{(n-1)/2}} q^{(1-n)/2} \begin{cases} \pmod{\Phi_n(q)^2\Phi_n(-q)^3} & \text{if } n \equiv 1 \pmod 4, \\ \pmod{\Phi_n(q)^3\Phi_n(-q)^3} & \text{if } n \equiv 3 \pmod 4. \end{cases} \end{aligned} \tag{1.9}$$

It is not hard to check that

$$\frac{(3/4)_{(p-1)/2}}{(5/4)_{(p-1)/2}} \equiv -\frac{p}{16}\Gamma_p\left(\tfrac{1}{4}\right)^4 \pmod{p^2}$$

for primes $p \equiv 3 \pmod 4$, where we have adopted Pochhammer's symbol: $(a)_0 = 1$ and $(a)_n = a(a+1)\cdots(a+n-1)$ for $n \geqslant 1$. Clearly, the q-congruence (1.9) may be considered as a common generalization of (1.2) and (1.3).

Using a q-analogue of Watson's ${}_3F_2$ summation, the author and Zudilin [13] also obtained the following weaker form of (1.9): modulo $\Phi_n(q)^2$,

$$\sum_{k=0}^{(n-1)/2} \frac{(1+q^{4k+1})(q^2;q^4)_k^3}{(1+q)\,(q^4;q^4)_k^3} q^k \equiv \begin{cases} \dfrac{(1+q^n)(q^2;q^4)_m^2}{(1+q)(q^4;q^4)_m^2} & \text{if } n=4m+1. \\ 0 & \text{if } n\equiv 3 \pmod 4, \end{cases} \tag{1.10}$$

(We have corrected a typo here.)

Motivated by the aforementioned work, we shall generalize the first part of (1.9) (or (1.10)) to the modulus $\Phi_n(q)^3$ case.

Theorem 1.1. *Let $n\equiv 1 \pmod 4$ be a positive integer. Then, modulo $\Phi_n(q)^3$,*

$$\begin{aligned} &\sum_{k=0}^{(n-1)/2} \frac{(1+q^{4k+1})(q^2;q^4)_k^3}{(1+q)\,(q^4;q^4)_k^3} q^k \\ &\quad \equiv \frac{(1+q^n)(q^2;q^4)_{(n-1)/4}^2}{(1+q)(q^4;q^4)_{(n-1)/4}^2} \left\{ 1+\frac{(n^2-1)(1-q^n)^2}{8} + 2[n]^2 \sum_{j=1}^{(n-1)/4} \frac{q^{4j-2}}{[4j-2]^2} \right\}. \end{aligned} \tag{1.11}$$

It is clear that (1.11) is a generalization of (1.10) for $n\equiv 1 \pmod 4$. Further, letting $n=p$ be a prime and taking the limits as q tends to 1 on both sides of (1.11), we arrive at the following supercongruence:

$$\sum_{k=0}^{(p-1)/2} \frac{1}{64^k}\binom{2k}{k}^3 \equiv \frac{1}{2^{p-1}}\binom{(p-1)/2}{(p-1)/4}^2 \left\{1+\frac{p^2}{2}H_{(p-1)/2}^{(2)} - \frac{p^2}{8}H_{(p-1)/4}^{(2)}\right\} \pmod{p^3},$$

which was already obtained by Wei [28]. Here the harmonic numbers of order 2 are defined as

$$H_n^{(2)} = \sum_{k=1}^{n} \frac{1}{k^2}.$$

Applying the well-known congruence

$$H_{(p-1)/2}^{(2)} \equiv 0 \pmod p \quad \text{for any prime } p>3,$$

we obtain the following conclusion, which was first noticed by Wei [28, Corollary 1.2].

Corollary 1.2. *Let $p\equiv 1 \pmod 4$ be a prime. Then*

$$\sum_{k=0}^{(p-1)/2} \frac{1}{64^k}\binom{2k}{k}^3 \equiv \frac{1}{2^{p-1}}\binom{(p-1)/2}{(p-1)/4}^2 \left\{1-\frac{p^2}{8}H_{(p-1)/4}^{(2)}\right\} \pmod{p^3}. \tag{1.12}$$

Wei [28, Propositon 1.3] has directly proved that the right-hand side of (1.12) is congruent to $-\Gamma_p(\frac{1}{4})^4$ modulo p^3. Therefore, the q-congruence (1.11) is indeed a q-analogue of (1.4) for $p \equiv 1 \pmod 4$.

