Some variations of a "divergent" Ramanujan-type q-supercongruence

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Abstract. Using the q-Wilf–Zeilberger method and a q-analogue of a "divergent" Ramanujan-type supercongruence, we give several q-supercongruences modulo the fourth power of a cyclotomic polynomial. One of them is a q-analogue of a supercongruence recently proved by Wang: for any prime $p > 3$,

$$\sum_{k=0}^{p-1}(3k-1)\frac{(\frac{1}{2})_k(-\frac{1}{2})_k^2}{k!^3}4^k \equiv p - 2p^3 \pmod{p^4},$$

where $(a)_k = a(a+1)\cdots(a+k-1)$ is the Pochhammer symbol.

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1. Introduction

By making use of the Wilf–Zeilberger (abbr. WZ) method [31,32], Guillera and Zudilin [3] established the following supercongruence: for any odd prime p,

$$\sum_{k=0}^{(p-1)/2}\frac{3k+1}{16^k}\binom{2k}{k}^3 \equiv p \pmod{p^3}. \tag{1.1}$$

We can also sum k in (1.1) up to $p-1$, since the p-adic order of $(\frac{1}{2})_k/k!$ is 1 for k in the range $(p+1)/2 \leqslant k \leqslant p-1$. In the spirit of [33], the supercongruence (1.1) corresponds to a divergent Ramanujan-type series for $1/\pi$:

$$\sum_{k=0}^{\infty}\frac{3k+1}{16^k}\binom{2k}{k}^3 \text{ “=” } \frac{-2i}{\pi} \tag{1.2}$$

(see [3, (47)]). Here the summation in (1.2) must be understood as the analytic continuation of the corresponding hypergeometric series.

Still using the WZ method and the divisibility result: for $n > 1$,

$$2n\binom{2n}{n} \Bigg| \sum_{k=0}^{n-1}(3k+1)\binom{2k}{k}^3 16^{n-k-1}, \tag{1.3}$$

which was conjectured by Z.-W. Sun [25] and confirmed by Mao and Zhang [19], B.Y. Sun [24] proved the following result: for $n > 1$,

$$2n\binom{2n}{n} \Bigg| \sum_{k=0}^{n-1}\frac{6k^4}{2k-1}\binom{2k}{k}^3 16^{n-k-1}. \tag{1.4}$$

Motivated by B.Y. Sun's work, we found the following supercongruence: for any prime $p > 3$,

$$\sum_{k=0}^{p-1}\frac{6k^4}{16^k(2k-1)}\binom{2k}{k}^3 \equiv p + 2p^3 \pmod{p^4}. \tag{1.5}$$

We shall prove the supercongruence (1.5) by establishing its q-analogue. Recall that the *q-shifted factorial* is defined by $(a;q)_0 = 1$ and $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n \geqslant 1$, the *q-integer* is defined as $[n] = [n]_q = (1-q^n)/(1-q)$ (see [2]), and the *q-binomial coefficients* are given by

$$\begin{bmatrix} M \\ N \end{bmatrix} = \begin{cases} \dfrac{(q;q)_M}{(q;q)_N(q;q)_{M-N}}, & \text{if } 0 \leqslant N \leqslant M, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, the n-th *cyclotomic polynomial* $\Phi_n(q)$ is defined as

$$\Phi_n(q) = \prod_{\substack{1\leqslant k\leqslant n \\ \gcd(k,n)=1}} (q - \zeta^k),$$

where ζ is an n-th primitive root of unity. Our q-analogue of (1.5) can be stated as follows.

Theorem 1.1. *Let $n > 1$ be an odd integer. Then*

$$\sum_{k=0}^{n-1}\frac{[3k][2k][k]^2}{[2k-1](-q;q)_k^4}\begin{bmatrix} 2k \\ k \end{bmatrix}^3 q^{-(k^2+3k)/2}$$
$$\equiv [n]q^{-(n+1)/2} + (1+q)[n]^3 + \frac{(n^2-1)(1-q)^2}{24}[n]^3 q^{-(n+1)/2} \pmod{[n]\Phi_n(q)^3}. \tag{1.6}$$

Here we say that two rational functions $A(q)$ and $B(q)$ are congruent modulo a polynomial $P(q)$ if and only if $P(q)$ divides the numerator of the reduced form of $A(q) - B(q)$ in the polynomial ring $\mathbb{Z}[q]$.

