q-ANALOGUES OF TWO SUPERCONGRUENCES IN THE (E.3) AND (F.3) CONJECTURES OF SWISHER

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ABSTRACT. Some supercongruences in the (E.3) and (F.3) conjectures of Swisher were proved by the author and Zudilin, and by Jana and Kalita. In this paper we confirm some of the remaining cases of these two conjectures by using the method of 'creative microscoping' in a new way. Meanwhile, we confirm two related supercongruences conjectured by the author early.

1. INTRODUCTION

In 1997, Van Hamme [17] developed p-adic analogues of Ramanujan's or Ramanujan-like series for $1/\pi$. He observed 13 supercongruences, such as

$$\sum_{k=0}^{(p-1)/2}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^3}{k!^3}\equiv(-1)^{(p-1)/2}p \pmod{p^3}, \tag{1.1}$$

$$\sum_{k=0}^{(p-1)/3}(-1)^k(6k+1)\frac{(\frac{1}{3})_k^3}{k!^3}\equiv p \pmod{p^3}, \quad \text{for} \quad p\equiv 1 \pmod 3, \tag{1.2}$$

$$\sum_{k=0}^{(p-1)/4}(-1)^k(8k+1)\frac{(\frac{1}{4})_k^3}{k!^3}\equiv(-1)^{(p-1)/4}p \pmod{p^3}, \quad \text{for} \quad p\equiv 1 \pmod 4, \tag{1.3}$$

where p is an odd prime, $(a)_n=a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol. Different proofs of (1.1) were later given in [12, 13, 21]. Swisher [16] proved 4 supercongruences of Van Hamme, including (1.2) and (1.3) (i.e., the tagged (E.2) and (F.2) supercongruences in [17]). He [7] established a generalization of (1.2).

In [16, Conjectures (E.3)], Swisher proposed the following conjectures: for any prime $p\equiv 1 \pmod 3$ and integer $r\geqslant 1$,

$$\sum_{k=0}^{(p^r-1)/3}(-1)^k(6k+1)\frac{(\frac{1}{3})_k^3}{k!^3}\equiv p\sum_{k=0}^{(p^{r-1}-1)/3}(-1)^k(6k+1)\frac{(\frac{1}{3})_k^3}{k!^3} \pmod{p^{3r}}, \tag{1.4}$$

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and for any prime $p\equiv 2 \pmod 3$ and even integer $r\geqslant 2$,

$$\sum_{k=0}^{(p^r-1)/3}(-1)^k(6k+1)\frac{(\frac13)_k^3}{k!^3}\equiv p^2\sum_{k=0}^{(p^{r-2}-1)/3}(-1)^k(6k+1)\frac{(\frac13)_k^3}{k!^3}\pmod{p^{3r-2}}. \tag{1.5}$$

Recently, the author and Zudilin [6] confirmed (1.4) by establishing its q-analogue and as a conclusion they also proved (1.5) modulo p^{2r}. In this note, we shall completely confirm (1.5) by showing the following q-analogue.

Theorem 1.1. *Let n be a positive integer with $n\equiv 5 \pmod 6$ and let $r\geqslant 2$ be even. Then, modulo $[n^r]_{q^2}\prod_{j=2}^r\Phi_{n^j}(q^2)^2$,*

$$\begin{aligned}\sum_{k=0}^{(n^r-1)/d}&(-1)^k[6k+1]_{q^2}\frac{(q^2;q^6)_k^3(-q^3;q^6)_k}{(q^6;q^6)_k^3(-q^5;q^6)_k}q^k\\
&\equiv q^{1-n^2}[n^2]_{q^2}\sum_{k=0}^{(n^{r-2}-1)/d}(-1)^k[6k+1]_{q^{2n^2}}\frac{(q^{2n^2};q^{6n^2})_k^3(-q^{3n^2};q^{6n^2})_k}{(q^{6n^2};q^{6n^2})_k^3(-q^{5n^2};q^{6n^2})_k}q^{n^2k},\end{aligned}\tag{1.6}$$

where $d=1,3$.

