Two families of q-congruences from Watson's ${}_8\phi_7$ transformation

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Abstract. By making use of Watson's ${}_8\phi_7$ transformation formula, we prove two families of q-congruences modulo the square of a cyclotomic polynomial. As conclusions, we confirm a supercongruence conjecture of the author and Wei, and also partially confirm another supercongruence conjecture of theirs. We put forward several conjectures on q-congruences for further study.

Keywords: cyclotomic polynomial; q-congruence; supercongruence; Watson's transformation

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1. Introduction

In 1997, Van Hamme [26] proposed 13 interesting supercongruences and proved three of them himself, such as

$$\sum_{k=0}^{(p-1)/2}\frac{(\frac{1}{2})_k^3}{k!^3}\equiv\begin{cases}-\Gamma_p(1/4)^4 \pmod{p^2}, & \text{if } p\equiv 1 \pmod 4,\\ 0 \pmod{p^2}, & \text{if } p\equiv 3 \pmod 4,\end{cases}\tag{1.1}$$

where p is a prime, $(a)_n=a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol, and $\Gamma_p(x)$ is the p-adic Gamma function [23]. In 2019, the author and Zudilin [15, Theorem 2] obtained a q-analogue of (1.1) as follows: modulo $\Phi_n(q)^2$,

$$\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{2k}\equiv\begin{cases}\dfrac{(q^2;q^4)_{(n-1)/4}^2}{(q^4;q^4)_{(n-1)/4}^2}q^{(n-1)/2} & \text{if } n\equiv 1 \pmod 4,\\ 0 & \text{if } n\equiv 3 \pmod 4,\end{cases}\tag{1.2}$$

which generalizes a result of the author and Zeng [12, Corollary 1.2]. Here and in what follows, the *q-shifted factorial* is defined as $(a;q)_0=1$ and $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n\geqslant 1$, the *q-integer* is defined as $[n]=1+q+\cdots+q^{n-1}$, and $\Phi_n(q)$ denotes the n-th *cyclotomic polynomial* in q, which can be written as

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(n,k)=1}}(q-\zeta^k),$$

where ζ is an n-th primitive root of unity. Moreover, two rational functions $A(q)$ and $B(q)$ in q are said to be congruent modulo a polynomial $P(q)$, denoted $A(q)\equiv B(q)$

(mod $P(q)$), if the numerator of the reduced form of $A(q)-B(q)$ is divisible by $P(q)$ in the polynomial ring $\mathbb{Z}[q]$.

Further generalizations of (1.2) modulo $\Phi_n(q)^3$ can be found in [5, 6, 28, 30]. Many other generalizations of (1.1) can be found in the literature now. For example, Liu [18] proved that, for any prime $p\equiv 3 \pmod 4$ and positive integer m,

$$\sum_{k=0}^{mp-1}\frac{(\frac{1}{2})_k^3}{k!^3}\equiv 0 \pmod{p^2}. \tag{1.3}$$

In 2015, Swisher [25] proved the following supercongruence:

$$\sum_{k=0}^{(p-1)/4}(8k+1)\frac{(\frac{1}{4})_k^4}{k!^4}\equiv p\frac{\Gamma_p(\frac{1}{2})\Gamma_p(\frac{1}{4})}{\Gamma_p(\frac{3}{4})} \pmod{p^3},\quad \text{if } p\equiv 1 \pmod 4, \tag{1.4}$$

which was originally observed by Van Hamme [26, (G.2)]. Liu and Wang [19] noticed that (1.4) can also be derived from the following q-supercongruence: for $n\equiv 1 \pmod 4$,

$$\sum_{k=0}^{(n-1)/4}[8k+1]\frac{(q;q^4)_k^4}{(q^4;q^4)_k^4}q^{2k}\equiv [n]\frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}q^{(1-n)/4} \pmod{[n]\Phi_n(q)^2} \tag{1.5}$$

(see [8, Theorem 4.3] for a more general result). On the other hand, the author and Schlosser [10, Theorem 2 with $d=4$] showed that, for $n\equiv 3 \pmod 4$,

$$\sum_{k=0}^{n-1}[8k+1]\frac{(q;q^4)_k^4}{(q^4;q^4)_k^4}q^{2k}\equiv 0 \pmod{\Phi_n(q)^2}. \tag{1.6}$$

The objective of this paper is to give some generalizations of (1.6). Our first result can be stated as follows.

Theorem 1.1. *Let m and n be positive integers with $n\equiv 3 \pmod 4$. Then*

$$\sum_{k=0}^{mn-1}[8k+1]\frac{(q;q^4)_k^4}{(q^4;q^4)_k^4}q^{2k}\equiv 0 \pmod{\Phi_n(q)^2}, \tag{1.7}$$

$$\sum_{k=0}^{mn+(3n-1)/4}[8k+1]\frac{(q;q^4)_k^4}{(q^4;q^4)_k^4}q^{2k}\equiv 0 \pmod{\Phi_n(q)^2}. \tag{1.8}$$

Letting $n=p$ be a prime and taking the limits as $q\to 1$ in (1.7), we obtain a supercongruence similar to (1.3): for any prime $p\equiv 3 \pmod 4$ and positive integer m,

$$\sum_{k=0}^{mp-1}(8k+1)\frac{(\frac{1}{4})_k^4}{k!^4}\equiv 0 \pmod{p^2}.$$

We shall also give the following refinement of Theorem 1.1 for certain m.