The paper is organized as follows. We shall employ the method of creative microscoping and the Chinese remainder theorem for polynomials to give a parametric form of Theorem 1.1 in Section 2. A proof of Theorem 1.1 by using a series of q-congruences modulo $\Phi_n(q)$ together with L'Hôpital's rule will be given in Section 3. Finally, in Section 4, we present a related problem with respect to further study.

2. A parametric form of Theorem 1.1

Following Gasper and Rahman [3], the *basic hypergeometric series* ${}_{r+1}\phi_r$ is defined as

$$
{}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix} ; q,\ z\right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_k z^k}{(q, b_1, \dots, b_r; q)_k}.
$$

We will make use of a q-analogue of Watson's ${}_3F_2$ summation [3, Appendix (II.16)]:

$$
\begin{aligned}
&{}_8\phi_7\left[\begin{matrix} \lambda, & q\lambda^{\frac{1}{2}}, & -q\lambda^{\frac{1}{2}}, & a, & b, & c, & -c, & \lambda q/c^2 \\ & \lambda^{\frac{1}{2}}, & -\lambda^{\frac{1}{2}}, & \lambda q/a, & \lambda q/b, & \lambda q/c, & -\lambda q/c, & c^2 \end{matrix} ; q,\ -\frac{\lambda q}{ab}\right] \\
&\quad = \frac{(\lambda q, c^2/\lambda; q)_\infty (aq, bq, c^2q/a, c^2q/b; q^2)_\infty}{(\lambda q/a, \lambda q/b; q)_\infty (q, abq, c^2q, c^2q/ab; q^2)_\infty},
\end{aligned} \tag{2.1}
$$

where $\lambda = -c(ab/q)^{\frac{1}{2}}$.

We now can state and prove the following parametric form of Theorem 1.1.

Theorem 2.1. *Let $n \equiv 1 \pmod 4$ be a positive integer. Let a and b be indeterminates. Then, modulo $(1-aq^n)(a-q^n)(b-q^n)$,*

$$
\begin{aligned}
&\sum_{k=0}^{(n-1)/2} \frac{(b+q^{4k+1})(aq, q/a, -bq, -q/b; q^2)_k (q^2/b^2; q^4)_k}{(b+q)(q^2, q^2/b^2, -aq^2/b, -q^2/ab; q^2)_k (q^4; q^4)_k} \left(\frac{q}{b}\right)^k \\
&\quad \equiv \frac{(-q^3/b; q^2)_{(n-1)/2} (q^2, b^2q^2; q^4)_{(n-1)/4}}{(-bq; q^2)_{(n-1)/2} (q^4, q^4/b^2; q^4)_{(n-1)/4}} \frac{(b-q^n)(ab-1-a^2+aq^n)}{(a-b)(1-ab)} \\
&\qquad + \frac{(aq^3, q^3/a; q^4)_{(n-1)/2} (-1; q^2)_{(n-1)/2}}{(-aq, -q/a, q^2; q^2)_{(n-1)/2} (q^2; q^4)_{(n-1)/2}} \frac{(1-aq^n)(a-q^n)}{(a-b)(1-ab)}.
\end{aligned} \tag{2.2}
$$

Proof. Letting $q \mapsto q^2$, $a \mapsto aq$, $b \mapsto q/a$, and $c \mapsto q/b$ (and so $\lambda = -q/b$) in (2.1), we get the following identity:

$$
\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(b+q^{4k+1})(aq, q/a, -bq, -q/b; q^2)_k (q^2/b^2; q^4)_k}{(b+q)(q^2, q^2/b^2, -aq^2/b, -q^2/ab; q^2)_k (q^4; q^4)_k} \left(\frac{q}{b}\right)^k \\
&\quad = \frac{(-q^3/b, -q/b; q^2)_\infty (aq^3, q^3/a, aq^3/b^2, q^3/ab^2; q^4)_\infty}{(-aq^2/b, -q^2/ab; q^2)_\infty (q^2, q^2/b^2; q^2)_\infty}.
\end{aligned} \tag{2.3}
$$