It is easy to see that, letting $n = p > 3$ be a prime and taking $q \to 1$ in (1.6), we obtain the supercongruence (1.5). Furthermore, we can also deduce from (1.6) that, for any prime $p > 3$ and integer $r \geqslant 2$,

$$\sum_{k=0}^{p^r-1} \frac{6k^4}{16^k(2k-1)} \binom{2k}{k}^3 \equiv p^r \pmod{p^{r+3}}.$$

Recently, via the WZ method and the summation package `Sigma` [22], Wang [27] proved the following supercongruence: for any prime $p > 3$,

$$\sum_{k=0}^{p-1} (3k-1) \frac{(\frac{1}{2})_k (-\frac{1}{2})_k^2}{k!^3} 4^k \equiv p - 2p^3 \pmod{p^4}, \tag{1.7}$$

where $(a)_k = a(a+1)\cdots(a+k-1)$ is the Pochhammer symbol. This also extends a conjectural result of the author and Schlosser [11, Conjecture 6.2].

In this paper, we shall give a q-analogue of (1.7) as follows:

Theorem 1.2. *Let $n > 1$ be an odd integer. Then*

$$\begin{aligned}&\sum_{k=0}^{n-1} [3k-1] \frac{(q;q^2)_k (q^{-1};q^2)_k^2}{(q;q)_k^2 (q^2;q^2)_k} q^{(3k-k^2)/2} \\ &\quad \equiv [n] q^{-(n+1)/2} - (1+q)[n]^3 + \frac{(n^2-1)(1-q)^2}{24} [n]^3 q^{-(n+1)/2} \pmod{[n]\Phi_n(q)^3}. \end{aligned} \tag{1.8}$$

As before, letting $n = p > 3$ be a prime and taking $q \to 1$ in (1.8), we are led to (1.7). Moreover, it follows from (1.8) that, for any prime $p > 3$ and integer $r \geqslant 2$,

$$\sum_{k=0}^{p^r-1} (3k-1) \frac{(\frac{1}{2})_k (-\frac{1}{2})_k^2}{k!^3} 4^k \equiv p^r \pmod{p^{r+3}}.$$

We shall prove Theorems 1.1 and 1.2 by making use of the q-WZ method [31, 32] and the following q-supercongruence: for odd n,

$$\begin{aligned}&\sum_{k=0}^{n-1} [3k+1] \frac{(q;q^2)_k^3 q^{-\binom{k+1}{2}}}{(q;q)_k^2 (q^2;q^2)_k} \\ &\quad \equiv q^{(1-n)/2}[n] + \frac{(n^2-1)(1-q)^2}{24} q^{(1-n)/2}[n]^3 \pmod{[n]\Phi_n(q)^3}. \end{aligned} \tag{1.9}$$

This q-supercongruence was originally conjectured in [4] and recently proved in [6] with the help of the "creative microscoping" method [14] and the Chinese reminder theorem. It is easy to see that (1.1) follows from (1.9) by taking $n = p$ and $q \to 1$. The $n = p^r$ being an odd prime power and $q \to 1$ case of (1.9) was conjectured by Z.-W. Sun [25].

We point out that some other interesting q-supercongruences were given in [4–11, 15–18, 20, 21, 23, 26, 28–30, 34].

The paper is organized as follows. We first give two lemmas in the next section. The proofs of Theorems 1.1 and 1.2 will be given in Sections 3 and 4, respectively. Two more similar q-supercongruences are given in Section 5. Finally, using a result of Ni and Pan [20], we give a q-analogue of (1.4) in Section 6.

