Some variations of a “divergent” Ramanujan-type \( q \)-supercongruence

Victor J. W. Guo
School of Mathematical Sciences, Huaiyin Normal University, Huai’an 223300, Jiangsu, People’s Republic of China
jwguo@hytc.edu.cn

Abstract. Using the \( q \)-Wilf–Zeilberger method and a \( q \)-analogue of a “divergent” Ramanujan-type supercongruence, we give several \( q \)-supercongruences modulo the fourth power of a cyclotomic polynomial. One of them is a \( q \)-analogue of a supercongruence recently proved by Wang: for any prime \( p > 3 \),
\[
\sum_{k=0}^{p-1} (3k - 1) \frac{\left(\frac{1}{2}\right)_k \left(-\frac{1}{2}\right)_k^2}{k!^3} 4^k \equiv p - 2p^3 \pmod{p^4},
\]
where \( (a)_k = a(a + 1) \cdots (a + k - 1) \) is the Pochhammer symbol.

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1. Introduction

By making use of the Wilf–Zeilberger (abbr. WZ) method [31,32], Guillera and Zudilin [3] established the following supercongruence: for any odd prime \( p \),
\[
\sum_{k=0}^{(p-1)/2} \frac{3k + 1}{16^k} \left(\begin{array}{c} 2k \\ k \end{array}\right)^3 \equiv p \pmod{p^3}.
\]
(1.1)

We can also sum \( k \) in (1.1) up to \( p - 1 \), since the \( p \)-adic order of \( \left(\frac{1}{2}\right)_k/k! \) is 1 for \( k \) in the range \( (p + 1)/2 \leq k \leq p - 1 \). In the spirit of [33], the supercongruence (1.1) corresponds to a divergent Ramanujan-type series for \( 1/\pi \):
\[
\sum_{k=0}^{\infty} \frac{3k + 1}{16^k} \left(\begin{array}{c} 2k \\ k \end{array}\right)^3 \equiv -\frac{2i}{\pi}
\]
(1.2)

(see [3, (47)]). Here the summation in (1.2) must be understood as the analytic continuation of the corresponding hypergeometric series.
Still using the WZ method and the divisibility result: for \( n > 1 \),
\[
2n {2n \choose n} \sum_{k=0}^{n-1} (3k + 1) {2k \choose k}^3 16^{n-k-1},
\]
which was conjectured by Z.-W. Sun [25] and confirmed by Mao and Zhang [19], B.Y. Sun [24] proved the following result: for \( n > 1 \),
\[
2n {2n \choose n} \sum_{k=0}^{n-1} \frac{6k^4}{2k-1} {2k \choose k}^3 16^{n-k-1}.
\]

Motivated by B.Y. Sun’s work, we found the following supercongruence: for any prime \( p > 3 \),
\[
\sum_{k=0}^{p-1} \frac{6k^4}{16k(2k-1)} {2k \choose k}^3 \equiv p + 2p^3 \pmod{p^4}.
\]

We shall prove the supercongruence (1.5) by establishing its \( q \)-analogue. Recall that the \( q \)-shifted factorial is defined by \((a; q)_0 = 1\) and \((a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})\) for \( n \geq 1 \), the \( q \)-integer is defined as \([n] = [n]_q = (1 - q^n)/(1 - q)\) (see [2]), and the \( q \)-binomial coefficients are given by
\[
\binom{M}{N}_q = \begin{cases} 
\frac{(q; q)_M}{(q; q)_N(q; q)_{M-N}}, & \text{if } 0 \leq N \leq M, \\
0, & \text{otherwise.}
\end{cases}
\]

Moreover, the \( n \)-th cyclotomic polynomial \( \Phi_n(q) \) is defined as
\[
\Phi_n(q) = \prod_{\gcd(k,n)=1, 1 \leq k \leq n} (q - \zeta^k),
\]
where \( \zeta \) is an \( n \)-th primitive root of unity. Our \( q \)-analogue of (1.5) can be stated as follows.