Theorem 1.2. *Let $n\equiv 3 \pmod 4$ be a positive integer. Then, modulo $\Phi_n(q)^2\Phi_{n^2}(q)^3$,*

$$\sum_{k=0}^{(n^2-1)/4}[8k+1]\frac{(q;q^4)_k^4}{(q^4;q^4)_k^4}q^{2k}\equiv [n^2]\frac{(q^2;q^4)_{(n^2-1)/4}}{(q^4;q^4)_{(n^2-1)/4}}q^{(1-n^2)/4}, \tag{1.9}$$

$$\sum_{k=0}^{n^2-1}[8k+1]\frac{(q;q^4)_k^4}{(q^4;q^4)_k^4}q^{2k}\equiv [n^2]\frac{(q^2;q^4)_{(n^2-1)/4}}{(q^4;q^4)_{(n^2-1)/4}}q^{(1-n^2)/4}. \tag{1.10}$$

Let $n=p$ be a prime and let $q\to 1$ in (1.9). Since $\Phi_p(1)=\Phi_{p^2}(1)=p$, and

$$\lim_{q\to 1}\frac{(q^2;q^4)_{(p^2-1)/4}}{(q^4;q^4)_{(p^2-1)/4}}=\frac{(\frac{1}{2})_{(p^2-1)/4}}{(1)_{(p^2-1)/4}}$$

and so on, we immediately get the following conclusion.

Corollary 1.3. *Let $p\equiv 3 \pmod 4$ be a prime. Then*

$$\sum_{k=0}^{(p^2-1)/4}(8k+1)\frac{(\frac{1}{4})_k^4}{k!^4}\equiv p^2\frac{(\frac{1}{2})_{(p^2-1)/4}}{(1)_{(p^2-1)/4}} \pmod{p^5}. \tag{1.11}$$

In 2020, Mao and Pan [21] (see also Sun [24, Theorem 1.3]) proved that, for any prime $p\equiv 1 \pmod 4$,

$$\sum_{k=0}^{(p+1)/2}\frac{(-\frac{1}{2})_k^3}{k!^3}\equiv 0 \pmod{p^2}. \tag{1.12}$$

Shortly afterwards, the author and Zudilin [14] gave a new q-analogue of (1.1): for any positive odd integer n,

$$\sum_{k=0}^{(n-1)/2}\frac{(1+q^{4k+1})(q^2;q^4)_k^3}{(1+q)(q^4;q^4)_k^3}q^k \equiv \frac{[n]_{q^2}(q^3;q^4)_{(n-1)/2}}{(q^5;q^4)_{(n-1)/2}}q^{(1-n)/2}\begin{cases}\pmod{\Phi_n(q)^2} & \text{if } n\equiv 1 \pmod 4,\\ \pmod{\Phi_n(q)^3} & \text{if } n\equiv 3 \pmod 4,\end{cases} \tag{1.13}$$

and a q-analogue of (1.12): for any odd integer $n>1$,

$$\sum_{k=0}^{(n+1)/2}\frac{(1+q^{4k-1})(q^{-2};q^4)_k^3}{(1+q)(q^4;q^4)_k^3}q^{7k} \equiv \frac{[n]_{q^2}(q;q^4)_{(n-1)/2}}{(q^7;q^4)_{(n-1)/2}}q^{(n-3)/2}\begin{cases}\pmod{\Phi_n(q)^3} & \text{if } n\equiv 1 \pmod 4,\\ \pmod{\Phi_n(q)^2} & \text{if } n\equiv 3 \pmod 4.\end{cases} \tag{1.14}$$

In this note, we shall prove the following generalization of (1.14) modulo $\Phi_n(q)^2$ for $n\equiv 1 \pmod 4$.

Theorem 1.4. *Let m and n be positive integers with $n \equiv 1 \pmod 4$ and $n > 1$. Then*

$$\sum_{k=0}^{mn-1} \frac{(1+q^{4k-1})(q^{-2};q^4)_k^3}{(1+q)(q^4;q^4)_k^3} q^{7k} \equiv 0 \pmod{\Phi_n(q)^2}, \tag{1.15}$$

$$\sum_{k=0}^{mn+(n+1)/2} \frac{(1+q^{4k-1})(q^{-2};q^4)_k^3}{(1+q)(q^4;q^4)_k^3} q^{7k} \equiv 0 \pmod{\Phi_n(q)^2}. \tag{1.16}$$

It is easy to see that, from (1.15) we can deduce the following result: for any prime $p \equiv 1 \pmod 4$ and positive integer m,

$$\sum_{k=0}^{mp-1} \frac{(-\frac{1}{2})_k^3}{k!^3} \equiv 0 \pmod{p^2},$$

which is very similar to (1.3).