Here and throughout the paper, $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ is the *q-shifted factorial*, $[n]=[n]_q=(1-q^n)/(1-q)$ is the *q-integer*, and $\Phi_n(q)$ denotes the n-th *cyclotomic polynomial* in q, i.e.,

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}}(q-\zeta^k),$$

where ζ is a n-th primitive root of unity. It is well known that $\Phi_n(q)$ is an irreducible polynomial in $\mathbb{Z}[q]$, and $\Phi_n(q)$ divides $1-q^m$ if and only if n divides m. For some recent work on q-congruences, we refer the reader to [4–6, 9–11, 14, 15, 18–20, 22].

It should be mentioned that the right-hand side of (1.6) is a rational function of q^{n^2}. There are no such q-supercongruences in the paper [6]. To the best of the author's knowledge, there are only two papers [15, 23] on q-supercongruences related to rational functions of q^{n^2}. But the results in those two papers do not have parametric generalizations (at least we do not know up to now), and the proofs of theorems are quite different from our proofs here.

Note that Theorem 1.1 also holds for $n>1$ and $n\equiv 1 \pmod 6$. This can be easily deduced from repeatedly using [6, Theorem 3.5] twice.

It is not difficult to see that, when $n=p$ and $q\to 1$, the q-supercongruence (1.6) for $d=3$ reduces to (1.5), and it for $d=1$ confirms the second supercongruence in [3, Conjecture 5.3]. Moreover, letting $n=p$ and $q\to -1$ in (1.6), we arrive at the following new Dwork-type supercongruence: for $p\equiv 2 \pmod 3$ and even $r\geqslant 2$,

$$\sum_{k=0}^{(p^r-1)/d}(6k+1)\frac{(\frac{1}{3})_k^3(\frac{1}{2})_k}{k!^3(\frac{5}{6})_k}\equiv p^2\sum_{k=0}^{(p^{r-2}-1)/d}(6k+1)\frac{(\frac{1}{3})_k^3(\frac{1}{2})_k}{k!^3(\frac{5}{6})_k} \pmod{p^{3r-2}}, \tag{1.7}$$

where $d=1,3$.

Swisher [16, Conjectures (F.3)] also proposed the following conjectures: for any prime $p\equiv 1 \pmod 4$ and integer $r\geqslant 1$,

$$\sum_{k=0}^{(p^r-1)/4}(-1)^k(8k+1)\frac{(\frac{1}{4})_k^3}{k!^3}\equiv (-1)^{(p^2-1)/8}p\sum_{k=0}^{(p^{r-1}-1)/4}(-1)^k(8k+1)\frac{(\frac{1}{4})_k^3}{k!^3} \pmod{p^{3r}}, \tag{1.8}$$

and for any prime $p\equiv 3 \pmod 4$ and even integer $r\geqslant 2$,

$$\sum_{k=0}^{(p^r-1)/4}(-1)^k(8k+1)\frac{(\frac{1}{4})_k^3}{k!^3}\equiv p^2\sum_{k=0}^{(p^{r-2}-1)/4}(-1)^k(8k+1)\frac{(\frac{1}{4})_k^3}{k!^3} \pmod{p^{3r-2}}. \tag{1.9}$$

Recently, the author and Zudilin [6] confirmed (1.8) through the method of 'creative microscoping' and also proved (1.9) modulo p^{2r}. In this note, we shall completely confirm (1.9) by establishing the following q-analogue.

Theorem 1.2. *Let n be a positive integer with $n\equiv 3 \pmod 4$ and let $r\geqslant 2$ be even. Then, modulo $[n^r]\prod_{j=2}^{r}\Phi_{n^j}(q)^2$,*

$$\begin{aligned}&\sum_{k=0}^{(n^r-1)/d}(-1)^k[8k+1]\frac{(q;q^4)_k^3(-q^2;q^4)_k}{(q^4;q^4)_k^3(-q^3;q^4)_k}q^k\\ &\quad\equiv q^{(1-n^2)/2}[n^2]\sum_{k=0}^{(n^{r-2}-1)/d}(-1)^k[8k+1]_{q^{n^2}}\frac{(q^{n^2};q^{4n^2})_k^3(-q^{2n^2};q^{4n^2})_k}{(q^{4n^2};q^{4n^2})_k^3(-q^{3n^2};q^{2n^2})_k}q^{n^2k},\end{aligned} \tag{1.10}$$

where $d=1,4$.