For $a = q^{-n}$ or $a = q^n$, the left-hand side of (2.2) can be written as the $a = q^{-n}$ case of (2.3). Therefore,

$$\sum_{k=0}^{(n-1)/2} \frac{(b+q^{4k+1})(q^{1-n}, q^{1+n}, -bq, -q/b; q^2)_k (q^2/b^2; q^4)_k}{(b+q)(q^2, q^2/b^2, -q^{2-n}/b, -q^{2+n}/b; q^2)_k (q^4; q^4)_k} \left(\frac{q}{b}\right)^k$$
$$= \frac{(-q^3/b, -q/b; q^2)_\infty (q^{3-n}, q^{3+n}, q^{3-n}/b^2, q^{3+n}/b^2; q^4)_\infty}{(-q^{2-n}/b, -q^{2+n}/b; q^2)_\infty (q^2, q^2/b^2; q^2)_\infty}$$
$$= \frac{(-q^3/b; q^2)_{(n-1)/2} (q^2, b^2q^2; q^4)_{(n-1)/4}}{(-bq; q^2)_{(n-1)/2} (q^4, q^4/b^2; q^4)_{(n-1)/4}}. \tag{2.4}$$

Namely, we have the following q-congruence:

$$\sum_{k=0}^{(n-1)/2} \frac{(b+q^{4k+1})(aq, q/a, -bq, -q/b; q^2)_k (q^2/b^2; q^4)_k}{(b+q)(q^2, q^2/b^2, -aq^2/b, -q^2/ab; q^2)_k (q^4; q^4)_k} \left(\frac{q}{b}\right)^k$$
$$\equiv \frac{(-q^3/b; q^2)_{(n-1)/2} (q^2, b^2q^2; q^4)_{(n-1)/4}}{(-bq; q^2)_{(n-1)/2} (q^4, q^4/b^2; q^4)_{(n-1)/4}} \pmod{(1-aq^n)(a-q^n)}. \tag{2.5}$$

For $b = q^n$, the left-hand side of (2.2) also has a closed form as follows:

$$\sum_{k=0}^{(n-1)/2} \frac{(q^n+q^{4k+1})(aq, q/a, -q^{1+n}, -q^{1-n}; q^2)_k (q^{2-2n}; q^4)_k}{(q^n+q)(q^2, q^{2-2n}, -aq^{2-n}, -q^{2-n}/a; q^2)_k (q^4; q^4)_k} q^{k-nk}$$
$$= \frac{(aq^3, q^3/a; q^4)_{(n-1)/2} (-1; q^2)_{(n-1)/2}}{(-aq, -q/a, q^2; q^2)_{(n-1)/2} (q^2; q^4)_{(n-1)/2}}. \tag{2.6}$$

The above identity cannot be obtained directly from (2.3) by taking $b = q^n$ because of the factor $(q^2/b^2; q^2)_\infty$ in the denominator. Multiplying (2.4) by $(-aq, -q/a, q^2; q^2)_{(n-1)/2} a^{(n-1)/2}$, we see that the two sides of the new identity are polynomials in a of degree less than or equal to $n-1$. It suffices to prove the identity (2.6) is true for n values of a. To this end, just notice that, for $a = q$, both sides of (2.6) are equal to 1; and for $a = q^{\pm(4m+3)}$ with $0 \leqslant m \leqslant (n-3)/2$, both sides of (2.6) are equal to 0 because $(aq^3, q^3/a; q^4)_\infty$ appears in the numerator on the right-hand side of (2.3). Clearly, the identity (2.6) implies the following q-congruence:

$$\sum_{k=0}^{(n-1)/2} \frac{(b+q^{4k+1})(aq, q/a, -bq, -q/b; q^2)_k (q^2/b^2; q^4)_k}{(b+q)(q^2, q^2/b^2, -aq^2/b, -q^2/ab; q^2)_k (q^4; q^4)_k} \left(\frac{q}{b}\right)^k$$
$$\equiv \frac{(aq^3, q^3/a; q^4)_{(n-1)/2} (-1; q^2)_{(n-1)/2}}{(-aq, -q/a, q^2; q^2)_{(n-1)/2} (q^2; q^4)_{(n-1)/2}} \pmod{b-q^n}. \tag{2.7}$$

