A new family of q-supercongruences from a quadratic summation of Gasper and Rahman

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Abstract. We establish a new family of q-congruences modulo the third power of a cyclotomic polynomial from a quadratic summation formula due to Gasper and Rahman. The main ingredient of our proof is the creative microscoping method developed by the author in collaboration with Zudilin. Two special cases of our result partially confirm a recent conjecture by He and Wang [Proc. Amer. Math. Soc. 152 (2024), 4775–4784]. A related conjecture on q-congruences is also proposed at the end of this paper.

Keywords: q-supercongruences; p-adic Gamma function; Gasper and Rahman's quadratic summation; creative microscoping

AMS Subject Classifications: 33D15, 11A07, 11B65

1. Introduction

Employing the Wilf–Zeilberger (abbr. WZ) method [20, 21], Guillera and Zudilin [3] succeeded in proving the following supercongruence: for any prime $p > 2$,

$$\sum_{k=0}^{(p-1)/2} \frac{3k+1}{16^k}\binom{2k}{k}^3 \equiv p \pmod{p^3}. \tag{1.1}$$

Note that, on the left-hand side of (1.1), we can also sum over k from 0 to $p-1$, since the p-adic order of $(\frac{1}{2})_k/k!$ is 1 for k satisfying $(p-1)/2 < k \leqslant p-1$. Still by the WZ method and the summation package `Sigma` [14], Wang [17] gave the following supercongruence: for any prime $p > 3$,

$$\sum_{k=0}^{p-1}(3k-1)\frac{(\frac{1}{2})_k(-\frac{1}{2})_k^2}{k!^3}4^k \equiv p-2p^3 \pmod{p^4}, \tag{1.2}$$

which generalizes a result conjectured by the author and Schlosser [9, Conjecture 6.2].

By making use of the method of creative microscoping introduced in [11] and the Chinese reminder theorem for coprime polynomials, the author [5] established the following q-supercongruence: for any positive odd integer n,

$$\sum_{k=0}^{n-1}[3k+1]\frac{(q;q^2)_k^3 q^{-\binom{k+1}{2}}}{(q;q)_k^2(q^2;q^2)_k}$$
$$\equiv q^{(1-n)/2}[n] + \frac{(n^2-1)(1-q)^2}{24}q^{(1-n)/2}[n]^3 \pmod{[n]\Phi_n(q)^3}. \tag{1.3}$$

which was originally conjectured in [4]. Letting $n = p^m$ be a prime power and $q \to 1$ in (1.3), we are led to the supercongruence:

$$\sum_{k=0}^{p^m-1} \frac{(\frac{1}{2})_k^3}{k!^3}(3k+1)2^{2k} \equiv p^m \pmod{p^{m+3}} \quad \text{for } p>3. \tag{1.4}$$

A stronger version of (1.4) modulo p^{m+4} was formulated by Sun [15, Conjecture 5.1(ii)], and has recently been confirmed by Wang and Hu [18]. On the basis of (1.3), using the q-WZ method again, the author [6] obtained a q-analogue of (1.2) as follows:

$$\begin{aligned}\sum_{k=0}^{n-1}&[3k-1]\frac{(q;q^2)_k(q^{-1};q^2)_k^2}{(q;q)_k^2(q^2;q^2)_k} q^{(3k-k^2)/2}\\ &\equiv [n]q^{-(n+1)/2} - (1+q)[n]^3 + \frac{(n^2-1)(1-q)^2}{24}[n]^3 q^{-(n+1)/2} \pmod{[n]\Phi_n(q)^3}.\end{aligned} \tag{1.5}$$

Here we need to recall the standard q-notation. The *q-shifted factorial* is defined as $(a;q)_0 = 1$ and $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n = 1, 2, \ldots,$ or $n = \infty$, and the *q-integer* is defined by $[n] = (1-q^n)/(1-q)$. For simplicity, we will often adopt the shorthand notation $(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n(a_2;q)_n \cdots (a_m;q)_n$. In addition, the n-th *cyclotomic polynomial* $\Phi_n(q)$ is given by

$$\Phi_n(q) = \prod_{\substack{1 \leqslant k \leqslant n\\ \gcd(k,n)=1}} (q-\zeta^k),$$

where ζ is an n-th primitive root of unity.

