Some q-supercongruences from the Gasper and Rahman quadratic summation

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Abstract. We give four families of q-supercongruences modulo the square and cube of a cyclotomic polynomial from Gasper and Rahman's quadratic summation. As conclusions, we obtain four new supercongruences modulo p^2 or p^3, such as: for $d \geqslant 2, r \geqslant 1$ with $\gcd(d,r)=1$ and $d+r$ odd, and any prime $p \equiv d+r \pmod{2d}$ with $p \geqslant d+r$,

$$\sum_{k=0}^{p-1}(3dk+r)\frac{(\frac{r}{d})_k(\frac{d-r}{d})_k(\frac{r}{2d})_k^2(\frac{1}{2})_k}{k!^4(\frac{d+2r}{2d})_k} \equiv 0 \pmod{p^3},$$

where $(x)_n = x(x+1)\cdots(x+n-1)$ is the Pochhammer symbol. We also propose three related conjectures on q-supercongruences.

Keywords: q-supercongruences; p-adic Gamma function; Gasper and Rahman's quadratic summation; creative microscoping

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1. Introduction

Following the work of [13, 14, 16], applying a ${}_7F_6$ summation of Gessel and Stanton [2], He [6] established the following supercongruence:

$$\sum_{k=0}^{p-1}(6k+1)\frac{(\frac{1}{2})_k^3(\frac{1}{4})_k^2}{k!^5} \equiv \begin{cases} -p\Gamma_p(\frac{1}{4})^4 \pmod{p^2}, & \text{if } p \equiv 1 \pmod{4}, \\ 0 \pmod{p^2}, & \text{if } p \equiv 3 \pmod{4}, \end{cases} \tag{1.1}$$

where $(x)_n = x(x+1)\cdots(x+n-1)$ is the Pochhammer symbol and $\Gamma_p(x)$ is Morita's p-adic Gamma function [15]. Soon afterwards, Liu [8] further proved that (1.1) holds modulo p^3 by employing another ${}_7F_6$ summation in [2].

Recently, using a summation formula of Gasper and Rahman (see (1.9)) and the method of 'creative microscoping' introduced by the author and Zudilin [5], among other things, Wei [18] gave the following q-analogue of Liu's generalization of (1.1) modulo p^3:

for any positive odd integer n, modulo $[n]\Phi_n(q)^2$,

$$\sum_{k=0}^{n-1}[6k+1]\frac{(q;q^2)_k^2(q,q,q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k^3}q^{2k}$$
$$\equiv\begin{cases}[n]\dfrac{(q^2;q^4)_{(n-1)/4}^2}{(q^4;q^4)_{(n-1)/4}^2}, & \text{if } n\equiv 1 \pmod 4,\\ 0, & \text{if } n\equiv 3 \pmod 4.\end{cases} \tag{1.2}$$

Here and throughout the paper, the *q-integer* is defined as $[n]=(1-q^n)/(1-q)$, the *q-shifted factorial* is defined as $(a;q)_0=1$ and $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n\geqslant 1$ or $n=\infty$. For convenience, we also adopt the abbreviated notation $(a_1,a_2,\dots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n$ for $n=0,1,\dots,$ or $n=\infty$. Moreover, let $\Phi_n(q)$ be the n-th *cyclotomic polynomial*, which can be written as

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}}(q-\zeta^k),$$

where ζ is an n-th primitive root of unity. For two rational functions $A(q)$ and $B(q)$, and a polynomial $P(q)$ with integer coefficients, we say that $A(q)$ is congruent to $B(q)$ modulo $P(q)$, denoted $A(q)\equiv B(q)\pmod{P(q)}$, if the numerator of the reduced fraction $A(q)-B(q)$ is divisible by $P(q)$ in the polynomial ring $\mathbb{Z}[q]$. If $A(q)\equiv 0\pmod{P(q)}$, then we will also say that $A(q)$ is divisible by $P(q)$. It should be mentioned that q-supercongruences have been widely studied in recent years. See, for example, [3, 4, 7, 9–12, 17, 19].