We point out that similar generalizations of (1.2) and (1.14) for $n \equiv 3$ were conjectured by the author and Zudilin [15] and confirmed by the author [4, 5]. Since $(n^r+1)/2 = (n^r - n^j)/2 + (n^j+1)/2$, from (1.15) and (1.16) it follows that, modulo $\prod_{j=1}^{r} \Phi_{n^j}(q)^2$,

$$\sum_{k=0}^{n^r-1} \frac{(1+q^{4k-1})(q^{-2};q^4)_k^3}{(1+q)(q^4;q^4)_k^3} q^{7k} \equiv 0, \tag{1.17}$$

$$\sum_{k=0}^{(n^r+1)/2} \frac{(1+q^{4k-1})(q^{-2};q^4)_k^3}{(1+q)(q^4;q^4)_k^3} q^{7k} \equiv 0, \tag{1.18}$$

where r is a positive integer. For n prime, letting $q \to 1$ in (1.17) and (1.18), we immediately obtain the following conclusion, confirming a conjecture of the author and Wei [11, Conjecture 1].

Corollary 1.5. *Let p be a prime with $p \equiv 1 \pmod 4$ and let $r \geqslant 1$. Then*

$$\sum_{k=0}^{p^r-1} \frac{(-\frac{1}{2})_k^3}{k!^3} \equiv 0 \pmod{p^{2r}} \quad \text{and} \quad \sum_{k=0}^{(p^r+1)/2} \frac{(-\frac{1}{2})_k^3}{k!^3} \equiv 0 \pmod{p^{2r}}.$$

Moreover, from Theorem 1.1 we can deduce the following result, which partially confirms another conjecture of the author and Wei [11, Conjecture 2] (the original conjecture asserts that r can be any positive integer).

Theorem 1.6. *Let p be a prime with $p \equiv 3 \pmod 4$ and let $r \geqslant 2$ be an even integer. Then*

$$\sum_{k=0}^{(p^r+1)/2} \frac{(-\frac{1}{2})_k^3}{k!^3} \equiv p^r \frac{(\frac{1}{4})_{(p^r-1)/2}}{(\frac{7}{4})_{(p^r-1)/2}} \pmod{p^{r+1}}, \tag{1.19}$$

$$\sum_{k=0}^{p^r-1} \frac{(-\frac{1}{2})_k^3}{k!^3} \equiv p^r \frac{(\frac{1}{4})_{(p^r-1)/2}}{(\frac{7}{4})_{(p^r-1)/2}} \pmod{p^{r+1}}. \tag{1.20}$$

2. Proof of Theorem 1.1

For brevity, we adopt the standard condensed notation

$$(a_1,\dots,a_m;q)_k=(a_1;q)_k\dots(a_m;q)_k \quad \text{for } k=0,1,2,\dots.$$

Following [2], the *basic hypergeometric* ${}_{r+1}\phi_r$ *series* with $r+1$ upper parameters $a_1,\dots,a_{r+1}$, r lower parameters $b_1,\dots,b_r$, base q and argument z is given by

$${}_{r+1}\phi_r\left[\begin{matrix}a_1,a_2,\dots,a_{r+1}\\ b_1,\dots,b_r\end{matrix};q,z\right]:=\sum_{k=0}^{\infty}\frac{(a_1,\dots,a_{r+1};q)_k}{(q,b_1,\dots,b_r;q)_k}z^k.$$

Then Watson's ${}_8\phi_7$ transformation formula (see [2, Appendix (III.18)]) can be stated as follows:

$$\begin{aligned}&{}_8\phi_7\left[\begin{matrix}a, & qa^{\frac12}, & -qa^{\frac12}, & b, & c, & d, & e, & q^{-n}\\ & a^{\frac12}, & -a^{\frac12}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1}\end{matrix};q,\ \frac{a^2q^{n+2}}{bcde}\right]\\ &\quad=\frac{(aq,aq/de;q)_n}{(aq/d,aq/e;q)_n}\,{}_4\phi_3\left[\begin{matrix}aq/bc,\ d,\ e,\ q^{-n}\\ aq/b,\ aq/c,\ deq^{-n}/a\end{matrix};q,\ q\right].\end{aligned}\tag{2.1}$$

It is easy to see that the left-hand side of (1.8) with $m\geqslant 0$ can be written as the following terminating ${}_8\phi_7$ series:

$${}_8\phi_7\left[\begin{matrix}q, & q^{\frac92}, & -q^{\frac92}, & q, & q, & q, & q^{4+(4m+3)n}, & q^{1-(4m+3)n}\\ & q^{\frac12}, & -q^{\frac12}, & q^4, & q^4, & q^4, & q^{1-(4m+3)n}, & q^{4+(4m+3)n}\end{matrix};q^4,\ q^2\right].\tag{2.2}$$