2. Two lemmas

In this section we give three simple q-congruences. The first one may be deemed a q-analogue of Fermat's little theorem $2^{p-1} \equiv 1 \pmod{p}$ for any odd prime p. The third one is a generalization of a recent result of Wang and Ni [28, Lemma 2.2].

Lemma 2.1. *Let $n > 1$ be an odd integer. Then*

$$(-q;q)_{n-1} \equiv 1 \pmod{\Phi_n(q)}. \tag{2.1}$$

Proof. It is well known that

$$\frac{x^n - 1}{x - 1} = \prod_{k=1}^{n-1}(x - \zeta^k), \tag{2.2}$$

where ζ is an n-th primitive root of unity. Letting $x = -1$ in (2.2), we get $(-\zeta;\zeta)_{n-1} = 1$, which is equivalent to (2.1). □

Lemma 2.2. *Let $n > 1$ be an odd integer. Then*

$$\frac{(q;q^2)_n}{(1-q)(q;q)_{n-1}} \equiv [n] \pmod{[n]\Phi_n(q)}, \tag{2.3}$$

$$\frac{(q;q^2)_{n-1}}{(q;q)_{n-1}} \equiv -[n]q \pmod{[n]\Phi_n(q)}. \tag{2.4}$$

Proof. It is easy to see that

$$\frac{(q;q^2)_n}{(1-q)(q;q)_{n-1}} = [n]\begin{bmatrix}2n\\n\end{bmatrix}\frac{1}{(-q;q)_n}, \tag{2.5}$$

and

$$\begin{aligned}\begin{bmatrix}2n\\n\end{bmatrix}\frac{1}{(-q;q)_n} &= \frac{(q;q^2)_{(n-1)/2}(q^{n+2};q^2)_{(n-1)/2}}{(q;q)_{n-1}}\\ &\equiv \frac{(q;q^2)_{(n-1)/2}(q^2;q^2)_{(n-1)/2}}{(q;q)_{n-1}} = 1 \pmod{\Phi_n(q)}\end{aligned} \tag{2.6}$$

in view of $q^n \equiv 1 \pmod{\Phi_n(q)}$. Since $\begin{bmatrix}2n\\n\end{bmatrix}$ is a polynomial in q and $[n]$ is relatively prime to $(-q;q)_n$ for odd n, the q-congruence (2.3) immediately follows from (2.5) and (2.6).

Observing that

$$\frac{(q;q^2)_{n-1}}{(q;q)_{n-1}} = \frac{1}{[2n-1]}\frac{(q;q^2)_n}{(1-q)(q;q)_{n-1}},$$

$$[2n-1] = (1-q^{2n-1})/(1-q) \equiv -q^{-1} \pmod{\Phi_n(q)},$$

and $[2n-1]$ and $[n]$ are relatively prime polynomials in q, we deduce (2.4) from (2.3). □

3. Proof of Theorem 1.1

Define two functions $F(n,k)$ and $G(n,k)$ as follows:

$$F(n,k) = [3n+2k+1]\frac{(q;q^2)_n(q^{2k+1};q^2)_n^2 q^{-\binom{n+1}{2}-(2n+1)k}}{(q;q)_n^2(q^2;q^2)_n}, \tag{3.1}$$

$$G(n,k) = -\frac{(1+q^{n+2k-1})(q;q^2)_n(q^{2k+1};q^2)_{n-1}^2 q^{-\binom{n}{2}-(2n-1)k}}{(1-q)(q;q)_{n-1}^2(q^2;q^2)_{n-1}}, \tag{3.2}$$

where we have assumed that $1/(q^2;q^2)_n = 0$ for any negative integer n. It is easy to check that

$$F(n,k-1) - F(n,k) = G(n+1,k) - G(n,k). \tag{3.3}$$

That is, the functions $F(n,k)$ and $G(n,k)$ form a q-WZ pair.