In this paper, we shall give some generalizations of (1.2), where the modulo $[n]\Phi_n(q)^2$ condition will be replaced by the weaker condition modulo $\Phi_n(q)^3$ or $\Phi_n(q)^2$. Our first result can be formulated as follows.

Theorem 1.1. *Let $d\geqslant 2$ and $r\geqslant 1$ be integers such that $d+r$ is odd and $\gcd(d,r)=1$. Let n be a positive integer satisfying $n\equiv d+r \pmod{2d}$ and $n\geqslant d+r$. Then*

$$\sum_{k=0}^{n-1}[3dk+r]\frac{(q^r,q^{d-r};q^d)_k(q^r,q^r,q^d;q^{2d})_k}{(q^d;q^d)_k^2(q^{2d},q^{2d},q^{d+2r};q^{2d})_k}q^{dk}\equiv 0 \pmod{\Phi_n(q)^3}. \tag{1.3}$$

It is easy to see that the $(d,r)=(2,1)$ case of (1.3) is just the second part of (1.2) modulo $\Phi_n(q)^3$. Besides, taking $n=p$ to be a prime and $q\to 1$ in (1.3), we arrive at the following result: for $d,r>0$ with $\gcd(d,r)=1$ and $d+r$ odd, and any prime $p\equiv d+r \pmod{2d}$ with $p\geqslant d+r$,

$$\sum_{k=0}^{p-1}(3dk+r)\frac{(\frac{r}{d})_k(\frac{d-r}{d})_k(\frac{r}{2d})_k^2(\frac12)_k}{k!^4(\frac{d+2r}{2d})_k}\equiv 0 \pmod{p^3},$$

which is a generalization of (1.1) for $p\equiv 3\pmod 4$.

We shall also give another two generalizations of the $n\equiv 3 \pmod 4$ case of (1.2) modulo $\Phi_n(q)^2$.

Theorem 1.2. *Let d and r be positive integers with $r < d \leqslant 2r$. Let n be a positive integer with $n \equiv -1 \pmod{2d}$. Then*

$$\sum_{k=0}^{n-1}[3dk+r]\frac{(q^r,q^{d-r};q^d)_k(q^r,q^r,q^d;q^{2d})_k}{(q^d;q^d)_k^2(q^{2d},q^{2d},q^{d+2r};q^{2d})_k}q^{dk} \equiv 0 \pmod{\Phi_n(q)^2}. \tag{1.4}$$

Likewise, taking $n = p$ to be a prime and $q \to 1$ in (1.4), we get the following result: for $0 < r < d \leqslant 2r$, and any prime $p \equiv -1 \pmod{2d}$,

$$\sum_{k=0}^{p-1}(3dk+r)\frac{(\frac{r}{d})_k(\frac{d-r}{d})_k(\frac{r}{2d})_k^2(\frac{1}{2})_k}{k!^4(\frac{d+2r}{2d})_k} \equiv 0 \pmod{p^2}. \tag{1.5}$$

Theorem 1.3. *Let d and r be positive integers such that r is odd, $d > r$ and $\gcd(d,r) = 1$. Let n be a positive integer satisfying $n \equiv -r \pmod{2d}$. Suppose that $(d,r) \notin \{(3,1),(4,3)\}$. Then*

$$\sum_{k=0}^{n-1}[3dk+r]\frac{(q^r,q^{d-r};q^d)_k(q^r,q^r,q^d;q^{2d})_k}{(q^d;q^d)_k^2(q^{2d},q^{2d},q^{d+2r};q^{2d})_k}q^{dk} \equiv 0 \pmod{\Phi_n(q)^2}. \tag{1.6}$$

It is easy to see that (1.6) implies the following result: for the same (d,r) in Theorem 1.3 and any prime $p \equiv -r \pmod{2d}$,

$$\sum_{k=0}^{p-1}(3dk+r)\frac{(\frac{r}{d})_k(\frac{d-r}{d})_k(\frac{r}{2d})_k^2(\frac{1}{2})_k}{k!^4(\frac{d+2r}{2d})_k} \equiv 0 \pmod{p^2}. \tag{1.7}$$

Both (1.5) and (1.7) are generalizations of (1.1) for $p \equiv 3 \pmod 4$.