**Theorem 1.1.** Let \( n > 1 \) be an odd integer. Then
\[
\sum_{k=0}^{n-1} \frac{[3k][2k][k]^2}{[2k-1](-q; q)^2_k} \binom{2k}{k}^3 q^{-(k^2+3k)/2} = [n]q^{-(n+1)/2} + (1 + q)[n]^3 + \frac{(n^2 - 1)(1 - q)^2}{24}[n]^3 q^{-(n+1)/2} \pmod{[n]\Phi_n(q)^3}.
\]

Here we say that two rational functions \( A(q) \) and \( B(q) \) are congruent modulo a polynomial \( P(q) \) if and only if \( P(q) \) divides the numerator of the reduced form of \( A(q) - B(q) \) in the polynomial ring \( \mathbb{Z}[q] \).
It is easy to see that, letting $n = p > 3$ be a prime and taking $q \to 1$ in (1.6), we obtain the supercongruence (1.5). Furthermore, we can also deduce from (1.6) that, for any prime $p > 3$ and integer $r \geq 2$,

$$\sum_{k=0}^{p^{r-1}} \frac{6k^4}{16^k (2k - 1)} \binom{2k}{k}^3 \equiv p^r \pmod{p^{r+3}}.$$ 

Recently, via the WZ method and the summation package Sigma [22], Wang [27] proved the following supercongruence: for any prime $p > 3$,

$$\sum_{k=0}^{p-1} (3k - 1) \frac{(1/2)_k (1/2)_k}{k!^3} 4^k \equiv p - 2p^3 \pmod{p^4}, \quad (1.7)$$

where $(a)_k = a(a+1) \cdots (a+k-1)$ is the Pochhammer symbol. This also extends a conjectural result of the author and Schlosser [11, Conjecture 6.2].

In this paper, we shall give a $q$-analogue of (1.7) as follows:

**Theorem 1.2.** Let $n > 1$ be an odd integer. Then

$$\sum_{k=0}^{n-1} (3k - 1) \frac{(q; q^2)_k(q^{-1}; q^2)_k^2}{(q; q^2)_k^3(q^2; q^2)_k} q^{(3k-k^2)/2}$$

$$\equiv [n]q^{-(n+1)/2} - (1 + q)[n]^3 + \frac{(n^2 - 1)(1 - q)^2}{24} [n]^3 q^{-(n+1)/2} \pmod{[n]\Phi_n(q)^3}. \quad (1.8)$$

As before, letting $n = p > 3$ be a prime and taking $q \to 1$ in (1.8), we are led to (1.7). Moreover, it follows from (1.8) that, for any prime $p > 3$ and integer $r \geq 2$,

$$\sum_{k=0}^{p^{r-1}} (3k - 1) \frac{(1/2)_k (1/2)_k}{k!^3} 4^k \equiv p^r \pmod{p^{r+3}}.$$ 

We shall prove Theorems 1.1 and 1.2 by making use of the $q$-WZ method [31,32] and the following $q$-supercongruence: for odd $n$,

$$\sum_{k=0}^{n-1} (3k + 1) \frac{(q; q^2)_k^3(q^{1-k})}{(q; q^2)_k^2} \frac{q^{1-k} - 1}{2} q^{-(k+1)/2} \equiv q^{(1-n)/2}[n] + \frac{(n^2 - 1)(1 - q)^2}{24} q^{(1-n)/2}[n]^3 \pmod{[n]\Phi_n(q)^3}. \quad (1.9)$$

This $q$-supercongruence was originally conjectured in [4] and recently proved in [6] with the help of the “creative microscoping” method [14] and the Chinese reminder theorem. It is easy to see that (1.1) follows from (1.9) by taking $n = p$ and $q \to 1$. The $n = p^r$ being an odd prime power and $q \to 1$ case of (1.9) was conjectured by Z.-W. Sun [25].
We point out that some other interesting $q$-supercongruences were given in [4–11, 15–18, 20, 21, 23, 26, 28–30, 34].