Recently, among other things, He and Wang [12] proved the following q-supercongruence: for any odd integer $n > 1$ and $\varepsilon = \pm 1$, modulo $[n]\Phi_n(q)^2$,

$$\sum_{k=0}^{n-1}[6k+\varepsilon]\frac{(q^\varepsilon, q^\varepsilon, q^2; q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k} q^{(1-\varepsilon)k-k^2} \equiv \begin{cases} [n]q^{(\varepsilon-n)/2}, & \text{if } n \equiv \varepsilon \pmod 4,\\ [3n]q^{(\varepsilon-3n)/2}, & \text{if } n \equiv -\varepsilon \pmod 4. \end{cases} \tag{1.6}$$

In this paper, we shall establish a new family of q-supercongruences similar to (1.3), (1.5), and (1.6) in the modulus $[n]\Phi_n(q)^2$ case as follows.

Theorem 1.1. *Let d be a positive integer and r an arbitrary integer. Let $n > 1$ be an odd integer with $\gcd(d, n) = 1$. Suppose that λ is an integer satisfying $\lambda n \equiv r \pmod{2d}$ and $(n-1)/2 \leqslant (\lambda n - r)/d \leqslant n-1$. Then*

$$\sum_{k=0}^{n-1}[3dk+r]\frac{(q^r, q^r, q^d; q^{2d})_k}{(q^d;q^d)_k^2(q^{2d};q^{2d})_k} q^{k(d-dk-2r)/2} \equiv [\lambda n]q^{(r-\lambda n)/2} \pmod{[n]\Phi_n(q)^2}.$$

For $n \equiv \pm r, \pm 1 \pmod{2d}$, we obtain the following four corollaries.

Corollary 1.2. *Let $d \geqslant 1$ and r be integers with r odd* $\gcd(d,r)=1$*. Let $n>1$ be an integer satisfying $n \equiv r \pmod{2d}$ and $(n-1)/2 \leqslant (n-r)/d \leqslant n-1$. Then*

$$\sum_{k=0}^{n-1}[3dk+r]\frac{(q^r,q^r,q^d;q^{2d})_k}{(q^d;q^d)_k^2(q^{2d};q^{2d})_k}q^{k(d-dk-2r)/2} \equiv [n]q^{(r-n)/2} \pmod{[n]\Phi_n(q)^2}. \tag{1.7}$$

Taking $n=p^m$ to be a prime power and $q\to 1$ in (1.7), we obtain the following supercongruence: for any integers d,m,r such that $d,m\geqslant 1$, r odd, and $\gcd(d,r)=1$, and any prime p with $p^m\equiv r \pmod{2d}$ and $(p^m-1)/2\leqslant (p^m-r)/d\leqslant p^m-1$, there holds

$$\sum_{k=0}^{p^m-1}(3dk+r)\frac{(\frac{r}{2d})_k^2(\frac12)_k}{k!^3}4^k \equiv p^m \pmod{p^{m+2}}.$$

Corollary 1.3. *Let $d \geqslant 1$ and r be integers with r odd* $\gcd(d,r)=1$*. Let $n>1$ be an integer satisfying $n \equiv -r \pmod{2d}$ and $(n-1)/2 \leqslant (2dn-n-r)/d \leqslant n-1$. Then*

$$\sum_{k=0}^{n-1}[3dk+r]\frac{(q^r,q^r,q^d;q^{2d})_k q^{k(d-dk-2r)/2}}{(q^d;q^d)_k^2(q^{2d};q^{2d})_k} \equiv [2dn-n]q^{(r+n-2dn)/2} \pmod{[n]\Phi_n(q)^2}. \tag{1.8}$$

Likewise, letting $n=p^m$ be a prime power and $q\to 1$ in (1.8), we are led to the following result: for any integers d,m,r such that $d,m\geqslant 1$, r odd, and $\gcd(d,r)=1$, and any prime p with $p^m\equiv -r \pmod{2d}$ and $(p^m-1)/2\leqslant (2dp^m-p^m-r)/d\leqslant p^m-1$, there holds

$$\sum_{k=0}^{p^m-1}(3dk+r)\frac{(\frac{r}{2d})_k^2(\frac12)_k}{k!^3}4^k \equiv (2d-1)p^m \pmod{p^{m+2}}.$$

Note that Corollaries 1.2 and 1.3 partially confirm a recent conjecture of He and Wang [12, Conjecture 2.6].