The last aim of this paper is to give a generalization of (1.2) modulo $\Phi_n(q)^3$ for $n \equiv 1 \pmod 4$.

Theorem 1.4. *Let d and r be positive integers such that r is odd and $\gcd(r,d) = 1$. Let n be a positive integer satisfying $n \equiv r \pmod{2d}$ with $n \geqslant r$. Then*

$$\sum_{k=0}^{n-1}[3dk+r]\frac{(q^r,q^{d-r};q^d)_k(q^r,q^r,q^d;q^{2d})_k}{(q^d;q^d)_k^2(q^{2d},q^{2d},q^{d+2r};q^{2d})_k}q^{dk} \equiv [n]\frac{(q^d,q^{2r};q^{2d})_{(n-r)/(2d)}}{(q^{2d},q^{d+2r};q^{2d})_{(n-r)/(2d)}} \pmod{\Phi_n(q)^3}. \tag{1.8}$$

It is clear that the $(d,r) = (2,1)$ case of (1.8) reduces to the first part of (1.2). Similarly as before, the q-supercongruence (1.8) leads to the following result: for the same (d,r) in Theorem 1.4 and any prime $p \equiv r \pmod{2d}$ with $p \geqslant r$,

$$\sum_{k=0}^{p-1}(3dk+r)\frac{(\frac{r}{d})_k(\frac{d-r}{d})_k(\frac{r}{2d})_k^2(\frac{1}{2})_k}{k!^4(\frac{d+2r}{2d})_k} \equiv p\frac{(\frac{1}{2})_{(p-r)/(2d)}(\frac{r}{d})_{(p-r)/(2d)}}{(1)_{(p-r)/(2d)}(\frac{d+2r}{2d})_{(p-r)/(2d)}} \pmod{p^2}.$$

Recall that the *basic hypergeometric series* ${}_{r+1}\phi_r$ (see [1]) is defined as

$$ {}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix} ; q,\, z\right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_k z^k}{(q, b_1, \dots, b_r; q)_k}. $$

The Gasper and Rahman quadratic summation (see [1, (3.8.12)]) can be stated as follows:

$$ \begin{aligned} &\sum_{k=0}^{\infty} \frac{1-aq^{3k}}{1-a} \frac{(a, b, q/b; q)_k (d, f, a^2q/df; q^2)_k}{(aq/d, aq/f, df/a; q)_k (q^2, aq^2/b, abq; q^2)_k} q^k \\ &\quad + \frac{(aq, f/a, b, q/b; q)_\infty (d, aq^2/df, fq^2/d, df^2q/a^2; q^2)_\infty}{(a/f, fq/a, aq/d, df/a; q)_\infty (aq^2/b, abq, fq/ab, bf/a; q^2)_\infty} \\ &\quad \times {}_3\phi_2\left[\begin{matrix} f, bf/a, fq/ab \\ fq^2/d, df^2q/a^2 \end{matrix} ; q^2, q^2\right] \\ &\quad = \frac{(aq, f/a; q)_\infty (aq^2/bd, abq/d, bdf/a, dfq/ab; q^2)_\infty}{(aq/d, df/a; q)_\infty (aq^2/b, abq, bf/a, fq/ab; q^2)_\infty}. \end{aligned} \tag{1.9} $$

We shall prove Theorems 1.1–1.4 by applying the 'creative microscoping' method and Gasper and Rahman's quadratic summation (1.9) again.

2. Proof of Theorem 1.1

We first give the following generalization of Theorem 1.1 with an extra parameter a.