The paper is organized as follows. We first give two lemmas in the next section. The proofs of Theorems 1.1 and 1.2 will be given in Sections 3 and 4, respectively. Two more similar $q$-supercongruences are given in Section 5. Finally, using a result of Ni and Pan [20], we give a $q$-analogue of (1.4) in Section 6.

2. Two lemmas

In this section we give three simple $q$-congruences. The first one may be deemed a $q$-analogue of Fermat’s little theorem $2^{p-1} \equiv 1 \pmod{p}$ for any odd prime $p$. The third one is a generalization of a recent result of Wang and Ni [28, Lemma 2.2].

**Lemma 2.1.** Let $n > 1$ be an odd integer. Then

$$(-q; q)_{n-1} \equiv 1 \pmod{\Phi_n(q)}. \quad (2.1)$$

**Proof.** It is well known that

$$\frac{x^n - 1}{x - 1} = \prod_{k=1}^{n-1} (x - \zeta^k), \quad (2.2)$$

where $\zeta$ is an $n$-th primitive root of unity. Letting $x = -1$ in (2.2), we get $(-\zeta; \zeta)_{n-1} = 1$, which is equivalent to (2.1). \Box

**Lemma 2.2.** Let $n > 1$ be an odd integer. Then

$$\frac{(q; q^2)_n}{(1-q)(q; q)_{n-1}} \equiv [n] \pmod{[n]\Phi_n(q)}, \quad (2.3)$$

$$\frac{(q; q^2)_{n-1}}{(q; q)_{n-1}} \equiv -[n]q \pmod{[n]\Phi_n(q)}. \quad (2.4)$$

**Proof.** It is easy to see that

$$\frac{(q; q^2)_n}{(1-q)(q; q)_{n-1}} = [n] \left[\frac{2n}{n}\right] \frac{1}{(-q; q)_n}, \quad (2.5)$$

and

$$\left[\frac{2n}{n}\right] \frac{1}{(-q; q)_n} = \frac{(q; q^2)(n-1)/2(q^{n+2}; q^2)(n-1)/2}{(q; q)_{n-1}} \equiv \frac{(q; q^2)(n-1)/2(q^2; q^2)(n-1)/2}{(q; q)_{n-1}} = 1 \pmod{\Phi_n(q)} \quad (2.6)$$
in view of $q^n \equiv 1 \pmod{\Phi_n(q)}$. Since $\binom{2n}{n}$ is a polynomial in $q$ and $[n]$ is relatively prime to $(-q; q)_n$ for odd $n$, the $q$-congruence (2.3) immediately follows from (2.5) and (2.6).

Observing that
\[
\frac{(q; q^2)_{n-1}}{(q; q)_n} = \frac{1}{[2n-1]} \frac{(q; q^2)_n}{(1 - q)(q; q)_{n-1}},
\]
we have $[2n-1] = (1 - q^{2n-1})/(1 - q) \equiv -q^{-1} \pmod{\Phi_n(q)}$.

and $[2n-1]$ and $[n]$ are relatively prime polynomials in $q$, we deduce (2.4) from (2.3).

\section{Proof of Theorem 1.1}

Define two functions $F(n, k)$ and $G(n, k)$ as follows:
\[
F(n, k) = [3n + 2k + 1] \frac{(q; q^2)_n(q^{2k+1}; q^2)_n^2q^{-(n+1) - (2n+1)k}}{(q; q^2)_n^2(q^2; q^2)_n},
\]
\[
G(n, k) = \frac{(1 + q^{n+2k-1})(q; q^2)_n(q^{2k+1}; q^2)_n^2q^{-(n+1) - (2n+1)k}}{(1 - q)(q; q^2)_n(q^2; q^2)_n},
\]
where we have assumed that $1/(q^2; q^2)_n = 0$ for any negative integer $n$. It is easy to check that
\[
F(n, k - 1) - F(n, k) = G(n + 1, k) - G(n, k).
\]

That is, the functions $F(n, k)$ and $G(n, k)$ form a $q$-WZ pair.