By Watson's transformation formula (2.1) with $q\mapsto q^4$, $a=b=c=d=q$, $e=q^{4+(4m+3)n}$, and $n\mapsto mn+(3n-1)/4$, we see that (2.2) is equal to

$$\begin{aligned}&\frac{(q^5,q^{-(4m+3)n};q^4)_{mn+(3n-1)/4}}{(q^4,q^{1-(4m+3)n};q^4)_{mn+(3n-1)/4}}\\ &\quad\times\sum_{k=0}^{mn+(3n-1)/4}\frac{(q^3;q^4)_k(q;q^4)_k(q^{4+(4m+3)n};q^4)_k(q^{1-(4m+3)n};q^4)_k}{(q^4;q^4)_k^3(q^5;q^4)_k}q^{4k}.\end{aligned}\tag{2.3}$$

Note that $n\equiv 3 \pmod 4$. There are exactly $m+1$ factors of the form $1-q^{an}$ (a is an integer) among the $mn+(3n-1)/4$ factors of $(q^5;q^4)_{mn+(3n-1)/4}$. So does $(q^{-(4m+3)n};q^4)_{mn+(3n-1)/4}$. But there are only m factors of the form $1-q^{an}$ (a is an integer) in each of $(q^4;q^4)_{mn+(3n-1)/4}$ and $(q^{1-(4m+3)n};q^4)_{mn+(3n-1)/4}$. Since $\Phi_n(q)$ is a factor of $1-q^N$ if and only if N is divisible by n, we conclude that the fraction before the summation in (2.3) is congruent to 0 modulo $\Phi_n(q)^2$. Moreover, the denominator of the reduced form of the fraction

$$\frac{(q^3;q^4)_k(q;q^4)_k(q^{4+(4m+3)n};q^4)_k(q^{1-(4m+3)n};q^4)_k}{(q^4;q^4)_k^3(q^5;q^4)_k}q^{4k}$$

is always coprime with $\Phi_n(q)$ for any non-negative integer k. This proves that the right-hand side of (2.3) (i.e. (2.2)) is congruent to 0 modulo $\Phi_n(q)^2$, thus establishing (1.8) for $m \geqslant 0$.

Observe that $(q;q^4)_k^4/(q^4;q^4)_k^4$ is congruent to 0 modulo $\Phi_n(q)^4$ for $mn+(3n-1)/4 < k \leqslant (m+1)n-1$. Thus, the q-congruence (1.7) with $m \mapsto m+1$ follows from (1.8) directly.

3. Proof of Theorem 1.2

Note that (1.5) has a accompanied q-supercongruence: for $n \equiv 1 \pmod 4$,

$$\sum_{k=0}^{n-1}[8k+1]\frac{(q;q^4)_k^4}{(q^4;q^4)_k^4}q^{2k} \equiv [n]\frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}q^{(1-n)/4} \pmod{[n]\Phi_n(q)^2} \tag{3.1}$$

(see [19]). Replacing n by n^2 in (1.5) and (3.1), we see that the q-congruences (1.9) and (1.10) hold modulo $\Phi_{n^2}(q)^3$.

It is not difficult to see that $[n^2]=(1-q^{n^2})/(1-q)$ is divisible by $\Phi_n(q)$. Moreover, for $n \equiv 3 \pmod 4$, $(q^2;q^4)_{(n^2-1)/4}$ contains $(n+1)/4$ factors of the form $1-q^{an}$ (a is an integer), while $(q^4;q^4)_{(n^2-1)/4}$ only has $(n-3)/4$ such factors. Hence, for $n \equiv 3 \pmod 4$,

$$[n^2]\frac{(q^2;q^4)_{(n^2-1)/4}}{(q^4;q^4)_{(n^2-1)/4}} \equiv 0 \pmod{\Phi_n(q)^2}.$$

On the other hand, in view of Theorem 1.1, the left-hand sides of (1.9) and (1.10) are both congruent to 0 modulo $\Phi_n(q)^2$ since $(n^2-1)/4 = n(n-3)/4+(3n-1)/4$. This indicates that the q-congruences (1.9) and (1.10) also hold modulo $\Phi_n(q)^2$. Since the polynomials $\Phi_n(q)$ and $\Phi_{n^2}(q)$ are relatively prime to each other, we complete the proof of the theorem.

4. A generalization of Theorem 1.2

Swisher [25, (H.3)] conjectured that, for any positive integer r and prime p with $p \equiv 3 \pmod 4$ and $p>3$,

$$\sum_{k=0}^{(p^{2r}-1)/2}\frac{(\frac{1}{2})_k^3}{k!^3} \equiv p^{2r} \pmod{p^{2r+3}}. \tag{4.1}$$

In 2020, the author [4] proved that (4.1) holds modulo p^{2r+2} by establishing a q-supercongruence.