Theorem 2.1. *Let $d \geqslant 2$ and $r \geqslant 1$ be integers such that $d+r$ is odd and $\gcd(d,r)=1$. Let n be a positive integer satisfying $n \equiv d+r \pmod{2d}$ and $n \geqslant d+r$, and let a be an indeterminate. Then, modulo $\Phi_n(q)(1-aq^{dn+n})(a-q^{dn+n})$,*

$$ \begin{aligned} &\sum_{k=0}^{(dn+n-r)/(2d)} [3dk+r] \frac{(q^r, q^{d-r}; q^d)_k (aq^r, q^r/a, q^d; q^{2d})_k}{(aq^d, q^d/a; q^d)_k (q^{2d}, q^{2d}, q^{d+2r}; q^{2d})_k} q^{dk} \\ &\quad \equiv [dn+n] \frac{(q^d, q^{2r}; q^{2d})_{(dn+n-r)/(2d)}}{(q^{2d}, q^{d+2r}; q^{2d})_{(dn+n-r)/(2d)}}. \end{aligned} \tag{2.1} $$

Proof. Letting $q \mapsto q^d, a = q^{r-n}, b = q^r$, $d = aq^r$, $f = q^r/a$ in (1.9) and noticing that $(q^{d+r-n}; q^d)_\infty = 0$, we get

$$ \sum_{k=0}^{(n-r)/d} \frac{(1-q^{3dk+r-n})(q^{r-n}, q^r, q^{d-r}; q^d)_k (aq^r, q^r/a, q^{d-2n}; q^{2d})_k}{(1-q^{r-n})(aq^{d-n}, q^{d-n}/a, q^{r+n}; q^d)_k (q^{2d}, q^{2d-n}, q^{d+2r-n}; q^{2d})_k} q^{dk} = 0, \tag{2.2} $$

where we have used $(q^{r-n}; q^d)_k = 0$ for $k > (n-r)/d$. Since $n \equiv d+r \pmod{2d}$, $d+r$ is odd, and $\gcd(d,r)=1$, we have $\gcd(2d,n)=1$. Note that $1-q^N \equiv 0 \pmod{\Phi_n(q)}$ if and only if N is a multiple of n. Thus, the smallest positive integer k such that $(q^{r+n}; q^d)_k \equiv 0$

(mod $\Phi_n(q)$) is $(n-r)/d+1$, and the smallest k for $(q^{d+2r-n};q^{2d})_k\equiv 0 \pmod{\Phi_n(q)}$ is $(dn+2n-d-2r)/(2d)+1$. This implies that the polynomial $(q^{r+n};q^d)_k(q^{d+2r};q^{2d})_k$ is coprime with $\Phi_n(q)$ for $0\leqslant k\leqslant (n-r)/d$ (since $0<(n-r)/d\leqslant (dn+2n-d-2r)/(2d)$). In view of $q^n\equiv 1 \pmod{\Phi_n(q)}$, we deduce from (2.2) that

$$\sum_{k=0}^{(n-r)/d}[3dk+r]\frac{(q^r,q^{d-r};q^d)_k(aq^r,q^r/a,q^d;q^{2d})_k}{(aq^d,q^d/a;q^d)_k(q^{2d},q^{2d},q^{d+2r};q^{2d})_k}q^{dk}\equiv 0 \pmod{\Phi_n(q)}, \tag{2.3}$$

and so (2.1) holds modulo $\Phi_n(q)$ since $(n-r)/d<(dn+n-r)/(2d)\leqslant (dn+2n-d-2r)/(2d)$.

On the other hand, performing the substitutions $q\mapsto q^d$, $a=q^r$, $b=q^r$, $d=q^{r-dn-n}$, and $f=q^{r+dn+n}$ in (1.9), and noticing that $(q^{r-dn-n};q^{2d})_\infty=0$, we obtain

$$\begin{aligned}
&\sum_{k=0}^{(dn+n-r)/(2d)}\frac{(1-q^{3dk+r})(q^r,q^{d-r};q^d)_k(q^{r-dn-n},q^{r+dn+n},q^d;q^{2d})_k}{(1-q^r)(q^{d+dn+n},q^{d-dn-n};q^d)_k(q^{2d},q^{2d},q^{d+2r};q^{2d})_k}q^{dk}\\
&=\frac{(q^{d+r},q^{dn+n};q^d)_\infty(q^{dn+n+2d-r},q^{dn+n+d+r},q^{2r},q^d;q^{2d})_\infty}{(q^{d+dn+n},q^r;q^d)_\infty(q^{2d},q^{d+2r},q^{dn+n+r},q^{dn+n+d-r};q^{2d})_\infty}\\
&=\frac{(1-q^{dn+n})(q^d,q^{2r};q^{2d})_{(dn+n-r)/(2d)}}{(1-q^r)(q^{2d},q^{d+2r};q^{2d})_{(dn+n-r)/(2d)}},
\end{aligned} \tag{2.4}$$