Corollary 1.4. *Let d and r be positive integers with* $\gcd(d,r)=1$ *and $r\leqslant d\leqslant 2r$. Let $n>1$ be an integer satisfying $n\equiv 1 \pmod{2d}$. Then*

$$\sum_{k=0}^{n-1}[3dk+r]\frac{(q^r,q^r,q^d;q^{2d})_k}{(q^d;q^d)_k^2(q^{2d};q^{2d})_k}q^{k(d-dk-2r)/2} \equiv [rn]q^{r(1-n)/2} \pmod{[n]\Phi_n(q)^2}. \tag{1.9}$$

It is not difficult to see that (1.9) indicates the following result: for any positive integers d,m,r such that $\gcd(d,r)=1$ and $r\leqslant d\leqslant 2r$, and any prime p with $p^m\equiv 1 \pmod{2d}$, there holds

$$\sum_{k=0}^{p^m-1}(3dk+r)\frac{(\frac{r}{2d})_k^2(\frac12)_k}{k!^3}4^k \equiv rp^m \pmod{p^{m+2}}.$$

Corollary 1.5. *Let d and r be positive integers with* $\gcd(d,r)=1$*. Let $n>1$ be an integer satisfying $n\equiv -1 \pmod{2d}$ and $(n-1)/2\leqslant (2dn-rn-r)/d\leqslant n-1$. Then*

$$\sum_{k=0}^{n-1}[3dk+r]\frac{(q^r,q^r,q^d;q^{2d})_k}{(q^d;q^d)_k^2(q^{2d};q^{2d})_k}q^{k(d-dk-2r)/2}\equiv [(2d-r)n]q^{(r+rn-2dn)/2}\pmod{[n]\Phi_n(q)^2}. \tag{1.10}$$

Similarly, the q-supercongruence (1.10) implies that, for any positive integers d,m,r such that $\gcd(d,r)=1$, and any prime p with $p^m\equiv -1\pmod{2d}$ and $(p^m-1)/2\leqslant (2dp^m-rp^m-r)/d\leqslant p^m-1$, there holds

$$\sum_{k=0}^{p^m-1}(3dk+r)\frac{(\frac{r}{2d})_k^2(\frac12)_k}{k!^3}4^k\equiv (2d-r)p^m\pmod{p^{m+2}}.$$

We can also give another four corollaries for $n\equiv d\pm r, d\pm 1\pmod{2d}$, which are left to the interested reader.

Recall that a quadratic summation of Gasper and Rahman (see [1, (3.8.12)]) can be written as follows:

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{1-aq^{3k}}{1-a}\frac{(a,b,q/b;q)_k(d,f,a^2q/df;q^2)_k}{(aq/d,aq/f,df/a;q)_k(q^2,aq^2/b,abq;q^2)_k}q^k\\
&\quad+\frac{(aq,f/a,b,q/b;q)_\infty(d,aq^2/df,fq^2/d,df^2q/a^2;q^2)_\infty}{(a/f,fq/a,aq/d,df/a;q)_\infty(aq^2/b,abq,fq/ab,bf/a;q^2)_\infty}\\
&\quad\times {}_3\phi_2\left[\begin{matrix}f,bf/a,fq/ab\\ fq^2/d,df^2q/a^2\end{matrix};q^2,q^2\right]\\
&\quad=\frac{(aq,f/a;q)_\infty(aq^2/bd,abq/d,bdf/a,dfq/ab;q^2)_\infty}{(aq/d,df/a;q)_\infty(aq^2/b,abq,bf/a,fq/ab;q^2)_\infty},\end{aligned}\tag{1.11}$$

where the *basic hypergeometric series* ${}_{r+1}\phi_r$ is defined as

$${}_{r+1}\phi_r\left[\begin{matrix}a_1,a_2,\dots,a_{r+1}\\ b_1,b_2,\dots,b_r\end{matrix};q,\ z\right]=\sum_{k=0}^{\infty}\frac{(a_1,a_2,\dots,a_{r+1};q)_kz^k}{(q,b_1,\dots,b_r;q)_k}.$$

