SOME NEW q-SUPERCONGRUENCES FROM JACKSON'S ${}_8\phi_7$ SUMMATION

VICTOR J. W. GUO

Abstract. We give some new q-supercongruences modulo the third, fourth and fifth powers of a cyclotomic polynomial, respectively. Two of them are partial q-analogues of supercongruences of Deines et al., and another one is a q-analogue of the following supercongruence due to Pan, Tauraso and Wang: for primes $p\equiv 2 \pmod 5$,

$$\sum_{k=0}^{p-1}(-1)^k\binom{-\frac{2}{5}}{k}^5 \equiv \frac{p}{10}\Gamma_p(\tfrac{1}{5})^5\Gamma_p(\tfrac{2}{5})^5 \pmod{p^5},$$

where $\Gamma_p(x)$ denotes the p-adic Gamma function. Our proof is mainly based on Jackson's ${}_8\phi_7$ summation and the method of "creative microscoping" developed in [Adv. Math. 346 (2019), 329–358].

1. Introduction

In 2016, Deines et al. [2] defined the higher dimensional analogues of Legendre curves as follows:

$$C_{n,\lambda}\colon\ y^n=(x_1x_2\cdots x_{n-1})^{n-1}(1-x_1)\cdots(1-x_{n-1})(x_1-\lambda x_2\cdots x_{n-1}).$$

Note that the curves $C_{2,\lambda}$ are the classical Legendre curves. Differing by at most a scalar multiple, the hypergeometric series

$${}_nF_{n-1}\left[\begin{matrix}\frac{j}{n},\frac{j}{n},\ldots,\frac{j}{n}\\ 1,\ldots,1\end{matrix};\lambda\right]$$

for any j in the range $1\leqslant j\leqslant n-1$ may be regarded as a period of $C_{n,\lambda}$ on condition that it is convergent. Deines et al. [2, Theorem 2] proved that the number of rational points on $C_{n,\lambda}$ over finite fields $\mathbb{F}_q$ can be given by employing Gaussian hypergeometric functions. They are also interested in disclosing how to employ truncated hypergeometric series to get information about the Galois representations and thereby local zeta functions of $C_{n,\lambda}$. Define the truncated hypergeometric series as follows:

$${}_{r+1}F_r\left[\begin{matrix}a_1,a_2,\ldots,a_{r+1}\\ b_1,\ldots,b_r\end{matrix};z\right]_m:=\sum_{k=0}^{m}\frac{(a_1)_k(a_2)_k\cdots(a_{r+1})_k}{(b_1)_k\cdots(b_r)_k k!}z^k,$$

1991 *Mathematics Subject Classification.* 33D15; 11A07, 11B65.

Key words and phrases. q-congruences; Jackson's ${}_8\phi_7$ summation; creative microscoping; Chinese remainder theorem for polynomials.

where $(x)_k=x(x+1)\cdots(x+k-1)$ is the *Pochhammer symbol*. Deines et al. [2, Theorem 6] established the following result: for any integer $n\geqslant 3$ and prime $p\equiv 1 \pmod n$,

$$
{}_nF_{n-1}\left[\begin{matrix}\frac{n-1}{n},\frac{n-1}{n},\dots,\frac{n-1}{n}\\ 1,\dots,1\end{matrix};1\right]_{p-1}\equiv -\Gamma_p(\tfrac{1}{n})^n \pmod{p^2}, \tag{1.1}
$$

where $\Gamma_p(x)$ denotes the *p-adic Gamma function*. The supercongruence (1.1) is also true for $n=2$, which is due to Mortenson [15]. Deines et al. further conjectured that (1.1) holds modulo p^3, which was later confirmed by Wang and Pan [18]. A q-analogue of (1.1) was given by the author [7].

Deines et al. [2, Theorem 7] also obtained another supercongruence on the truncated hypergeometric series related to $C_{4,1}$: for any prime $p\equiv 1 \pmod 4$,

$$
{}_4F_3\left[\begin{matrix}\frac14,\frac14,\frac14,\frac14\\ 1,1,1\end{matrix};1\right]_{p-1}\equiv (-1)^{(p-1)/4}\Gamma_p(\tfrac12)\Gamma_p(\tfrac14)^6 \pmod{p^4}. \tag{1.2}
$$

The first purpose of this paper is to give the following q-analogue of (1.2) modulo p^3.