where we have used $(q^{r-dn-n};q^{2d})_k=0$ for $k>(dn+n-r)/(2d)$. This proves that the left-hand side of (2.1) is also equal to 0 for $a=q^{-dn-n}$ or $a=q^{dn+n}$. Namely, the q-congruence (2.1) is true modulo $1-aq^{dn+n}$ and $a-q^{dn+n}$.

Since $\Phi_n(q)$, $1-aq^{dn+n}$, and $a-q^{dn+n}$ are pairwise coprime polynomials in q, we complete the proof of the theorem. □

Proof of Theorem 1.1. Since $\gcd(d,n)=1$, the polynomial $(q^d;q^d)_k$ is coprime with $\Phi_n(q)$ for any $0\leqslant k\leqslant n-1$. Further, the polynomial $(1-q^n)^2$ contains the factor $\Phi_n(q)^2$. Letting $a=1$ in (2.1), we get

$$\begin{aligned}
&\sum_{k=0}^{(dn+n-r)/(2d)}[3dk+r]\frac{(q^r,q^{d-r};q^d)_k(q^r,q^r,q^d;q^{2d})_k}{(q^d,q^d;q^d)_k(q^{2d},q^{2d},q^{d+2r};q^{2d})_k}q^{dk}\\
&\equiv [dn+n]\frac{(q^d,q^{2r};q^{2d})_{(dn+n-r)/(2d)}}{(q^{2d},q^{d+2r};q^{2d})_{(dn+n-r)/(2d)}} \pmod{\Phi_n(q)^3}.
\end{aligned} \tag{2.5}$$

It is easy to see that $(q^d;q^{2d})_{(dn+n-r)/(2d))}$ contains the factor $1-q^{dn}$, and $(q^{2r};q^{2d})_{(dn+n-r)/(2d)}$ contains the factor $1-q^{2n}$, and so the right-hand side of (2.5) is divisible by $\Phi_n(q)^3$. Moreover, for $(dn+n-r)/(2d)<k\leqslant n-1$, the k-th summand on the left-hand side of (1.3) is divisible by $\Phi_n(q)^3$ too. This completes the proof. □

3. Proof of Theorem 1.2

Similarly as before, we first give the following parametric generalization of Theorem 1.2.

Theorem 3.1. *Let d and r be positive integers with $r < d \leqslant 2r$. Let n be a positive integer with $n \equiv -1 \pmod{2d}$, and let a be an indeterminate. Then, modulo $(1-aq^{(2d-r)n})(a-q^{(2d-r)n})$,*

$$\sum_{k=0}^{(2dn-rn-r)/(2d)} [3dk+r]\frac{(q^r, q^{d-r}; q^d)_k (aq^r, q^r/a, q^d; q^{2d})_k}{(aq^d, q^d/a; q^d)_k (q^{2d}, q^{2d}, q^{d+2r}; q^{2d})_k} q^{dk}$$
$$\equiv [2dn-rn]\frac{(q^d, q^{2r}; q^{2d})_{(2dn-rn-r)/(2d)}}{(q^{2d}, q^{d+2r}; q^{2d})_{(2dn-rn-r)/(2d)}}. \tag{3.1}$$