We shall give a similar generalization of Theorem 1.2 as follows.

Theorem 4.1. *Let n and r be positive integers with $n \equiv 3 \pmod 4$. Then, modulo $\Phi_{n^{2r}}(q)^3\prod_{j=1}^{r}\Phi_{n^{2j-1}}(q)^2$, we have*

$$\sum_{k=0}^{(n^{2r}-1)/4}[8k+1]\frac{(q;q^4)_k^4}{(q^4;q^4)_k^4}q^{2k} \equiv [n^{2r}]\frac{(q^2;q^4)_{(n^{2r}-1)/4}}{(q^4;q^4)_{(n^{2r}-1)/4}}q^{(1-n^{2r})/4}, \tag{4.2}$$

$$\sum_{k=0}^{n^{2r}-1}[8k+1]\frac{(q;q^4)_k^4}{(q^4;q^4)_k^4}q^{2k}\equiv [n^{2r}]\frac{(q^2;q^4)_{(n^{2r}-1)/4}}{(q^4;q^4)_{(n^{2r}-1)/4}}q^{(1-n^{2r})/4}. \tag{4.3}$$

Proof. Replacing n by n^{2r} in (1.5) and (3.1), one sees that (4.2) and (4.3) are true modulo $\Phi_{n^{2r}}(q)^3$. Similarly as before, there holds

$$[n^{2r}]\frac{(q^2;q^4)_{(n^{2r}-1)/4}}{(q^4;q^4)_{(n^{2r}-1)/4}}q^{(1-n^{2r})/4}\equiv 0 \pmod{\prod_{j=1}^{r}\Phi_{n^{2j-1}}(q)^2}.$$

Furthermore, from Theorem 1.1 one can easily deduce that the left-hand sides of (4.2) and (4.3) are also congruent to 0 modulo $\prod_{j=1}^{r}\Phi_{n^{2j-1}}(q)^2$. □

Letting $n=p$ be a prime and taking $q\to 1$ in (4.2) and (4.3), we arrive at the following supercongruences.

Corollary 4.2. *Let $p\equiv 3 \pmod 4$ be a prime and let $r\geqslant 1$. Then*

$$\sum_{k=0}^{(p^{2r}-1)/4}(8k+1)\frac{(\frac14)_k^4}{k!^4}\equiv p^{2r}\frac{(\frac12)_{(p^{2r}-1)/4}}{(1)_{(p^{2r}-1)/4}} \pmod{p^{2r+3}}, \tag{4.4}$$

$$\sum_{k=0}^{p^{2r}-1}(8k+1)\frac{(\frac14)_k^4}{k!^4}\equiv p^{2r}\frac{(\frac12)_{(p^{2r}-1)/4}}{(1)_{(p^{2r}-1)/4}} \pmod{p^{2r+3}}. \tag{4.5}$$

5. Proof of Theorem 1.4

The left-hand side of (1.16) with $m\geqslant 0$ can be written as

$$q^{-1}{}_8\phi_7\left[\begin{matrix} q^{-2}, & q^3, & -q^3, & q^{-2}, & q^{-1}, & q^{-2}, & q^{4+(4m+2)n}, & q^{-2-(4m+2)n} \\ & q^{-1}, & -q^{-1}, & q^4, & q^3, & q^4, & q^{-2-(4m+2)n}, & q^{4+(4m+2)n}\end{matrix};q^4,\,q^7\right]. \tag{5.1}$$

In view of Watson's transformation (2.1) with $q\mapsto q^4$, $a=b=d=q^{-2}$, $c=q^{-1}$, $e=q^{4+(4m+2)n}$, and $n\mapsto mn+(n+1)/2$, one sees that (5.1) is equal to

$$q^{-1}\frac{(q^2,q^{-(4m+2)n};q^4)_{mn+(n+1)/2}}{(q^4,q^{-2-(4m+2)n};q^4)_{mn+(n+1)/2}}{}_4\phi_3\left[\begin{matrix} q^5, & q^{-2}, & q^{4+(4m+2)n}, & q^{-2-(4m+2)n} \\ & q^4, & q^3, & q^2\end{matrix};q^4,\,q^4\right]. \tag{5.2}$$

It is easy to see that there are exactly $2m+2$ factors of the form $1-q^{an}$ (a is an integer) among the $2mn+n+1$ factors of $(q^2,q^{-(4m+2)n};q^4)_{mn+(n+1)/2}$. But there are only $2m$ factors of the form $1-q^{an}$ in the polynomial $(q^4,q^{-2-(4m+2)n};q^4)_{mn+(n+1)/2}$. Note that $\Phi_n(q)$ is a factor of $1-q^N$ if and only if n divides N. Hence, the fraction before the ${}_4\phi_3$ series in (5.2) is congruent to 0 modulo $\Phi_n(q)^2$. For any integer x, let $f_n(x)$ denote the least non-negative integer k such that $(q^x;q^4)_k\equiv 0$ modulo $\Phi_n(q)$. Since $n\equiv 1 \pmod 4$, we obtain $f_n(-2)=(n+3)/2$, $f_n(2)=(n+1)/2$, $f_n(3)=(3n+1)/4$, $f_n(4)=n$, and