We now let $m > 1$ be an odd integer. Summing (3.3) over n from 0 to $m-1$, we obtain

$$\sum_{n=0}^{m-1} F(n,k-1) - \sum_{n=0}^{m-1} F(n,k) = G(m,k) - G(0,k) = G(m,k). \tag{3.4}$$

In light of (2.1) and (2.3), we have

$$\begin{aligned} G(m,1) &= -\frac{(1+q^{m+1})(q;q^2)_m(q^3;q^2)_{m-1}^2 q^{-\binom{m}{2}-2m+1}}{(1-q)(q;q)_{m-1}^3(-q;q)_{m-1}} \\ &= -\frac{(1+q^{m+1})(q;q^2)_m^3 q^{-\binom{m}{2}-2m+1}}{(1-q)^3(q;q)_{m-1}^3(-q;q)_{m-1}} \\ &\equiv -(1+q)q[m]^3 \pmod{[m]^3\Phi_m(q)}, \end{aligned} \tag{3.5}$$

since $q^m \equiv 1 \pmod{\Phi_m(q)}$. Combining (3.4) and (3.5), we conclude that

$$\sum_{n=0}^{m-1} F(n,1) \equiv \sum_{n=0}^{m-1} F(n,0) + (1+q)q[m]^3 \pmod{[m]^3\Phi_m(q)}. \tag{3.6}$$

It is easy to see that

$$\begin{aligned}\sum_{n=0}^{m-1} F(n,1) &= \sum_{n=0}^{m-1}[3n+3]\frac{(q;q^2)_n(q^3;q^2)_n^2 q^{-\binom{n+1}{2}-(2n+1)}}{(q;q)_n^2(q^2;q^2)_n}\\ &= \sum_{n=1}^{m}[3n]\frac{(q;q^2)_{n-1}(q^3;q^2)_{n-1}^2 q^{-\binom{n}{2}-(2n-1)}}{(q;q)_{n-1}^2(q^2;q^2)_{n-1}}\\ &= \sum_{n=1}^{m}\frac{[3n][2n][n]^2}{[2n-1](-q;q)_n^4}\begin{bmatrix}2n\\ n\end{bmatrix}^3 q^{-(n^2+3n)/2+1}. \qquad (3.7)\end{aligned}$$

On the other hand, by (1.9) we have

$$\begin{aligned}\sum_{n=0}^{m-1} F(n,0) &= \sum_{n=0}^{m-1}[3n+1]\frac{(q;q^2)_n^3 q^{-\binom{n+1}{2}}}{(q;q)_n^2(q^2;q^2)_n}\\ &\equiv q^{(1-m)/2}[m] + \frac{(m^2-1)(1-q)^2}{24}q^{(1-m)/2}[m]^3 \pmod{[m]\Phi_m(q)^3}. \qquad (3.8)\end{aligned}$$

Substituting (3.7) and (3.8) into (3.6), and noticing the m-th summand on the right-hand side of (3.7) is congruent to 0 modulo $[m]^4$, we are led to (1.6) with $n \mapsto m$ differing only by a factor q.

Remark. Usually the basic hypergeometric functions satisfying the condition

$$F(n,k+1) - F(n,k) = G(n+1,k) - G(n,k)$$

are called a q-WZ pair. It is also reasonable to call the basic hypergeometric functions satisfying (3.3) a q-WZ pair (see Zudilin [33]). The q-WZ pair in the proof of Theorem 1.1 was found by the author [4] in his proof of a weaker form of (1.9) modulo $[n]\Phi_n(q)^2$. But the corresponding WZ pair (the limiting case $q \to 1$) was first given by Guillera and Zudilin [3] in their proof of (1.1).