Wei [19] first utilized Gasper and Rahman's summation (1.11) to give a q-analogue of a supercongruence modulo p^3 of Liu [13]. Inspired by Wei's work, the author [7] deduced more supercongruences from (1.11). He and Wang [12] made use of this summation to prove the aforementioned q-supercongruence (1.6). Gu and Wang [2] also utilized (1.11) to confirm two conjectures in [8, 16]. Here we shall give a proof of Theorem 1.1 by using the creative microscoping method and Gasper and Rahman's summation (1.11) once more. We shall consider a new special case of this summation, which is different from those in [7, 12, 19].

2. Proof of Theorem 1.1

We first give the following parametric version of Theorem 1.1.

Lemma 2.1. *Let d be a positive integer and r an arbitrary integer. Let $n>1$ be an odd integer with* $\gcd(d,n)=1$, *and let a be an indeterminate. Suppose that λ is an integer satisfying* $\lambda n\equiv r \pmod{2d}$ *and* $(n-1)/2\leqslant (\lambda n-r)/d\leqslant n-1$. *Then, modulo* $\Phi_n(q)(1-aq^{\lambda n})(a-q^{\lambda n})$,

$$\sum_{k=0}^{n-1}[3dk+r]\frac{(aq^r,q^r/a,q^d;q^{2d})_k}{(aq^d,q^d/a;q^d)_k(q^{2d};q^{2d})_k}q^{k(d-dk-2r)/2}\equiv[\lambda n]q^{(r-\lambda n)/2}. \tag{2.1}$$

Proof. Putting $b=q^{-2m}$ in (1.11) and multiplying both sides by $1-a$, we obtain

$$\begin{aligned}&\sum_{k=0}^{2m}\frac{(1-aq^{3k})(a,q^{-2m},q^{1+2m};q)_k(d,f,a^2q/df;q^2)_kq^k}{(aq/d,aq/f,df/a;q)_k(q^2,aq^{2+2m},aq^{1-2m};q^2)_k}\\&\quad=(1-a)\frac{(aq,aq^2,f/a,fq/a,aq^{2+2m}/d,aq^{1-2m}/d,dfq^{-2m}/a,dfq^{1+2m}/a;q^2)_\infty}{(aq/d,aq^2/d,df/a,dfq/a,aq^{2+2m},aq^{1-2m},fq^{-2m}/a,fq^{1+2m}/a;q^2)_\infty}\\&\quad=(1-a)\frac{(aq^2,dq/a,fq/a,aq^2/df;q^2)_{2m}}{(q/a,aq^2/d,aq^2/f,dfq/a;q^2)_{2m}},\end{aligned} \tag{2.2}$$

which was implicitly noticed by He and Wang [12].

Performing the parameter substitutions $a=q^{r-\lambda n}$, $q\mapsto q^d$, $d=aq^r$, and $f=q^r/a$ in (2.2), and then taking $m\to\infty$, we obtain

$$\begin{aligned}&\sum_{k=0}^{(\lambda n-r)/d}(1-q^{3dk+r-\lambda n})\frac{(q^{r-\lambda n};q^d)_k(aq^r,q^r/a,q^{d-2\lambda n};q^{2d})_k}{(q^{d-\lambda n}/a,aq^{d-\lambda n},q^{r+\lambda n};q^d)_k(q^{2d};q^{2d})_k}q^{d(k-k^2)/2-k(r-\lambda n)}\\&\quad=\frac{(q^{r-\lambda n},aq^{d+\lambda n},q^{d+\lambda n}/a,q^{2d-r-\lambda n};q^{2d})_\infty}{(q^{d-r+\lambda n},q^{2d-\lambda n}/a,aq^{2d-\lambda n},q^{d+r+\lambda n};q^{2d})_\infty}\\&\quad=0,\end{aligned} \tag{2.3}$$