Proof. The proof is similar to that of Theorem 2.1. Taking $q \mapsto q^d$, $a = q^r$, $b = q^r$, $d = q^{r-(2d-r)n}$, and $f = q^{r+(2d-r)n}$ in (1.9), and noticing that $(q^{r-(2d-r)n}; q^{2d})_\infty = 0$, we obtain

$$\sum_{k=0}^{(2dn-rn-r)/(2d)} \frac{(1-q^{3dk+r})(q^r, q^{d-r}; q^d)_k (q^{r-(2d-r)n}, q^{r+(2d-r)n}, q^d; q^{2d})_k}{(1-q^r)(q^{d+(2d-r)n}, q^{d-(2d-r)n}; q^d)_k (q^{2d}, q^{2d}, q^{d+2r}; q^{2d})_k} q^{dk}$$
$$= \frac{(q^{d+r}, q^{(2d-r)n}; q^d)_\infty (q^{(2d-r)n+2d-r}, q^{(2d-r)n+d+r}, q^{2r}, q^d; q^{2d})_\infty}{(q^{d+(2d-r)n}, q^r; q^d)_\infty (q^{2d}, q^{d+2r}, q^{(2d-r)n+r}, q^{(2d-r)n+d-r}; q^{2d})_\infty}$$
$$= \frac{(1-q^{dn+n})(q^d, q^{2r}; q^{2d})_{(2dn-rn-r)/(2d)}}{(1-q^r)(q^{2d}, q^{d+2r}; q^{2d})_{(2dn-rn-r)/(2d)}}, \tag{3.2}$$

where we have used $(q^{r-(2d-r)n}; q^{2d})_k = 0$ for $k > (2dn-rn-r)/(2d)$. Thus, we have proved that the left-hand side of (3.1) is also equal to 0 for $a = q^{-(2d-r)n}$ or $a = q^{(2d-r)n}$. This means that (3.1) is true modulo $1-aq^{(2d-r)n}$ and $a-q^{(2d-r)n}$. □

Proof of Theorem 1.2. Since $n \equiv -1 \pmod{2d}$, we have $\gcd(2d, n) = 1$. Thus, the smallest positive integer k such that $(q^m; q^{2d})_k \equiv 0 \pmod{\Phi_n(q)}$ is $(2d-m)(n+1)/(2d)$ for m in the range $0 < m < 2d$. By the condition $r < d \leqslant 2r$, we get $0 \leqslant (d+2r)-2d < r$, and therefore $(q^{d+2r}; q^{2d})_k$ is coprime with $\Phi_n(q)$ for k in the range $0 \leqslant k \leqslant (2dn-rn-r)/(2d)$. Meanwhile, the polynomials $(q^d; q^d)_k$ and $(q^{2d}; q^{2d})_k$ are both coprime with $\Phi_n(q)$ for $0 \leqslant k \leqslant n-1$. Hence, specializing $a = 1$ in (3.1), we are led to

$$\sum_{k=0}^{(2dn-rn-r)/(2d)} [3dk+r]\frac{(q^r, q^{d-r}; q^d)_k (q^r, q^r, q^d; q^{2d})_k}{(q^d, q^d; q^d)_k (q^{2d}, q^{2d}, q^{d+2r}; q^{2d})_k} q^{dk}$$
$$\equiv [2dn-rn]\frac{(q^d, q^{2r}; q^{2d})_{(2dn-rn-r)/(2d)}}{(q^{2d}, q^{d+2r}; q^{2d})_{(2dn-rn-r)/(2d)}} \pmod{\Phi_n(q)^2}.$$

Since the right-hand of the above q-supercongruence is clearly divisible by $\Phi_n(q)^3$ and so is the k-th summand on the left-hand side of (1.4) for $(2dn-rn-r)/(2d) < k \leqslant n-1$, we complete the proof of the theorem. □

4. Proof of Theorem 1.3

We first present a parametric generalization of Theorem 1.3.