4. Proof of Theorem 1.2

Let the functions $F(n,k)$ and $G(n,k)$ be given by (3.1) and (3.2), respectively. Again, let m be an odd integer greater than 1. In view of (2.1), (2.3), (2.4) and $q^m \equiv 1 \pmod{\Phi_m(q)}$, we have

$$\begin{aligned}G(m,0) &= -\frac{(1+q^{m-1})(q;q^2)_m(q;q^2)_{m-1}^2 q^{-\binom{m}{2}}}{(1-q)(q;q)_{m-1}^3(-q;q)_{m-1}}\\ &\equiv -(1+q^{-1})q^2[m]^3 \pmod{[m]^3\Phi_m(q)}. \qquad (4.1)\end{aligned}$$

It follows from (3.4) and (4.1) that

$$\sum_{n=0}^{m-1} F(n,-1) \equiv \sum_{n=0}^{m-1} F(n,0) - (1+q)q[m]^3 \pmod{[m]^3\Phi_m(q)}. \tag{4.2}$$

By the definition of (3.1), we have

$$\sum_{n=0}^{m-1} F(n,-1) = \sum_{n=0}^{m-1} [3n-1] \frac{(q;q^2)_n (q^{-1};q^2)_n^2 q^{-\binom{n+1}{2}+(2n+1)}}{(q;q)_n^2 (q^2;q^2)_n}. \tag{4.3}$$

Finally, substituting (3.8) and (4.3) into (4.2), and dividing both sides by q, we arrive at (1.6) with $n \mapsto m$.

5. More similar q-supercongruences

From (1.9) and (3.4) we can deduce more q-supercongruences besides (1.6) and (1.8). Here we give two such examples.

Theorem 5.1. *Let $n > 3$ be an odd integer. Then, modulo $[n]\Phi_n(q)^3$,*

$$\sum_{k=0}^{n-1} [3k+5] \frac{(q;q^2)_k (q^5;q^2)_k^2}{(q;q)_k^2 (q^2;q^2)_k} q^{-(k^2+9k)/2}$$
$$\equiv [n] q^{(5-n)/2} + (1+q)q^3[n]^3 + \frac{(1+q^3)q^4}{(1+q+q^2)^2}[n]^3 + \frac{(n^2-1)(1-q)^2}{24}[n]^3 q^{(5-n)/2}. \tag{5.1}$$

Proof. Let $m > 3$ be an odd integer. Then

$$\sum_{n=0}^{m-1} F(n,2) = \sum_{k=0}^{m-1} [3k+5] \frac{(q;q^2)_k (q^5;q^2)_k^2}{(q;q)_k^2 (q^2;q^2)_k} q^{-(k^2+9k)/2-2}. \tag{5.2}$$

By (3.4), we get

$$\sum_{n=0}^{m-1} F(n,2) = \sum_{n=0}^{m-1} F(n,0) - G(m,1) - G(m,2). \tag{5.3}$$

Moreover, in view of (2.3), we have

$$\begin{aligned} G(m,2) &= -\frac{(1+q^{m+3})(q;q^2)_m (q^5;q^2)_{m-1}^2 q^{-\binom{m}{2}-4m+2}}{(1-q)(q;q)_{m-1}^3 (-q;q)_{m-1}} \\ &= -\frac{(1-q^{2m+1})^2 (1+q^{m+3})(q;q^2)_m^3 q^{-\binom{m}{2}-4m+2}}{(1-q^3)^2 (1-q)^3 (q;q)_{m-1}^3 (-q;q)_{m-1}} \\ &\equiv -\frac{(1+q^3)q^2}{(1+q+q^2)^2}[m]^3 \pmod{[m]\Phi_m(q)^3}. \end{aligned} \tag{5.4}$$

Substituting (3.5), (3.8), (5.2) and (5.4) into (5.3), we arrive at (5.1) with $n \mapsto m$ differing only by a factor q^{-2}. □

Theorem 5.2. *Let $n > 3$ be an odd integer. Then, modulo $[n]\Phi_n(q)^3$,*

$$\sum_{k=0}^{n-1}[3k-3]\frac{(q;q^2)_k(q^{-3};q^2)_k^2}{(q;q)_k^2(q^2;q^2)_k}q^{(7k-k^2)/2}$$
$$\equiv [n]q^{-(n+3)/2} - \frac{1+q}{q}[n]^3 - \frac{1+q^3}{(1+q+q^2)^2}[n]^3 + \frac{(n^2-1)(1-q)^2}{24}[n]^3q^{-(n+3)/2}. \quad (5.5)$$