We now let $m > 1$ be an odd integer. Summing (3.3) over $n$ from 0 to $m - 1$, we obtain
\[
\sum_{n=0}^{m-1} F(n, k - 1) - \sum_{n=0}^{m-1} F(n, k) = G(m, k) - G(0, k) = G(m, k).
\]

In light of (2.1) and (2.3), we have
\[
G(m, 1) = \frac{(1 + q^{m+1})(q; q^2)_m(q^3; q^2)_m^2q^{-(m) - 2m+1}}{(1 - q)(q; q^3)_m(-q; q)_m}
\]
\[
= \frac{(1 + q^{m+1})(q^3; q^2)_m^2q^{-(m) - 2m+1}}{(1 - q)^3(q; q^3)_m(-q; q)_m}
\]
\[
\equiv -(1 + q)q[m]^3 \pmod{[m]^3\Phi_m(q)},
\]

since $q^m \equiv 1 \pmod{\Phi_m(q)}$. Combining (3.4) and (3.5), we conclude that
\[
\sum_{n=0}^{m-1} F(n, 1) \equiv \sum_{n=0}^{m-1} F(n, 0) + (1 + q)q[m]^3 \pmod{[m]^3\Phi_m(q)}.
\]
It is easy to see that
\[
\sum_{n=0}^{m-1} F(n, 1) = \sum_{n=0}^{m-1} [3n + 3] \frac{(q; q^2)_n (q^3; q^2)_n^2 q^{-(n+1)-(2n+1)}}{(q; q^2)_n^2 (q^2; q^2)_n}
\]
\[
= \sum_{n=1}^{m} [3n] \frac{(q; q^2)_{n-1} (q^3; q^2)_{n-1}^2 q^{-(n+1)-(2n+1)}}{(q; q^2)_{n-1}^2 (q^2; q^2)_{n-1}}
\]
\[
= \sum_{n=1}^{m} [3n] [2n] [n]^2 \left[ 2n \right]^3 \frac{q^{(n^2+3n)/2+1}}{n}.
\] (3.7)

On the other hand, by (1.9) we have
\[
\sum_{n=0}^{m-1} F(n, 0) = \sum_{n=0}^{m-1} [3n + 1] \frac{(q; q^2)_n^3 q^{-(n+1)}}{(q; q)_n^2 (q^2; q^2)_n}
\]
\[
\equiv q^{(1-m)/2} [m] + \frac{(m^2 - 1)(1 - q)^2}{24} q^{(1-m)/2} [m]^3 \mod [m] \Phi_m(q)^3.
\] (3.8)

Substituting (3.7) and (3.8) into (3.6), and noticing the m-th summand on the right-hand side of (3.7) is congruent to 0 modulo [m]^3, we are led to (1.6) with n \rightarrow m differing only by a factor q.

Remark. Usually the basic hypergeometric functions satisfying the condition
\[
F(n, k + 1) - F(n, k) = G(n + 1, k) - G(n, k)
\]
are called a q-WZ pair. It is also reasonable to call the basic hypergeometric functions satisfying (3.3) a q-WZ pair (see Zudilin [33]). The q-WZ pair in the proof of Theorem 1.1 was found by the author [4] in his proof of a weaker form of (1.9) modulo [n] \Phi_n(q)^2. But the corresponding WZ pair (the limiting case q \rightarrow 1) was first given by Guillera and Zudilin [3] in their proof of (1.1).