Theorem 1.2 also holds for $n>1$ and $n\equiv 1 \pmod 4$, which can be deduced from applying [6, Theorem 3.6] twice.

It is easy to see that, when $n=p$ and $q\to 1$, the q-supercongruence (1.10) reduces to (1.9) when $d=4$, and confirms the fourth supercongruence in [3, Conjecture 5.3] when $d=1$.

2. Proof of Theorem 1.1

In order to prove Theorem 1.1, we need to establish the following parametric generalization. The proof is similar to that of [6, Theorem 3.5]. In order to make the paper self-contained, we give the whole proof here.

Theorem 2.1. *Let n be a positive integer with $n \equiv 5 \pmod 6$ and let $r \geqslant 2$ be even. Then, modulo*

$$[n^r]_{q^2} \prod_{j=0}^{(n^{r-2}-1)/d} (1-aq^{(6j+2)n^2})(a-q^{(6j+2)n^2}),$$

we have

$$\sum_{k=0}^{(n^r-1)/d} (-1)^k [6k+1]_{q^2} \frac{(aq^2, q^2/a, q^2, -q^3; q^6)_k}{(aq^6, q^6/a, q^6, -q^5; q^6)_k} q^k$$
$$\equiv q^{1-n^2}[n^2]_{q^2} \sum_{k=0}^{(n^{r-2}-1)/d} (-1)^k [6k+1]_{q^{2n^2}} \frac{(aq^{2n^2}, q^{2n^2}/a, q^{2n^2}, -q^{3n^2}; q^{6n^2})_k}{(aq^{6n^2}, q^{6n^2}/a, q^{6n^2}, -q^{5n^2}; q^{6n^2})_k} q^{n^2k}, \quad (2.1)$$

where $d = 1, 3$.

Proof. First note that [5, Theorem 4.2] can be generalized as follows: modulo $[n](1-aq^n)(a-q^n)$,

$$\sum_{k=0}^{(n-1)/d} [2mk+1] \frac{(aq, q/a, q/c, q; q^m)_k}{(aq^m, q^m/a, cq^m, q^m; q^m)_k} c^k q^{(m-2)k}$$
$$\equiv \frac{(c/q)^{(n-1)/m}(q^2/c; q^m)_{(n-1)/m}}{(cq^m; q^m)_{(n-1)/m}} [n] \quad \text{for } n \equiv 1 \pmod m, \quad (2.2)$$

where $d = 1$ or m. It is worth mentioning that, in order to prove (2.2) modulo $[n]$, we need to show that

$$\sum_{k=0}^{n-1} [2mk+1] \frac{(aq, q/a, q/c, q; q^m)_k}{(aq^m, q^m/a, cq^m, q^m; q^m)_k} c^k q^{(m-2)k} \equiv 0 \pmod{\Phi_n(q)}$$

is true for all integers $n > 1$ satisfying $\gcd(m, n) = 1$. Then we use the same arguments as [5, Theorems 1.2 and 1.3] to handle the modulus $[n]$ case (see [4, Lemma 2.2] for a clearer interpretation).

We put $m = 3$, $q \mapsto q^2$ and $c = -q^{-1}$ in (2.2) to get

$$\sum_{k=0}^{(n-1)/d} (-1)^k [6k+1]_{q^2} \frac{(aq^2, q^2/a, q^2, -q^3; q^6)_k}{(aq^6, q^6/a, q^6, -q^5; q^6)_k} q^k$$
$$\equiv (-q)^{1-n}[n]_{q^2} \pmod{[n]_{q^2}(1-aq^{2n})(a-q^{2n})} \quad \text{for } n \equiv 1 \pmod 3, \quad (2.3)$$

where $d = 1, 3$.