Theorem 4.1. *Let d and r be positive integers such that r is odd, $d > r$ and* $\gcd(d,r)=1$. *Let n be a positive integer satisfying $n \equiv -r \pmod{2d}$, and let a be an indeterminate. Then, modulo $(1-aq^{2dn-n})(a-q^{2dn-n})$,*

$$\sum_{k=0}^{(2dn-n-r)/(2d)} [3dk+r]\frac{(q^r, q^{d-r}; q^d)_k (aq^r, q^r/a, q^d; q^{2d})_k}{(aq^d, q^d/a; q^d)_k (q^{2d}, q^{2d}, q^{d+2r}; q^{2d})_k} q^{dk}$$
$$\equiv [2dn-n]\frac{(q^d, q^{2r}; q^{2d})_{(2dn-n-r)/(2d)}}{(q^{2d}, q^{d+2r}; q^{2d})_{(2dn-n-r)/(2d)}}. \tag{4.1}$$

Proof. The proof is again very similar to that of Theorem 2.1. This time we take $q \mapsto q^d$, $a=q^r$, $b=q^r$, $d=q^{r-(2d-1)n}$, and $f=q^{r+(2d-1)n}$ in (1.9) to get

$$\sum_{k=0}^{(2dn-n-r)/(2d)} \frac{(1-q^{3dk+r})(q^r, q^{d-r}; q^d)_k (q^{r-(2d-1)n}, q^{r+(2d-1)n}, q^d; q^{2d})_k}{(1-q^r)(q^{d+(2d-1)n}, q^{d-(2d-1)n}; q^d)_k (q^{2d}, q^{2d}, q^{d+2r}; q^{2d})_k} q^{dk}$$
$$= \frac{(1-q^{dn-n})(q^d, q^{2r}; q^{2d})_{(2dn-n-r)/(2d)}}{(1-q^r)(q^{2d}, q^{d+2r}; q^{2d})_{(2dn-n-r)/(2d)}}, \tag{4.2}$$

This proves that the q-congruence (4.1) is true modulo $1-aq^{(2d-1)n}$ and $a-q^{(2d-1)n}$. □

Proof of Theorem 1.3. Note that $d \neq 3$. For $d=2$ (and so $r=1$), the q-supercongruence (1.4) follows from (1.2) immediately. It is easy to see that $\gcd(2d,n)=1$, and the smallest positive integer k such that $(q^{d-r}; q^d)_k \equiv 0 \pmod{\Phi_n(q)}$ is $(n+r-d)/d+1$, while the smallest k satisfying $(q^{d+2r}; q^{2d})_k \equiv 0 \pmod{\Phi_n(q)}$ is $(dn-2n-d-2r)/(2d)+1$ (We can verify that $dn-2n-d-2r \geqslant 0$ according to $d=4$, and $d \geqslant 5$, respectively).

It is not difficult to see that $0 < (n+r-d)/d \leqslant (dn-2n-d-2r)/(2d)$ by the conditions in the theorem. Hence, the denominator of reduced form of $(q^{d-r}; q^d)_k/(q^{d+2r}; q^{2d})_k$ is coprime with $\Phi_n(q)$ for $0 \leqslant k \leqslant (dn-n-r)/d$. Thus, taking the value $a=1$ in (4.1), we arrive at

$$\sum_{k=0}^{(2dn-n-r)/(2d)} [3dk+r]\frac{(q^r, q^{d-r}; q^d)_k (q^r, q^r, q^d; q^{2d})_k}{(q^d, q^d; q^d)_k (q^{2d}, q^{2d}, q^{d+2r}; q^{2d})_k} q^{dk}$$
$$\equiv [2dn-n]\frac{(q^d, q^{2r}; q^{2d})_{(2dn-n-r)/(2d)}}{(q^{2d}, q^{d+2r}; q^{2d})_{(2dn-n-r)/(2d)}} \pmod{\Phi_n(q)^2},$$

which is equivalent to (1.6) for the same reason as before. □

5. Proof of Theorem 1.4

Likewise, we have the following parametric generalization of Theorem 1.4.