Proof. The proof is similar to that of Theorem 5.1. This time we need to use

$$\sum_{n=0}^{m-1}F(n,-2) = \sum_{n=0}^{m-1}F(n,0) + G(m,0) + G(m,-1),$$

and

$$G(m,-1) = -\frac{(1+q^{m-3})(q;q^2)_m(q^{-1};q^2)_{m-1}^2q^{-\binom{m}{2}+2m-1}}{(1-q)(q;q)_{m-1}^3(-q;q)_{m-1}}$$
$$\equiv -\frac{(1+q^3)q^2}{(1+q+q^2)^2}[m]^3 \pmod{[m]\Phi_m(q)^3}.$$

□

6. A q-analogue of (1.4)

A q-analogue of (1.3) was conjectured by the author [4, Conjecture 1.7], and was recently confirmed by the author and Wang [13]. In this section we shall give a q-analogue of (1.4). To this end, we need some lemmas. The first one is a special case of [20, Lemma 3.2].

Lemma 6.1. *Let s and t be non-negative integers with $0 \leqslant t \leqslant d-1$. Then*

$$\frac{(q;q^2)_{sd+t}}{(q^2,q^2)_{sd+t}} \equiv \frac{1}{4^s}\binom{2s}{s}\frac{(q;q^2)_t}{(q,q^2)_t} \pmod{\Phi_d(q)}.$$

The second one is the following q-congruence.

Lemma 6.2. *Let d be a positive odd integer. Let s and t be non-negative integers. Then*

$$(-q;q)_{sd+t} \equiv 2^s(-q;q)_t \pmod{\Phi_d(q)}.$$

Proof. It is easy to see that $(-q;q)_d \equiv 2 \pmod{\Phi_d(q)}$ (see [12, Lemma 3.2]). The proof then follows from the equality $(-q;q)_{sd+t} = (-q^{sd+1};q)_t\prod_{j=0}^{s-1}(-q^{jd+1};q)_d$ and the q-congruence $q^d \equiv 1 \pmod{\Phi_d(q)}$. □

To state the third lemma, we have to introduce some notation. For any positive integer n, let

$$\mathcal{S}(n) = \left\{ d \geqslant 3 : d \text{ is odd and } \left\lfloor \frac{n - \frac{d+1}{2}}{d} \right\rfloor = \left\lfloor \frac{n}{d} \right\rfloor \right\},$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. It is clear that, for any integer $d > 2n-1$, we have $(d+1)/2 > n$, and so $d \notin \mathcal{S}(n)$. That is, the cardinality of $\mathcal{S}(n)$ is finite. Let

$$A_n(q) = \prod_{d \in \mathcal{S}(n)} \Phi_d(q),$$

$$C_n(q) = \prod_{\substack{d \,|\, n,\, d>1 \\ d \text{ is odd}}} \Phi_d(q),$$

It is easily seen that, if $d \,|\, n$, then $d \notin \mathcal{S}(n)$. Hence, the polynomials $A_n(q)$ and $C_n(q)$ are relatively prime. We need the following result, which is a special case of [20, Theorem 2.1].

Lemma 6.3. *Let $\nu_0(q), \nu_1(q), \ldots$ be a sequence of rational functions in q such that, for any odd integer $d > 1$,*

(i) *$\nu_k(q)$ is $\Phi_d(q)$-integral for each $k \geqslant 0$, i.e., the denominator of $\nu_k(q)$ is relatively prime to $\Phi_d(q)$;*

(ii) *for any non-negative integers s and t with $t \leqslant d-1$,*

$$\nu_{sd+t}(q) \equiv \mu_s(q)\nu_t(q) \pmod{\Phi_d(q)},$$

where $\mu_s(q)$ is a $\Phi_d(q)$-integral rational function only dependent on s;

(iii)

$$\sum_{k=0}^{d-1} \frac{(q;q^2)_k}{(q^2;q^2)_k} \nu_k(q) \equiv 0 \pmod{\Phi_d(q)}.$$

Then, for all positive integers n,

$$\sum_{k=0}^{n-1} \frac{(q;q^2)_k}{(q^2;q^2)_k} \nu_k(q) \equiv 0 \pmod{A_n(q)C_n(q)}. \tag{6.1}$$

For Sun's divisibility result (1.4), we find that the following stronger version holds:

$$4n\binom{2n}{n} \,\Bigg|\, \sum_{k=0}^{n-1} \frac{6k^4}{2k-1} \binom{2k}{k}^3 16^{n-k-1}. \tag{6.2}$$

Here we shall prove the following q-analogue of (6.2).