where we have used the fact that $(q^{r-\lambda n};q^d)_k=0$ for $k>(\lambda n-r)/d$ and $(q^{r-\lambda n};q^{2d})_\infty=0$. It is easy to see that all the denominators on the left-hand side of (2.3) are coprime with $\Phi_n(q)$. In view of $q^n\equiv 1 \pmod{\Phi_n(q)}$, we conclude that

$$\sum_{k=0}^{(\lambda n-r)/d}[3dk+r]\frac{(aq^r,q^r/a,q^d;q^{2d})_k}{(aq^d,q^d/a;q^d)_k(q^{2d};q^{2d})_k}q^{k(d-dk-2r)/2}\equiv 0 \pmod{\Phi_n(q)}. \tag{2.4}$$

Since $(n-1)/2\leqslant(\lambda n-r)/d\leqslant n-1$, and $(q^d;q^{2d})_k/(q^{2d};q^{2d})_k\equiv 0 \pmod{\Phi_n(q)}$ for $(n-1)/2\leqslant k\leqslant n-1$, we deduce from (2.4) that

$$\sum_{k=0}^{n-1}[3dk+r]\frac{(aq^r,q^r/a,q^d;q^{2d})_k}{(aq^d,q^d/a;q^d)_k(q^{2d};q^{2d})_k}q^{k(d-dk-2r)/2}\equiv 0 \pmod{\Phi_n(q)}. \tag{2.5}$$

Namely, the q-congruence (2.1) holds modulo $\Phi_n(q)$.

We now consider the q-congruence (2.1) modulo $(1-aq^{\lambda n})(a-q^{\lambda n})$. For $a=q^{-\lambda n}$ or $a=q^{\lambda n}$, the left-hand side of (2.1) can be written as

$$\sum_{k=0}^{(\lambda n-r)/(2d)}[3dk+r]\frac{(q^{r-\lambda n},q^{r+\lambda n},q^d;q^{2d})_k}{(q^{d-\lambda n},q^{d+\lambda n};q^d)_k(q^{2d};q^{2d})_k}q^{k(d-dk-2r)/2},$$

where we have used the fact that $(q^{r-\lambda n};q^{2d})_k=0$ for $k>(\lambda n-r)/(2d)$. Making the parameter substitutions $a=q^r$, $q\mapsto q^d$, $d=q^{r-\lambda n}$, and $f=q^{r+\lambda n}$ in (2.2), we see that the above sum is equal to

$$\begin{aligned}\frac{(q^r,q^{d-\lambda n},q^{d+\lambda n},q^{2d-r};q^{2d})_\infty}{(1-q)(q^{d-r},q^{2d+\lambda n},q^{2d-\lambda n},q^{d+r};q^{2d})_\infty}&=[r]\frac{(q^{d-\lambda n},q^{2d+r};q^{2d})_{(\lambda n-r)/(2d)}}{(q^{d+r},q^{2d-\lambda n};q^{2d})_{(\lambda n-r)/(2d)}}\\&=[\lambda n]q^{(r-\lambda n)/2},\end{aligned}$$

which is just the right-hand side of (2.1). This proves the truth of (2.1) modulo $(1-aq^{\lambda n})$ and $(a-q^{\lambda n})$. Since $\Phi_n(q)$, $(1-aq^{\lambda n})$, and $(a-q^{\lambda n})$ are pairwise coprime polynomials, we complete the proof of (2.1). $\square$

We also need to build the following lemma.