For $n \equiv 5 \pmod 6$, we have $n^r \equiv n^{r-2} \equiv 1 \pmod 6$. In view of (2.3) with n replaced by n^r, we see that the left-hand side of (2.1) is congruent to 0 modulo $[n^r]_{q^2}$. On the other hand, replacing n by n^{r-2} and q by q^{n^2} in (2.3), we conclude that the summation on the right-hand side of (2.1) is congruent to 0 modulo $[n^{r-2}]_{q^{2n^2}}$. Further, since n is odd, it is easy to see that $[n^2]_{q^2}$ is relatively prime to the denominators of the sum on the right-hand side of (2.1). Thus we have proved that the right-hand side of (2.1) is congruent to 0 modulo $[n^2]_{q^2}[n^{r-2}]_{q^{2n^2}} = [n^r]_{q^2}$. Therefore, the q-congruence (2.1) is true modulo $[n^r]$.

To show it also holds modulo

$$\prod_{j=0}^{(n^{r-2}-1)/d} (1-aq^{(6j+2)n^2})(a-q^{(6j+2)n^2}), \tag{2.4}$$

it suffices to prove that both sides of (2.1) are equal when we take the value $a=q^{-(6j+2)n^2}$ or $a=q^{(6j+2)n^2}$ for any j with $0\leqslant j\leqslant (n^{r-2}-1)/d$, that is,

$$\begin{aligned}
&\sum_{k=0}^{(n^r-1)/d}(-1)^k[6k+1]_{q^2}\frac{(q^{2-(6j+2)n^2},q^{2+(6j+2)n^2},q^2,-q^3;q^6)_k}{(q^{6-(6j+2)n^2},q^{6+(6j+2)n^2},q^6,-q^5;q^6)_k}q^k\\
&\quad\equiv q^{1-n^2}[n^2]_{q^2}\sum_{k=0}^{(n^{r-2}-1)/d}(-1)^k[6k+1]_{q^{2n^2}}\frac{(q^{-6jn^2},q^{(6j+4)n^2},q^{2n^2},-q^{3n^2};q^{6n^2})_k}{(q^{(4-6j)n^2},q^{(6j+8)n^2},q^{6n^2},-q^{5n^2};q^{6n^2})_k}q^{n^2k}.
\end{aligned} \tag{2.5}$$

It is easy to see that $(n^r-1)/d\geqslant((3j+1)n^2-1)/3$ for $0\leqslant j\leqslant(n^{r-2}-1)/d$, and $(q^{2-(6j+2)n^2};q^6)_k=0$ for $k>((3j+1)n^2-1)/3$. By (2.3), the left-hand side of (2.5) is equal to

$$(-q)^{1-(3j+1)n^2}[(3j+1)n^2]_{q^2}.$$

For the same reason, the right-hand side of (2.5) is equal to

$$q^{1-n^2}[n^2]_{q^2}\cdot(-q^{n^2})^{1-(3j+1)}[3j+1]_{q^{2n^2}}=(-q)^{1-(3j+1)n^2}[(3j+1)n^2]_{q^2}.$$

This proves (2.5). Namely, the q-congruence (2.1) holds modulo (2.4). Since $[n^r]_{q^2}$ is relatively prime to (2.4), we complete the proof of (2.1). $\square$

Proof of Theorem 1.1. Let $\lfloor x\rfloor$ denote the integer part of a real number x. It is not hard to see that the $a=1$ case of (2.4) has the factor

$$\begin{cases}\prod_{j=2}^{r}\Phi_{n^j}(q^2)^{2n^{r-j}} & \text{if } d=1,\\ \prod_{j=2}^{r}\Phi_{n^j}(q^2)^{2\lfloor(n^{r-j}+2)/3\rfloor} & \text{if } d=3.\end{cases}$$