Theorem 5.1. *Let d and r be positive integers such that r is odd and* $\gcd(d,r)=1$. *Let n be a positive integer satisfying $n\equiv r \pmod{2d}$ with $n\geqslant r$, and let a be an indeterminate. Then, modulo $\Phi_n(q)(1-aq^n)(a-q^n)$,*

$$\sum_{k=0}^{(n-r)/d}[3dk+r]\frac{(q^r,q^{d-r};q^d)_k(aq^r,q^r/a,q^d;q^{2d})_k}{(aq^d,q^d/a;q^d)_k(q^{2d},q^{2d},q^{d+2r};q^{2d})_k}q^{dk} \equiv [n]\frac{(q^d,q^{2r};q^{2d})_{(n-r)/(2d)}}{(q^{2d},q^{d+2r};q^{2d})_{(n-r)/(2d)}}. \tag{5.1}$$

Proof. When $n=r$, both sides of (4.1) are equal to $[r]$ and so (5.1) holds. We now suppose that $n\geqslant 2d+r$. Letting $q\mapsto q^d, a=q^{r-n}, b=q^r,\ d=aq^r,\ f=q^r/a$ in (1.9) and noticing that $(q^{d+r-n};q^d)_\infty=0$ leads to (2.2) again. By the condition in the theorem, we have $\gcd(2d,n)=1$. Moreover, the same arguments imply that (2.3) holds, or equivalently (5.1) is true modulo $\Phi_n(q)$.

On the other hand, making the substitutions $q\mapsto q^d$, $a=q^r$, $b=q^r$, $d=q^{r-n}$, and $f=q^{r+n}$ in (1.9), we get

$$\sum_{k=0}^{(n-r)/(2d)}\frac{(1-q^{3dk+r})(q^r,q^{d-r};q^d)_k(q^{r-n},q^{r+n},q^d;q^{2d})_k}{(1-q^r)(q^{d+n},q^{d-n};q^d)_k(q^{2d},q^{2d},q^{d+2r};q^{2d})_k}q^{dk} = \frac{(1-q^n)(q^d,q^{2r};q^{2d})_{(n-r)/(2d)}}{(1-q^r)(q^{2d},q^{d+2r};q^{2d})_{(n-r)/(2d)}}. \tag{5.2}$$

Since $(q^{r-n};q^{2d})_k=0$ for $k>(n-r)/(2d)$, one sees that both sides of (5.1) are equal for $a=q^{-n}$ and $a=q^n$. In other words, the q-congruence (5.1) holds modulo $1-aq^n$ and $a-q^n$. □

Proof of Theorem 1.3. Since $\gcd(2d,n)=1$, the polynomial $(q^{2d};q^{2d})_k$ is coprime with $\Phi_n(q)$ for any $0\leqslant k\leqslant (n-r)/d$. Putting $a=1$ in (4.1) and noticing that the k-th summand on the left-hand side of (1.8) is divisible by $\Phi_n(q)^3$ too, we obtain (1.8). □

6. Concluding remarks

We point out that Wei [18] gave a generalization of (1.2) modulo $[n]\Phi_n(q)^4$ by using the method of 'creative microscoping' together with the Chinese remainder theorem for coprime polynomials. However, it seems difficult to give such generalizations of Theorems 1.1–1.4, even in the modulus $\Phi_n(q)^4$ case.

Although the condition $r<d\leqslant 2r$ is necessary in the proof of Theorem 1.2, we believe that this condition can be weakened as only $r<d$, which we formulate as follows.

Conjecture 6.1. *Let d and r be positive integers with $d > r$. Let n be a positive integer with $n \equiv -1 \pmod{2d}$. Then*

$$\sum_{k=0}^{n-1}[3dk+r]\frac{(q^r, q^{d-r}; q^d)_k (q^r, q^r, q^d; q^{2d})_k}{(q^d; q^d)_k^2 (q^{2d}, q^{2d}, q^{d+2r}; q^{2d})_k} q^{dk} \equiv 0 \pmod{\Phi_n(q)^2}. \tag{6.1}$$

It is easy to see that we may truncate the left-hand side of (6.1) at $k = (2dn - rn - r)/(2d)$, just like the left-hand side of (1.4).

Similarly, Theorem 1.3 has the following generalization.

Conjecture 6.2. *Let d and r be positive integers such that r is odd and $\gcd(d, r) = 1$. Let n be a positive integer satisfying $n \equiv -r \pmod{2d}$ and $dn \geqslant n + r$. Then (6.1) holds.*

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