It is obvious that the polynomials $(1-aq^n)(a-q^n)$ and $b-q^n$ are coprime. Noticing the following q-congruences:

$$\frac{(b-q^n)(ab-1-a^2+aq^n)}{(a-b)(1-ab)}\equiv 1 \pmod{(1-aq^n)(a-q^n)},$$

$$\frac{(1-aq^n)(a-q^n)}{(a-b)(1-ab)}\equiv 1 \pmod{b-q^n}.$$

and making use of the Chinese remainder theorem for polynomials, we obtain the desired q-congruence (2.2). □

3. Proof of Theorem 1.1

Note that $1-q^n$ contains the factor $\Phi_n(q)$. Taking $b=1$ in (2.2), we conclude that, modulo $(1-aq^n)(a-q^n)\Phi_n(q)$,

$$\begin{aligned}
&\sum_{k=0}^{(n-1)/2}\frac{(1+q^{4k+1})(aq,q/a,-q,-q/;q^2)_k(q^2;q^4)_k}{(1+q)(q^2,q^2,-aq^2,-q^2/a;q^2)_k(q^4;q^4)_k}q^k\\
&\quad\equiv\frac{(1+q^n)(q^2;q^4)_{(n-1)/4}^2}{(1+q)(q^4;q^4)_{(n-1)/4}^2}\frac{(1-q^n)(1+a^2-a-aq^n)}{(1-a)^2}\\
&\qquad-\frac{(aq^3,q^3/a;q^4)_{(n-1)/2}(-1;q^2)_{(n-1)/2}}{(-aq,-q/a,q^2;q^2)_{(n-1)/2}(q^2;q^4)_{(n-1)/2}}\frac{(1-aq^n)(a-q^n)}{(1-a)^2}.
\end{aligned}\tag{3.1}$$

It is easy to see that

$$\begin{aligned}
(aq^3;q^4)_{(n-1)/2}&=(aq^3;q^4)_{(n-1)/4}(aq^{n+2};q^4)_{(n-1)/4}\\
&\equiv(aq^{3-n};q^4)_{(n-1)/4}(aq^2;q^4)_{(n-1)/4}\\
&=(-a)^{(n-1)/4}q^{-(n-1)^2/8}(q^2/a,aq^2;q^4)_{(n-1)/4}\pmod{\Phi_n(q)},\\
(q^3/a;q^4)_{(n-1)/2}&\equiv(-a)^{-(n-1)/4}q^{-(n-1)^2/8}(aq^2,q^2/a;q^4)_{(n-1)/4}\pmod{\Phi_n(q)},\\
(q^2;q^2)_{(n-1)/2}&=(q^2,q^4;q^4)_{(n-1)/4}\\
(q^2;q^4)_{(n-1)/2}&=(q^2;q^4)_{(n-1)/4}(q^{n+1};q^4)_{(n-1)/4}\\
&\equiv(q^2;q^4)_{(n-1)/4}(q^{1-n};q^4)_{(n-1)/4}\\
&=(-1)^{(n-1)/4}q^{-(n-1)(n+3)/8}(q^2,q^4;q^4)_{(n-1)/4}\pmod{\Phi_n(q)}.
\end{aligned}$$

Moreover, by [7, eq. (3.1)],

$$(-1;q^2)_{(n-1)/2}=\frac{2(-q^2;q^2)_{(n-1)/2}}{1+q^{n-1}}\equiv\frac{2q}{1+q}(-1)^{(n^2-1)/8}q^{(n^2-1)/8}\pmod{\Phi_n(q)},$$

and by [5, Lemma 2.1] there holds

$$(-aq,-q/a;q^2)_{(n-1)/2}\equiv\frac{(1+a^n)q^{(1-n^2)/4}}{(1+a)a^{(n-1)/2}}\pmod{\Phi_n(q)}.$$