$f_n(5) = (n-1)/4$. It follows that the denominator of the reduced form of the k-th summand
$$\frac{(q^5, q^{-2}, q^{4+(4m+2)n}, q^{-2-(4m+2)n}; q^4)_k}{(q^4, q^4, q^3, q^2; q^4)_k} q^{4k}$$
in the ${}_4\phi_3$ series is always relatively prime to $\Phi_n(q)$ for any $k \geqslant 0$. This implies that (5.2) (i.e., (5.1)) is congruent to 0 modulo $\Phi_n(q)^2$, thus building (1.16) for $m \geqslant 0$.

It is clear that $(q^{-2}; q^4)_k^3/(q^4; q^4)_k^3$ is congruent to 0 modulo $\Phi_n(q)^3$ for $mn+(n+1)/2 < k \leqslant (m+1)n-1$. So, the q-congruence (1.15) with m replaced by $m+1$ immediately follows from (1.16).

6. Proof of Theorem 1.6

Let $r = 2s$. We first prove the following q-congruences.

Theorem 6.1. *Let n and s be positive integers with $n \equiv 3 \pmod 4$. Then, modulo $\Phi_{n^{2s}}(q) \prod_{j=1}^{s} \Phi_{n^{2j}}(q)^2$,*

$$\sum_{k=0}^{(n^{2s}+1)/2} \frac{(1+q^{4k-1})(q^{-2}; q^4)_k^3}{(1+q)(q^4; q^4)_k^3} q^{7k} \equiv \frac{[n^{2s}]_{q^2}(q; q^4)_{(n^{2s}-1)/2}}{(q^7; q^4)_{(n^{2s}-1)/2}} q^{(n^{2s}-3)/2}, \tag{6.1}$$

$$\sum_{k=0}^{n^{2s}-1} \frac{(1+q^{4k-1})(q^{-2}; q^4)_k^3}{(1+q)(q^4; q^4)_k^3} q^{7k} \equiv \frac{[n^{2s}]_{q^2}(q; q^4)_{(n^{2s}-1)/2}}{(q^7; q^4)_{(n^{2s}-1)/2}} q^{(n^{2s}-3)/2}. \tag{6.2}$$

Proof. It is clear that $n^{2s} \equiv 1 \pmod 4$. Replacing n by n^{2s} in (1.14), we see that the q-congruence (6.1) holds modulo $\Phi_{n^{2s}}(q)^3$. For k in the range $(n^{2s}+1)/2 < k \leqslant n^{2s}-1$, the k-th summand on the left-hand side of (6.2) is congruent to 0 modulo $\Phi_{n^{2s}}(q)^3$. So, the q-congruence (6.2) also holds modulo $\Phi_{n^{2s}}(q)^3$.

Moreover, we observe that, for $n \equiv 3 \pmod 4$, and $1 \leqslant j \leqslant s-1$,
$$\frac{[n^{2s}]_{q^2}(q; q^4)_{(n^{2s}-1)/2}}{(q^7; q^4)_{(n^{2s}-1)/2}} q^{(n^{2s}-3)/2} \equiv 0 \pmod{\Phi_{n^{2j}}(q)^2}.$$
This is because $[n^{2s}]_{q^2} = (1-q^{2n^{2s}})/(1-q^2)$ is divisible by $\Phi_{n^{2j}}(q)$, and the polynomial $(q; q^4)_{(n^{2s}-1)/2}$ contains $(n^{2s-2j}+1)/2$ factors of the form $1-q^{an^{2j}}$ (a is an integer), while the polynomial $(q^7; q^4)_{(n^{2s}-1)/2}$ only has $(n^{2s-2j}-1)/2$ such factors. Meanwhile, by Theorem 1.1, the left-hand sides of (6.1) and (6.2) are both congruent to 0 modulo $\Phi_{n^{2j}}(q)^2$ for $1 \leqslant j \leqslant s-1$, since $(n^{2s}-1)/2 = (n^{2s-2j}-1)n^{2j}/2 + (n^{2j}-1)/2$ and $n^{2j} \equiv 1 \pmod 4$. This means that the q-congruences (6.1) and (6.2) also hold modulo $\Phi_{n^{2j}}(q)^2$ for $1 \leqslant j \leqslant s-1$. Since the moduli $\Phi_{n^2}(q), \Phi_{n^4}(q), \ldots \Phi_{n^{2s}}(q)$ are pairwise relatively prime polynomials in q, we complete the proof of the theorem. □

Proof of Theorem 1.6. Letting $n = p$ and $q \to 1$ in (6.1) and (6.2) and noticing that $r = 2s$, we obtain the desired supercongruences (1.19) and (1.20). □