Theorem 6.4. *Let $n > 1$ be an integer. Then, modulo $(1+q^{n-1})^3[n]\begin{bmatrix}2n-1\\n-1\end{bmatrix}$,*

$$\sum_{k=0}^{n-1} \frac{[3k][2k][k]^2}{[2k-1]} \begin{bmatrix}2k\\k\end{bmatrix}^3 (-q^{k+1};q)_{n-k-1}^4 q^{-(k^2+3k)/2} \equiv 0. \tag{6.3}$$

Note that $\frac{1}{[2k-1]}\begin{bmatrix}2k\\k\end{bmatrix}$ is always a polynomial in q with integer coefficients.

Proof. It is clear that, for $0 \leqslant k \leqslant n-2$, we have

$$(-q^{k+1};q)_{n-k-1} \equiv 0 \pmod{1+q^{n-1}}.$$

Moreover, for $k = n-1$, there holds $\begin{bmatrix}2k\\k\end{bmatrix} \equiv 0 \pmod{1+q^{n-1}}$. This means that the left-hand side of (6.3) is divisible by $(1+q^{n-1})^4$.

We now need to prove that the left-hand side of (6.3) is divisible by $[n]\begin{bmatrix}2n-1\\n-1\end{bmatrix}$. It is well known that the q-binomial coefficient $\begin{bmatrix}n\\k\end{bmatrix}_q$ can be factorized into (see [1, Lemma 1]):

$$\begin{bmatrix}n\\k\end{bmatrix}_q = \prod_{d\in\mathcal{D}_{n,k}} \Phi_d(q),$$

where

$$\mathcal{D}_{n,k} := \left\{ d \geqslant 2 : \left\lfloor \frac{k}{d} \right\rfloor + \left\lfloor \frac{n-k}{d} \right\rfloor < \left\lfloor \frac{n}{d} \right\rfloor \right\}.$$

It is not hard to see that $1 < d \in \mathcal{D}_{2n-1,n-1}$ is odd if and only $d \in \mathcal{S}(n)$, and so

$$[n]\begin{bmatrix}2n-1\\n-1\end{bmatrix} = A_n(q)C_n(q) \prod_{\substack{d\mid n\\ d\geqslant 2 \text{ is even}}} \Phi_d(q) \cdot \prod_{\substack{d\in\mathcal{D}_{2n-1,n-1}\\ d \text{ is even}}} \Phi_d(q).$$

For $k \geqslant 0$, let

$$\nu_k(q) = \frac{[3k][2k][k]^2(q;q^2)_k^2(-q;q)_k^2}{[2k-1](q^2;q^2)_k^2} q^{-(k^2+3k)/2}.$$

In view of Lemmas 6.1 and 6.2, for any positive odd integer d and non-negative s and t with $t \leqslant d-1$, we have

$$\begin{aligned}\nu_{sd+t}(q) &= \frac{[3(sd+t)][2(sd+t)][sd+t]^2(q;q^2)_{sd+t}^2(-q;q)_{sd+t}^2}{[2(sd+t)-1](q^2;q^2)_{sd+t}^2} q^{-((sd+t)^2+3(sd+t))/2}\\ &\equiv \frac{1}{4^s}\binom{2s}{s}^2 \nu_t(q) \pmod{\Phi_d(q)}.\end{aligned}$$

Moreover, by (1.6), we get

$$\sum_{k=0}^{d-1} \frac{[3k][2k][k]^2(q;q^2)_k^3(-q;q)_k^2}{[2k-1](q^2;q^2)_k^3} q^{-(k^2+3k)/2} \equiv 0 \pmod{\Phi_d(q)}.$$