Note that the denominator of the left-hand side of (2.1) is divisible by that of the right-hand side of (2.1). Since $\gcd(n,6)=1$, the factor related to a of the former is

$$(aq^6;q^6)_{(n^r-1)/d}(q^6/a;q^6)_{(n^r-1)/d}.$$

When $a=1$ it only has the factor

$$\begin{cases}\prod_{j=2}^{r}\Phi_{n^j}(q^2)^{2n^{r-j}-2} & \text{if } d=1,\\ \prod_{j=2}^{r}\Phi_{n^j}(q^2)^{2\lfloor(n^{r-j}-1)/3\rfloor} & \text{if } d=3,\end{cases}$$

related to $\Phi_{n^2}(q^2), \Phi_{n^3}(q^2), \dots, \Phi_{n^r}(q^2)$. Hence, letting $a \to 1$ in (2.1) we conclude that (1.6) is true modulo $\Phi_n(q^2)\prod_{j=2}^r \Phi_{n^j}(q^2)^3$, where one product $\prod_{j=1}^r \Phi_{n^j}(q^2)$ comes from $[n^r]_{q^2}$.

Finally, the proof of (2.1) modulo $[n^r]_{q^2}$ is valid for $a=1$ as well. Since the least common multiple of $\Phi_n(q^2)\prod_{j=2}^r \Phi_{n^j}(q^2)^3$ and $[n^r]_{q^2}$ is just $[n^r]_{q^2}\prod_{j=2}^r \Phi_{n^j}(q^2)^2$, we finish the proof of (1.6). □

3. Proof of Theorem 1.2

The proof is similar to that of Theorem 1.1. This time we take $m=4$ and $c=-q^{-1}$ in (2.2) to get

$$\sum_{k=0}^{(n-1)/d}(-1)^k[8k+1]\frac{(aq,q/a,q,-q^2;q^4)_k}{(aq^4,q^4/a,q^4,-q^3;q^4)_k}q^k \\ \equiv (-1)^{(n-1)/4}q^{(1-n)/2}[n] \pmod{\Phi_n(q)(1-aq^n)(a-q^n)} \quad \text{for } n\equiv 1 \pmod 4,$$

where $d=1,4$. Employing this q-congruence, we can produce a generalization of (1.10) with an extra parameter a: for $n\equiv 3 \pmod 4$ and even $r\geqslant 2$, modulo

$$[n^r]\prod_{j=0}^{(n^{r-2}-1)/d}(1-aq^{(4j+1)n^2})(a-q^{(4j+1)n^2}),$$

we have

$$\sum_{k=0}^{(n^r-1)/d}(-1)^k[8k+1]\frac{(aq,q/a,q,-q^2;q^4)_k}{(aq^4,q^4/a,q^4,-q^3;q^4)_k}q^k \\ \equiv q^{(1-n^2)/2}[n^2]\sum_{k=0}^{(n^{r-2}-1)/d}(-1)^k[8k+1]_{q^{n^2}}\frac{(aq^{n^2},q^{n^2}/a,q^{n^2},-q^{2n^2};q^{4n^2})_k}{(aq^{4n^2},q^{4n^2}/a,q^{4n^2},-q^{3n^2};q^{2n^2})_k}q^{n^2k}, \tag{3.1}$$

where $d=1,4$.

Furthermore, the $a=1$ case of

$$\prod_{j=0}^{(n^{r-2}-1)/d}(1-aq^{(4j+1)n^2})(a-q^{(4j+1)n^2}),$$

contains the factor

$$\begin{cases} \prod_{j=2}^r \Phi_{n^j}(q)^{2n^{r-j}} & \text{if } d=1, \\ \prod_{j=2}^r \Phi_{n^j}(q)^{2\lfloor (n^{r-j}+3)/4\rfloor} & \text{if } d=4. \end{cases}$$

The factor related to a of the denominator of the left-hand side of (3.1) is

$$(aq^4;q^4)_{(n^r-1)/d}(q^4/a;q^4)_{(n^r-1)/d}$$

of which the $a=1$ case contains the factor

$$\begin{cases} \prod_{j=2}^{r} \Phi_{n^j}(q)^{2n^{r-j}-2} & \text{if } d=1, \\ \prod_{j=2}^{r} \Phi_{n^j}(q)^{2\lfloor (n^{r-j}-1)/4\rfloor} & \text{if } d=4, \end{cases}$$

related to $\Phi_{n^2}(q), \Phi_{n^3}(q), \dots, \Phi_{n^r}(q)$. The proof then follows by taking $a=1$ in (3.1).