7. Concluding remarks

In 2017, He [16] proved the following supercongruences: modulo p^2,

$$\sum_{k=0}^{p-1}(6k+1)\frac{(\frac{1}{2})_k^3(\frac{1}{4})_k}{k!^4 4^k} \equiv \begin{cases} (-1)^{(p+3)/4}p\Gamma_p(\frac{1}{2})\Gamma_p(\frac{1}{4})^2, & \text{if } p\equiv 1 \pmod 4,\\ 0, & \text{if } p\equiv 3 \pmod 4,\end{cases} \tag{7.1}$$

and

$$\sum_{k=0}^{p-1}(6k+1)\frac{(\frac{1}{2})_k^3(\frac{1}{4})_k^2}{k!^5} \equiv \begin{cases} -p\Gamma_p(\frac{1}{4})^4, & \text{if } p\equiv 1 \pmod 4,\\ 0, & \text{if } p\equiv 3 \pmod 4.\end{cases} \tag{7.2}$$

Liu [17] further proved that (7.1) and (7.2) are true modulo p^3. In 2022, Liu and Wang [20] noticed that (7.1) can be deduced from the following q-supercongruence: for any positive odd integer n, modulo $[n]\Phi_n(q)^2$,

$$\begin{aligned}&\sum_{k=0}^{n-1}[6k+1]\frac{(q;q^4)_k(q;q^2)_k^3}{(q^2;q^2)_k(q^4;q^4)_k^3}q^{k^2+k}\\ &\quad\equiv \begin{cases} \dfrac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}[n]q^{(1-n)/4}, & \text{if } n\equiv 1 \pmod 4,\\ 0, & \text{if } n\equiv 3 \pmod 4.\end{cases}\end{aligned} \tag{7.3}$$

See [3] and [13, Theorem 4.5] for generalizations of (7.3).

We conjecture that for $n\equiv 3 \pmod 4$ the modulus $\Phi_n(q)^2$ case of (7.3) has the following generalization like Theorem 1.1.

Conjecture 7.1. *Let m and n be positive integers with $n\equiv 3 \pmod 4$. Then*

$$\sum_{k=0}^{mn-1}[6k+1]\frac{(q;q^4)_k(q;q^2)_k^3}{(q^2;q^2)_k(q^4;q^4)_k^3}q^{k^2+k}\equiv 0 \pmod{\Phi_n(q)^2}, \tag{7.4}$$

$$\sum_{k=0}^{mn+(n-1)/2}[6k+1]\frac{(q;q^4)_k(q;q^2)_k^3}{(q^2;q^2)_k(q^4;q^4)_k^3}q^{k^2+k}\equiv 0 \pmod{\Phi_n(q)^2}. \tag{7.5}$$

Wei [31] gave a q-analogue of (7.2): for any positive odd integer n, modulo $[n]\Phi_n(q)^2$,

$$\begin{aligned}&\sum_{k=0}^{n-1}[6k+1]\frac{(q;q^2)_k^2(q,q,q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k^3}q^{2k}\\ &\quad\equiv \begin{cases} [n]\dfrac{(q^2;q^4)_{(n-1)/4}^2}{(q^4;q^4)_{(n-1)/4}^2}, & \text{if } n\equiv 1 \pmod 4,\\ 0, & \text{if } n\equiv 3 \pmod 4.\end{cases}\end{aligned} \tag{7.6}$$

Similarly, we have the following two conjectural generalizations of (7.6) for $n\equiv 3$ (mod 4). One is modulo $\Phi_n(q)^3$ and the other is modulo $\Phi_n(q)^2$.

Conjecture 7.2. *Let m and n be positive integers with $n\equiv 3 \pmod 4$. Then*

$$\sum_{k=0}^{mn-1}[6k+1]\frac{(q;q^2)_k^2(q,q,q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k^3}q^{2k}\equiv 0 \pmod{\Phi_n(q)^3}, \tag{7.7}$$

$$\sum_{k=0}^{mn+(n-1)/2}[6k+1]\frac{(q;q^2)_k^2(q,q,q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k^3}q^{2k}\equiv 0 \pmod{\Phi_n(q)^2}.$$

Van Hamme [26, (A.2)] also conjectured that

$$\sum_{k=0}^{p-1}(-1)^k(4k+1)\frac{(\frac12)_k^5}{k!^5}\equiv\begin{cases}-\dfrac{p}{\Gamma_p(\frac34)^4} \pmod{p^3}, & \text{if } p\equiv 1 \pmod 4,\\ 0 \pmod{p^3}, & \text{if } p\equiv 3 \pmod 4,\end{cases} \tag{7.8}$$

which was later proved by McCarthy and Osburn [22]. Wang and Yue [27], together with the author [7], gave a q-analogue of (7.8) as follows: for any positive odd integer n, modulo $[n]\Phi_n(q)^2$,

$$\sum_{k=0}^{n-1}(-1)^k[4k+1]\frac{(q;q^2)_k^4(q^2;q^4)_k}{(q^2;q^2)_k^4(q^4;q^4)_k}q^k$$
$$\equiv\begin{cases}[n]\dfrac{(q^2;q^4)_{(n-1)/4}^2}{(q^4;q^4)_{(n-1)/4}^2}, & \text{if } n\equiv 1 \pmod 4,\\ 0, & \text{if } n\equiv 3 \pmod 4.\end{cases} \tag{7.9}$$

Moreover, a stronger version of (7.9) modulo $[n]\Phi_n(q)^3$ was recently presented by Wei [29].