Therefore, applying Lemma 6.3 we conclude that

$$\sum_{k=0}^{n-1} \frac{[3k][2k][k]^2(q;q^2)_k^3(-q;q)_k^2}{[2k-1](q^2;q^2)_k^3} q^{-(k^2+3k)/2} \equiv 0 \pmod{A_n(q)C_n(q)}. \tag{6.4}$$

Multiplying the left-hand side of (6.4) by $(-q;q)_{n-1}^4$, and noticing the relation

$$\frac{(q;q^2)_k}{(q^2;q^2)_k}(-q;q)_{n-1}^2 = \begin{bmatrix} 2k \\ k \end{bmatrix}(-q^{k+1};q)_{n-k-1}^2.$$

we see that (6.3) is true modulo $A_n(q)C_n(q)$.

It remains to show that (6.3) is also true modulo

$$\prod_{\substack{d\,|\,n \\ d\geqslant 2 \text{ is even}}} \Phi_d(q) \cdot \prod_{\substack{d\in\mathcal{D}_{2n-1,n-1} \\ d \text{ is even}}} \Phi_d(q).$$

Firstly, let $d\,|\,n$ be an even integer. Then

$$1+q^{d/2} = \frac{1-q^d}{1-q^{d/2}} \equiv 0 \pmod{\Phi_d(q)}.$$

It follows that, for $0 \leqslant k < n-d/2$, the q-shifted factorial $(-q^{k+1};q)_{n-k-1}$ incorporates the factor $1+q^{n-d/2}$ and is therefore congruent to 0 modulo $\Phi_d(q)$. On the other hand, for $n-d/2 \leqslant k \leqslant n-1$, we have $d \in \mathcal{D}_{2k,k}$, i.e.,

$$\begin{bmatrix} 2k \\ k \end{bmatrix} \equiv 0 \pmod{\Phi_d(q)}. \tag{6.5}$$

Thus, for $0 \leqslant k \leqslant n-1$, we always have

$$\begin{bmatrix} 2k \\ k \end{bmatrix}(-q^{k+1};q)_{n-k-1} \equiv 0 \pmod{\Phi_d(q)}. \tag{6.6}$$

This means that (6.3) is true modulo $\prod_{\substack{d\,|\,n \\ d\geqslant 2 \text{ is even}}} \Phi_d(q)$. Secondly, let $d \in \mathcal{D}_{2n-1,n-1}$ be even. Write $n = ud+v$ with $0 \leqslant v \leqslant d-1$. Then $v > d/2$, and so the polynomial $(-q^{k+1};q)_{n-k-1}$ contains the factor $1+q^{ud+d/2}$ (which is divisible by $\Phi_d(q)$) for $0 \leqslant k < ud+d/2$. Moreover, for $ud+d/2 \leqslant k \leqslant n-1$, we have $d \in \mathcal{D}_{2k,k}$, and i.e., (6.5) holds. Therefore, for $0 \leqslant k \leqslant n-1$, the q-congruence (6.6) always holds. This proves that (6.3) is true modulo $\prod_{\substack{d\in\mathcal{D}_{2n-1,n-1} \\ d \text{ is even}}} \Phi_d(q)$.

Since $[n]\begin{bmatrix} 2n-1 \\ n-1 \end{bmatrix} = (1+q^{n-1})[2n-1]\begin{bmatrix} 2n-3 \\ n-2 \end{bmatrix}$, and $(1+q^{n-1})$ is relatively prime to $[2n-1]\begin{bmatrix} 2n-3 \\ n-2 \end{bmatrix}$, the least common multiple of $(1+q^{n-1})^4$ and $[n]\begin{bmatrix} 2n-1 \\ n-1 \end{bmatrix}$ is just $(1+q^{n-1})^3[n]\begin{bmatrix} 2n-1 \\ n-1 \end{bmatrix}$. This proves the theorem. □

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