4. Proof of Theorem 1.2

Let the functions F(n, k) and G(n, k) be given by (3.1) and (3.2), respectively. Again, let m be an odd integer greater than 1. In view of (2.1), (2.3), (2.4) and \(q^m \equiv 1 \mod \Phi_m(q)\), we have
\[
G(m, 0) = - \frac{(1 + q^{m-1}) (q; q^2)_m (q; q^2)^2_{m-1} q^{-(\frac{m}{2})}}{(1 - q)(q; q)_m^3 (-q; q)_{m-1}}
\]
\[
\equiv -(1 + q^{-1}) q^2 [m]^3 \mod [m] \Phi_m(q)^3.
\] (4.1)
It follows from (3.4) and (4.1) that
\[
\sum_{n=0}^{m-1} F(n, -1) \equiv \sum_{n=0}^{m-1} F(n, 0) - (1 + q)[m]^3 \pmod{[m]^3 \Phi_m(q)). \quad (4.2)
\]
By the definition of (3.1), we have
\[
\sum_{n=0}^{m-1} F(n, -1) = \frac{[3n - 1](q; q^2)_n(q^{-1}; q^2)_n q^{-(n+1)+(2n+1)}}{(q; q^2)_n(q^2; q^2)_n}. \quad (4.3)
\]
Finally, substituting (3.8) and (4.3) into (4.2), and dividing both sides by \(q\), we arrive at (1.6) with \(n \mapsto m\).

5. More similar \(q\)-supercongruences

From (1.9) and (3.4) we can deduce more \(q\)-supercongruences besides (1.6) and (1.8). Here we give two such examples.

**Theorem 5.1.** Let \(n > 3\) be an odd integer. Then, modulo \([n]\Phi_u(q)^3\),
\[
\sum_{k=0}^{n-1} [3k + 5] \frac{(q; q^2)_k(q^5; q^2)_k^2}{(q; q^2)_k^2(q^2; q^2)_k} q^{-(k^2 + 9k)/2} \equiv [n]q^{(5-n)/2} + (1 + q)[n]^3 + \frac{(1 + q^3)q^4}{(1 + q + q^2)^2}[n]^3 + \frac{(n^2 - 1)(1 - q)^2}{24}[n]^3 q^{(5-n)/2}. \quad (5.1)
\]
**Proof.** Let \(m > 3\) be an odd integer. Then
\[
\sum_{n=0}^{m-1} F(n, 2) = \sum_{k=0}^{m-1} [3k + 5] \frac{(q; q^2)_k(q^5; q^2)_k^2}{(q; q^2)_k^2(q^2; q^2)_k} q^{-(k^2 + 9k)/2 - 2}. \quad (5.2)
\]
By (3.4), we get
\[
\sum_{n=0}^{m-1} F(n, 2) = \sum_{n=0}^{m-1} F(n, 0) - G(m, 1) - G(m, 2). \quad (5.3)
\]
Moreover, in view of (2.3), we have
\[
G(m, 2) = -\frac{(1 + q^{m+3})(q; q^3)_{m-1}q^{-(\phi(n)/2)-4m+2}}{(1 - q)(q; q^3)_{m-1}(-q; q)_{m-1}}
= -\frac{(1 - q^{2m+1})^2(1 + q^{m+3})(q; q^3)_{m-1}q^{-(\phi(n)/2)-4m+2}}{(1 - q^3)^2(1 - q)^3(q; q^3)_{m-1}(-q; q)_{m-1}}
\equiv -\frac{(1 + q^3)q^2}{(1 + q + q^2)^2}[m]^3 \pmod{[m]\Phi_m(q)^3}. \quad (5.4)
\]
Substituting (3.5), (3.8), (5.2) and (5.4) into (5.3), we arrive at (5.1) with \( n \mapsto m \) differing only by a factor \( q^{-2} \).