Thus, observing the relation

$$(1-q^n)(1+a^2-a-aq^n)=(1-a)^2+(1-aq^n)(a-q^n),$$

and using $q^{n(n+3)/4}\equiv 1 \pmod{\Phi_n(q)}$, we may write the q-congruence (3.1) as: modulo $(1-aq^n)(a-q^n)\Phi_n(q)$,

$$\begin{aligned}
&\sum_{k=0}^{(n-1)/2}\frac{(1+q^{4k+1})(aq,q/a,-q,-q/;q^2)_k(q^2;q^4)_k}{(1+q)(q^2,q^2,-aq^2,-q^2/a;q^2)_k(q^4;q^4)_k}q^k\\
&\quad\equiv\frac{(1+q^n)(q^2;q^4)_{(n-1)/4}^2}{(1+q)(q^4;q^4)_{(n-1)/4}^2}+\frac{2(1-aq^n)(a-q^n)}{(1+q)(1-a)^2}\\
&\qquad\times\left\{\frac{(q^2;q^4)_{(n-1)/4}^2}{(q^4;q^4)_{(n-1)/4}^2}-\frac{(1+a)a^{(n-1)/2}(aq^2,q^2/a;q^4)_{(n-1)/4}^2}{(1+a^n)(q^2,q^4;q^4)_{(n-1)/4}^2}\right\}.
\end{aligned}\tag{3.2}$$

By L'Hôpital's rule, we have

$$\begin{aligned}
&\lim_{a\to 1}\frac{(1-aq^n)(a-q^n)}{(1-a)^2}\left\{\frac{(q^2;q^4)_{(n-1)/4}^2}{(q^4;q^4)_{(n-1)/4}^2}-\frac{(1+a)a^{(n-1)/2}(aq^2,q^2/a;q^4)_{(n-1)/4}^2}{(1+a^n)(q^2,q^4;q^4)_{(n-1)/4}^2}\right\}\\
&\quad=[n]^2\frac{(q^2;q^4)_{(n-1)/4}^2}{(q^4;q^4)_{(n-1)/4}^2}\left\{\frac{(n^2-1)(1-q)^2}{8}+2\sum_{j=1}^{(n-1)/4}\frac{q^{4j-2}}{[4j-2]^2}\right\}.
\end{aligned}\tag{3.3}$$

Letting $a\to 1$ in (3.2), using the above limit and the fact that $2/(1+q)\equiv(1+q^n)/(1+q) \pmod{\Phi_n(q)}$, we are led to the desired q-supercongruence (1.11).

4. Concluding remarks

In 2020, Mao and Pan [18] (see also [24, Theorem 1.3]) proved the following result: for any prime $p\equiv 1 \pmod 4$,

$$\sum_{k=0}^{(p+1)/2}\frac{(-\frac{1}{2})_k^3}{k!^3}\equiv 0 \pmod{p^2}.\tag{4.1}$$

The author and Zudilin [13] established a q-analogue of (4.1): for any odd integer n greater than 1,

$$\begin{aligned}
&\sum_{k=0}^{(n+1)/2}\frac{(1+q^{4k-1})\,(q^{-2};q^4)_k^3}{(1+q)\,(q^4;q^4)_k^3}\,q^{7k}\\
&\quad\equiv\frac{[n]_{q^2}(q;q^4)_{(n-1)/2}}{(q^7;q^4)_{(n-1)/2}}\,q^{(n-3)/2}\begin{cases}\pmod{\Phi_n(q)^3\Phi_n(-q)^3} & \text{if } n\equiv 1 \pmod 4,\\[2ex] \pmod{\Phi_n(q)^2\Phi_n(-q)^3} & \text{if } n\equiv 3 \pmod 4.\end{cases}
\end{aligned}$$

It would be interesting if one can generalize the above q-congruence to the modulus $\Phi_n(q)^3$ case for $n\equiv 3 \pmod 4$.

Declarations

Conflict of interest. The author declares no conflict of interest.

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