4. Concluding remarks

Recently, Wang and Yue [18] gave a uniform generalization of (1.4) and (1.7) by using the same method given in [6]. We also have such a generalization of Theorems 1.1 and 1.2 as follows:

Let $m \geqslant 2$ be an integer. Let $n>1$ be an odd integer with $n \equiv -1 \pmod m$ and let $r \geqslant 2$ be even. Then, modulo $[n^r]_{q^2} \prod_{j=2}^{r} \Phi_{n^j}(q^2)^2$,

$$\begin{aligned} &\sum_{k=0}^{(n^r-1)/d} (-1)^k [2mk+1]_{q^2} \frac{(q^2;q^{2m})_k^3(-q^m;q^{2m})_k}{(q^{2m};q^{2m})_k^3(-q^{m+2};q^{2m})_k} q^{(m-2)k} \\ &\equiv q^{1-n^2}[n^2]_{q^2} \sum_{k=0}^{(n^{r-2}-1)/d} (-1)^k [2mk+1]_{q^{2n^2}} \frac{(q^{2n^2};q^{2mn^2})_k^3(-q^{mn^2};q^{2mn^2})_k}{(q^{2mn^2};q^{2mn^2})_k^3(-q^{(m+2)n^2};q^{2mn^2})_k} q^{(m-2)n^2k}, \end{aligned} \tag{4.1}$$

where $d=1,m$.

For n prime, letting $q \to 1$ in (4.1), we get the following supercongruence: for any integer $m>1$, prime $p \equiv -1 \pmod m$, and even $r \geqslant 2$,

$$\sum_{k=0}^{(p^r-1)/d} (-1)^k (2mk+1) \frac{(\frac{1}{m})_k^3}{k!^3} \equiv p^2 \sum_{k=0}^{(p^{r-2}-1)/d} (-1)^k (2mk+1) \frac{(\frac{1}{m})_k^3}{k!^3} \pmod{p^{3r-2}},$$

where $d=1,m$. Moreover, for odd m, letting n be a prime and $q \to -1$ in (4.1), we are led to the following generalization of (1.7): for any odd integer $m>1$, prime $p \equiv -1 \pmod m$, and even $r \geqslant 2$,

$$\sum_{k=0}^{(p^r-1)/d} (2mk+1) \frac{(\frac{1}{m})_k^3(\frac{1}{2})_k}{k!^3(\frac{m+2}{2m})_k} \equiv p^2 \sum_{k=0}^{(p^{r-2}-1)/d} (2mk+1) \frac{(\frac{1}{m})_k^3(\frac{1}{2})_k}{k!^3(\frac{m+2}{2m})_k} \pmod{p^{3r-2}},$$

where $d=1,m$.

In [16, Conjectures (E.3)], Swisher also conjectured that, and for any prime $p \equiv 2 \pmod 3$ and odd integer $r \geqslant 3$,

$$\sum_{k=0}^{(p^r-2)/3} (-1)^k (6k+1) \frac{(\frac{1}{3})_k^3}{k!^3} \equiv p^2 \sum_{k=0}^{(p^{r-2}-2)/3} (-1)^k (6k+1) \frac{(\frac{1}{3})_k^3}{k!^3} \pmod{p^{3r-1}},$$

which is still open so far. On the other hand, we notice that the third supercongruence in [16, (F.3)] was confirmed by Jana and Kalita [8]. Therefore, all the supercongruences in [16, (F.3)] have been proven.

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