**Theorem 5.2.** Let \( n > 3 \) be an odd integer. Then, modulo \([n]\Phi_n(q)^3\),

\[
\sum_{k=0}^{n-1} (3k - 3) \frac{(q; q^2)_k(q^{-3}; q^2)_k q^{(7k-k^2)/2}}{(q; q^2)_k(q^2; q^2)_k} = [n]q^{-(n+3)/2} - \frac{1 + q^3}{1 + q + q^2} [n]^3 + \frac{(n^2 - 1)(1 - q)^2}{24} [n]^3 q^{-(n+3)/2}. \tag{5.5}
\]

**Proof.** The proof is similar to that of Theorem 5.1. This time we need to use

\[
\sum_{n=0}^{m-1} F(n, -2) = \sum_{n=0}^{m-1} F(n, 0) + G(m, 0) + G(m, -1),
\]

and

\[
G(m, -1) = -\frac{(1 + q^{m-3})(q; q^2)_m(q^{-1}; q^2)_m q^{-(m)+2m-1}}{(1 - q)(q; q^2)_m(-q; q^m-1)} = -\frac{(1 + q^3)q^2}{(1 + q + q^2)^2} [m]^3 \pmod{[m]\Phi_m(q)^3}. \]

\[
\square
\]

**6. A \( q \)-analogue of (1.4)**

A \( q \)-analogue of (1.3) was conjectured by the author [4, Conjecture 1.7], and was recently confirmed by the author and Wang [13]. In this section we shall give a \( q \)-analogue of (1.4). To this end, we need some lemmas. The first one is a special case of [20, Lemma 3.2].

**Lemma 6.1.** Let \( s \) and \( t \) be non-negative integers with \( 0 \leq t \leq d - 1 \). Then

\[
\frac{(q; q^2)_s d + t}{(q^2, q^2)_s d + t} \equiv \frac{1}{4s} \left( \frac{2s}{s} \right) (q; q^2)_s \pmod{\Phi_d(q)}. \]

The second one is the following \( q \)-congruence.

**Lemma 6.2.** Let \( d \) be a positive odd integer. Let \( s \) and \( t \) be non-negative integers. Then

\[
(-q; q)_s d + t \equiv 2^s (-q; q)_t \pmod{\Phi_d(q)}.
\]

**Proof.** It is easy to see that \((-q; q)_d \equiv 2 \pmod{\Phi_d(q)} \) (see [12, Lemma 3.2]). The proof then follows from the equality \((-q; q)_s d + t = (-q^{s d + 1}; q)_t \prod_{j=0}^{s-1} (-q^{j d + 1}; q)_d \) and the \( q \)-congruence \( q^d \equiv 1 \pmod{\Phi_d(q)}). \boxed
To state the third lemma, we have to introduce some notation. For any positive integer $n$, let

$$S(n) = \left\{ d \geq 3 : d \text{ is odd and } \left\lfloor \frac{n - \frac{d+1}{2}}{d} \right\rfloor = \left\lfloor \frac{n}{d} \right\rfloor \right\},$$

where $[x]$ denotes the greatest integer less than or equal to $x$. It is clear that, for any integer $d > 2n - 1$, we have $(d + 1)/2 > n$, and so $d \not\in S(n)$. That is, the cardinality of $S(n)$ is finite. Let

$$A_n(q) = \prod_{d \in S(n)} \Phi_d(q),$$

$$C_n(q) = \prod_{\substack{d \mid n, \ d > 1 \ d \text{ is odd}}} \Phi_d(q),$$

It is easily seen that, if $d \mid n$, then $d \not\in S(n)$. Hence, the polynomials $A_n(q)$ and $C_n(q)$ are relatively prime. We need the following result, which is a special case of [20, Theorem 2.1].

**Lemma 6.3.** Let $\nu_0(q), \nu_1(q), \ldots$ be a sequence of rational functions in $q$ such that, for any odd integer $d > 1$,

(i) $\nu_k(q)$ is $\Phi_d(q)$-integral for each $k \geq 0$, i.e., the denominator of $\nu_k(q)$ is relatively prime to $\Phi_d(q)$;

(ii) for any non-negative integers $s$ and $t$ with $t \leq d - 1$,

$$\nu_{sd+t}(q) \equiv \mu_s(q)\nu_t(q) \pmod{\Phi_d(q)},$$

where $\mu_s(q)$ is a $\Phi_d(q)$-integral rational function only dependent on $s$;