Lemma 2.2. *Let d be a positive integer and r an arbitrary integer. Let $n>1$ be an odd integer with $\gcd(d,n)=1$. Then*

$$\sum_{k=0}^{n-1}[3dk+r]\frac{(q^r,q^r,q^d;q^{2d})_k}{(q^d;q^d)_k^2(q^{2d};q^{2d})_k}q^{k(d-dk-2r)/2}\equiv 0\pmod{[n]}.\tag{2.6}$$

Proof. The proof is similar to that of [10, Lemma 2.2]. Let $\zeta\neq 1$ be an n-th root of unity. In other words, ζ is a primitive root of unity of odd degree n_1 subject to $n_1\mid n$. Let $c_q(k)$ denote the k-th term on the left-hand side of (2.6), i.e.,

$$c_q(k)=[3dk+r]\frac{(q^r,q^r,q^d;q^{2d})_k}{(q^d;q^d)_k^2(q^{2d};q^{2d})_k}q^{k(d-dk-2r)/2}.$$

If $r\equiv 0\pmod{n_1}$, then $c_\zeta(k)=0$ for any $k\geqslant 0$, and so

$$\sum_{k=0}^{n-1}c_\zeta(k)=0.$$

We now consider the case where $r\not\equiv 0\pmod{n_1}$. The q-congruence (2.1) modulo $\Phi_n(q)$ with $n=n_1$ and $a=1$ indicates that

$$\sum_{k=0}^{n_1-1}c_\zeta(k)=0.$$

It is not difficult to see that, for all non-negative integers ℓ and k,

$$\frac{c_\zeta(\ell n_1+k)}{c_\zeta(\ell n_1)}=\lim_{q\to\zeta}\frac{c_q(\ell n_1+k)}{c_q(\ell n_1)}=\frac{c_\zeta(k)}{[r]_\zeta},$$

and so

$$\sum_{k=0}^{n-1}c_\zeta(k)=\sum_{\ell=0}^{n/n_1-1}\sum_{k=0}^{n_1-1}c_\zeta(\ell n_1+k)=\frac{1}{[r]}\sum_{\ell=0}^{n/n_1-1}c_\zeta(\ell n_1)\sum_{k=0}^{n_1-1}c_\zeta(k)=0.$$

This means that the sum $\sum_{k=0}^{n-1}c_q(k)$ is congruent to 0 modulo $\Phi_{n_1}(q)$. Letting n_1 range over all divisors of n greater than 1, we conclude that this sum is congruent to 0 modulo

$$\prod_{n_1\mid n,\, n_1>1}\Phi_{n_1}(q)=[n],$$

thus completing the proof of (2.6). □

3. An open problem

Note that the condition $(n-1)/2\leqslant(\lambda n-r)/d$ is necessary in our proof of (2.5). Inspired by [12, Conjecture 2.6], we believe that this condition in Theorem 1.1 can be weakened as $0\leqslant(\lambda n-r)/d$. Namely, the following conjecture should be true.

Conjecture 3.1. *Let d be a positive integer and r an arbitrary integer. Let $n>1$ be an odd integer with $\gcd(d,n)=1$. Suppose that λ is an integer satisfying $\lambda n\equiv r \pmod{2d}$ and $0\leqslant(\lambda n-r)/d\leqslant n-1$. Then*

$$\sum_{k=0}^{n-1}[3dk+r]\frac{(q^r,q^r,q^d;q^{2d})_k}{(q^d;q^d)_k^2(q^{2d};q^{2d})_k}q^{k(d-dk-2r)/2}\equiv[\lambda n]q^{(r-\lambda n)/2}\pmod{[n]\Phi_n(q)^2}.$$

It is easy to see that the $n\equiv\pm r\pmod{2d}$ case of Conjecture 3.1 reduces to [12, Conjecture 2.6]. On the other hand, the $n\equiv 1\pmod{2d}$ case of Conjecture 3.1, which is also a generalization of Corollary 1.4, can be restated as follows:

Conjecture 3.2. *Let d and r be positive integers with $\gcd(d,r)=1$ and $r\leqslant d$. Let $n>1$ be an integer satisfying $n\equiv 1\pmod{2d}$. Then*

$$\sum_{k=0}^{n-1}[3dk+r]\frac{(q^r,q^r,q^d;q^{2d})_k}{(q^d;q^d)_k^2(q^{2d};q^{2d})_k}q^{k(d-dk-2r)/2}\equiv[rn]q^{r(1-n)/2}\pmod{[n]\Phi_n(q)^2}.\qquad(3.1)$$

Clearly, the $r=d=1$ case of (3.1) immediately follows from (1.3).

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