Likewise, we have two conjectural generalizations of for $n\equiv 3 \pmod 4$.

Conjecture 7.3. *Let m and n be positive integers with $n\equiv 3 \pmod 4$. Then*

$$\sum_{k=0}^{mn-1}(-1)^k[4k+1]\frac{(q;q^2)_k^4(q^2;q^4)_k}{(q^2;q^2)_k^4(q^4;q^4)_k}q^k\equiv 0 \pmod{\Phi_n(q)^3}, \tag{7.10}$$

$$\sum_{k=0}^{mn+(n-1)/2}(-1)^k[4k+1]\frac{(q;q^2)_k^4(q^2;q^4)_k}{(q^2;q^2)_k^4(q^4;q^4)_k}q^k\equiv 0 \pmod{\Phi_n(q)^2}. \tag{7.11}$$

We point out that neither (7.4) nor (7.5) holds modulo $\Phi_n(q)^3$ in general. This means that the q-supercongruences (7.7) and (7.10) might be very difficult to prove. Numerical calculation implies that the following conjecture related to (7.1) and (7.2) seems to be true.

Conjecture 7.4. *Let $p\equiv 3 \pmod 4$ be a prime and let $r\geqslant 1$. Then*

$$\sum_{k=0}^{(p^{2r}-1)/d}(6k+1)\frac{(\frac12)_k^3(\frac14)_k}{k!^4 4^k}\equiv p^{2r}\frac{(\frac12)_{(p^{2r}-1)/4}}{(1)_{(p^{2r}-1)/4}} \pmod{p^{2r+3}},$$

$$\sum_{k=0}^{(p^{2r}-1)/d}(6k+1)\frac{(\frac12)_k^3(\frac14)_k^2}{k!^5}\equiv p^{2r}\frac{(\frac12)_{(p^{2r}-1)/4}^2}{(1)_{(p^{2r}-1)/4}^2} \pmod{p^{3r+3}},$$

where $d=1$ or 2.

There is a similar generalization of (7.8) for $p\equiv 3 \pmod 4$. But this is just a special case of Conjecture (A.3) in [25], and is omitted here.

The author, Schlosser, and Zudilin [9] established a new q-analogue of (1.12) as follows: for any integer $n>1$ satisfying $n\equiv 1 \pmod 4$,

$$\sum_{k=0}^{(n+1)/2}\frac{(q^{-1};q^2)_k^2(q^{-2};q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{6k}\equiv 0 \pmod{\Phi_n(q)^2}. \tag{7.12}$$

A further generalization of (7.12) modulo $\Phi_n(q)^3$ was later given by the author and Wei [11].

We believe that the following generalization of (7.12) should be true.

Conjecture 7.5. *Let m and n be positive integers with $n\equiv 1 \pmod 4$ and $n>1$. Then*

$$\sum_{k=0}^{mn-1}\frac{(q^{-1};q^2)_k^2(q^{-2};q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{6k}\equiv 0 \pmod{\Phi_n(q)^2},$$

$$\sum_{k=0}^{mn+(n+1)/2}\frac{(q^{-1};q^2)_k^2(q^{-2};q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{6k}\equiv 0 \pmod{\Phi_n(q)^2}.$$

It is clear that Corollary 1.5 can also be deduced from Conjecture 7.5. Although the author [4] proved a similar generalization of (1.2) for $n\equiv 3$ by using Bailey's ${}_{10}\phi_9$ transformation (see [2, Appendix (III.28)]), it seems difficult to confirm Conjecture 7.5 by using Bailey's transformation again.

It should be mentioned that we did not use the method of 'creative microscoping' introduced in [13] in our proof of Theorem 1.1, though this method is very useful in proving q-supercongruences. In fact, we only utilized Watson's ${}_8\phi_7$ transformation formula. However, it seems rather difficult to find the corresponding transformation formulas to confirm Conjectures 7.1 and 7.2. No doubt that we can apply Andrews' multi-series generalization [1] of Watson's ${}_8\phi_7$ transformation to the left-hand side of (7.11). But there is still a big obstacle to confirming (7.11). We hope that an interested reader can make progress on the conjectures in this paper and settle at least one of them.

8. Declarations

Conflict of interest. The author declares no conflict of interest.

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