(iii)

$$\sum_{k=0}^{d-1} \frac{(q; q^2)_k}{(q^2; q^2)_k} \nu_k(q) \equiv 0 \pmod{\Phi_d(q)}.$$

Then, for all positive integers $n$,

$$\sum_{k=0}^{n-1} \frac{(q; q^2)_k}{(q^2; q^2)_k} \nu_k(q) \equiv 0 \pmod{A_n(q)C_n(q)}. \tag{6.1}$$

For Sun’s divisibility result (1.4), we find that the following stronger version holds:

$$4n \binom{2n}{n} \sum_{k=0}^{n-1} \frac{6k^4}{2k-1} \binom{2k}{k}^3 16^{n-k-1}. \tag{6.2}$$

Here we shall prove the following $q$-analogue of (6.2).
Theorem 6.4. Let \( n > 1 \) be an integer. Then, modulo \((1 + q^{n-1})^3[n][2n-1]_n\),

\[
\sum_{k=0}^{n-1} \frac{[3k][2k][k]^2}{[2k - 1]} \binom{2k}{k}^3 \left( -q^{k+1}; q \right)_n q^{-(k^2+3k)/2} \equiv 0. \tag{6.3}
\]

Note that \( \frac{1}{[2k-1]} \binom{2k}{k} \) is always a polynomial in \( q \) with integer coefficients.

Proof. It is clear that, for \( 0 \leq k \leq n-2 \), we have

\[
(-q^{k+1}; q)_{n-k-1} \equiv 0 \pmod{1 + q^{n-1}}.
\]

Moreover, for \( k = n-1 \), there holds \( \frac{[2k]}{[k]} \equiv 0 \pmod{1 + q^{n-1}} \). This means that the left-hand side of (6.3) is divisible by \((1 + q^{n-1})^4\).

We now need to prove that the left-hand side of (6.3) is divisible by \([n][2n-1]_n\). It is well known that the \( q \)-binomial coefficient \( [n]_q \) can be factorized into (see [1, Lemma 1]):

\[
[n]_q = \prod_{d \in D_{n,k}} \Phi_d(q),
\]

where

\[
D_{n,k} := \left\{ d \geq 2 : \left\lfloor \frac{k}{d} \right\rfloor + \left\lfloor \frac{n-k}{d} \right\rfloor < \left\lfloor \frac{n}{d} \right\rfloor \right\}.
\]

It is not hard to see that \( 1 < d \in D_{2n-1,n-1} \) is odd if and only \( d \in S(n) \), and so

\[
[n][2n-1]_n = A_n(q)C_n(q) \prod_{\substack{d \mid n \quad d \geq 2 \text{ is even}}} \Phi_d(q) \cdot \prod_{\substack{d \not\mid n \quad d \text{ is even}}} \Phi_d(q).
\]

For \( k \geq 0 \), let

\[
\nu_k(q) = \frac{[3k][2k][k]^2(q; q^2)^2_k(-q; q)_k^2}{[2k - 1](q^2; q^2)_k^2} q^{-(k^2+3k)/2}.
\]

In view of Lemmas 6.1 and 6.2, for any positive odd integer \( d \) and non-negative \( s \) and \( t \) with \( t \leq d-1 \), we have

\[
\nu_{sd+t}(q) = \frac{[3(sd + t)][2(sd + t)][sd + t]^2(q; q^2)^2_{sd+t}(-q; q)_{sd+t}^2}{[2(sd + t) - 1](q^2; q^2)_{sd+t}^2} q^{-(sd+t)^2+3(sd+t)/2} \\
\equiv \frac{1}{4^s} \binom{2s}{s}^2 \nu_s(q) \pmod{\Phi_d(q)}.
\]

Moreover, by (1.6), we get

\[
\sum_{k=0}^{d-1} \frac{[3k][2k][k]^2(q; q^2)_k^3(-q; q)_k^2}{[2k - 1](q^2; q^2)_k^3} q^{-(k^2+3k)/2} \equiv 0 \pmod{\Phi_d(q)}.
\]
Therefore, applying Lemma 6.3 we conclude that

$$\sum_{k=0}^{n-1} \frac{[3k][2k][k^2](q; q^2)_{k}^3(-q; q)_{k}^2}{[2k-1](q^2; q^2)_k^3} q^{-(k^2+3k)/2} \equiv 0 \pmod{A_n(q)C_n(q)}.$$  \hspace{1cm} (6.4)

Multiplying the left-hand side of (6.4) by \((-q; q)_{n-1}^4\), and noticing the relation

$$\frac{(q; q^2)_{k}}{(q^2; q^2)_k} (-q; q)_{n-1}^2 = \left[\frac{2k}{k}\right] (-q^{k+1}; q)_{n-k-1}^2.$$

we see that (6.3) is true modulo \(A_n(q)C_n(q)\).

It remains to show that (6.3) is also true modulo

$$\prod_{d|n \; d \geq 2 \; \text{is even}} \Phi_d(q) \cdot \prod_{d \in \mathcal{D}_{2n-1, n-1} \; d \; \text{is even}} \Phi_d(q).$$

Firstly, let \(d|n\) be an even integer. Then

$$1 + q^{d/2} = \frac{1 - q^d}{1 - q^{d/2}} \equiv 0 \pmod{\Phi_d(q)}.$$

It follows that, for \(0 \leq k < n - d/2\), the \(q\)-shifted factorial \((-q^{k+1}; q)_{n-k-1}\) incorporates the factor \(1 + q^{n-d/2}\) and is therefore congruent to 0 modulo \(\Phi_d(q)\). On the other hand, for \(n - d/2 \leq k \leq n - 1\), we have \(d \in \mathcal{D}_{2k,k}\), i.e.,

$$\left[\frac{2k}{k}\right] \equiv 0 \pmod{\Phi_d(q)}.$$  \hspace{1cm} (6.5)

Thus, for \(0 \leq k \leq n - 1\), we always have

$$\left[\frac{2k}{k}\right] (-q^{k+1}; q)_{n-k-1} \equiv 0 \pmod{\Phi_d(q)}.$$  \hspace{1cm} (6.6)

This means that (6.3) is true modulo \(\prod_{d|n \; d \geq 2 \; \text{is even}} \Phi_d(q)\). Secondly, let \(d \in \mathcal{D}_{2n-1, n-1}\) be even. Write \(n = ud + v\) with \(0 \leq v \leq d - 1\). Then \(v > d/2\), and so the polynomial \((-q^{k+1}; q)_{n-k-1}\) contains the factor \(1 + q^{ud+1/2}\) (which is divisible by \(\Phi_d(q)\)) for \(0 \leq k < ud+d/2\). Moreover, for \(ud + d/2 \leq k \leq n - 1\), we have \(d \in \mathcal{D}_{2k,k}\), and i.e., (6.5) holds. Therefore, for \(0 \leq k \leq n - 1\), the \(q\)-congruence (6.6) always holds. This proves that (6.3) is true modulo

$$\prod_{d \in \mathcal{D}_{2n-1, n-1} \; d \; \text{is even}} \Phi_d(q).$$

Since \([n]\begin{bmatrix}2n-1\end{bmatrix}_{n-1} = (1 + q^{n-1})[2n - 1][2n-3]_{n-2}\), and \((1 + q^{n-1})\) is relatively prime to \([2n-1][2n-3]_{n-2}\), the least common multiple of \((1 + q^{n-1})\) and \([n]\begin{bmatrix}2n-1\end{bmatrix}_{n-1}\) is just \((1 + q^{n-1})\) \([n]\begin{bmatrix}2n-1\end{bmatrix}_{n-1}\). This proves the theorem. \(\square\)

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References


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