Theorem 1.1. *Let $n>1$ be an integer with $n\equiv 1 \pmod 4$. Then, modulo $\Phi_n(q)^3$,*

$$
\begin{aligned}
&\sum_{k=0}^{n-1}\frac{(1+q^{8k+1})(q^2;q^8)_k^4}{(1+q)(q^8;q^8)_k^4}q^{8k}\\
&\quad\equiv \frac{(q^{10},q^4,q^3,q^7;q^8)_{(n-1)/4}}{(q^9,q^5,q^2,q^8;q^8)_{(n-1)/4}}\left\{1+[2n]^2\sum_{j=1}^{(n-1)/4}\left(\frac{q^{8j-1}}{[8j-1]^2}-\frac{q^{8j}}{[8j]^2}\right)\right\}.
\end{aligned} \tag{1.3}
$$

Here we need to familiarize ourselves with the standard q-notation. For indeterminates a and q, $[n]=[n]_q=(1-q^n)/(1-q)$ denotes the *q-integer*; $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ stands for the *q-shifted factorial*, and we also adopt the condensed symbol $(a_1,\dots,a_m;q)_n=(a_1;q)_n\cdots(a_m;q)_n$. Moreover, let $\Phi_n(q)$ be the n-th *cyclotomic polynomial* in q, which is irreducible over the ring of integers and can be factorized over the field of complex numbers as

$$
\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}}(q-\zeta^k),
$$

where ζ is an nth primitive root of unity.

Letting $n=p^r$ be a prime power with $p^r\equiv 1 \pmod 4$ and taking $q\to 1$ in (1.3), we obtain the following supercongruence.

Corollary 1.2. *Let p be a prime and r a positive integer with $p^r\equiv 1 \pmod 4$. Then*

$$
{}_4F_3\left[\begin{matrix}\frac14,\frac14,\frac14,\frac14\\ 1,1,1\end{matrix};1\right]_{p^r-1}\equiv p^r\frac{(\frac12)_{(p^r-1)/4}(\frac38)_{(p^r-1)/4}(\frac78)_{(p^r-1)/4}}{(\frac98)_{(p^r-1)/4}(\frac58)_{(p^r-1)/4}(1)_{(p^r-1)/4}}
$$

$$\times\left\{1+4p^{2r}\sum_{j=1}^{(p^r-1)/4}\left(\frac{1}{(8j-1)^2}-\frac{1}{64j^2}\right)\right\}\pmod{p^3}.\quad(1.4)$$

Comparing the supercongruence (1.2) modulo p^3 with the $r=1$ case of (1.4), we arrive at the following conclusion.

Corollary 1.3. *Let $p\equiv 1\pmod 4$ be a prime. Then*

$$p\frac{(\frac12)_{(p-1)/4}(\frac38)_{(p-1)/4}(\frac78)_{(p-1)/4}}{(\frac98)_{(p-1)/4}(\frac58)_{(p-1)/4}(1)_{(p-1)/4}}\left\{1+4p^2\sum_{j=1}^{(p-1)/4}\left(\frac{1}{(8j-1)^2}-\frac{1}{64j^2}\right)\right\}$$
$$\equiv(-1)^{(p-1)/4}\Gamma_p(\tfrac12)\Gamma_p(\tfrac14)^6\pmod{p^3}.$$

Deines et al. [2] mentioned that the following supercongruence on the truncated hypergeometric series: for any prime $p\equiv 1\pmod 5$,

$${}_5F_4\left[\begin{matrix}\frac25,\frac25,\frac25,\frac25,\frac25\\ 1,1,1,1\end{matrix};1\right]_{p-1}\equiv-\Gamma_p(\tfrac15)^5\Gamma_p(\tfrac25)^5\pmod{p^4},\quad(1.5)$$

can be deduced from Dougall's ${}_7F_6$ summation. They further conjectured that (1.5) holds modulo p^5, which was confirmed by Pan, Tauraso, and Wang [16].

The second purpose of this paper is to give a q-analogue of (1.5).

Theorem 1.4. *Let $n>1$ be an integer with $n\equiv 1\pmod 5$. Then, modulo $\Phi_n(q)^4$,*

$$\sum_{k=0}^{n-1}\frac{(1+q^{5k+1})(q^2;q^5)_k^5}{(1+q)(q^5;q^5)_k^5}q^{5k}$$
$$\equiv\frac{(q^3,q^4,q^4,q^7;q^5)_{(2n-2)/5}}{(q^2,q^5,q^5,q^6;q^5)_{(2n-2)/5}}\left\{1+[2n]^2q^{-2n}\sum_{j=1}^{(2n-2)/5}\left(\frac{q^{5j-1}}{[5j-1]^2}-\frac{q^{5j}}{[5j]^2}\right)\right\}.\quad(1.6)$$

For prime powers $n=p^r$ with $p^r\equiv 1\pmod 5$, letting $q\to 1$ in (1.6), we get the following conclusion.

Corollary 1.5. *Let p be a prime and r a positive integer with $p^r\equiv 1\pmod 5$. Then, modulo p^4,*

$$\sum_{k=0}^{p^r-1}\frac{(\frac25)_k^5}{k!^5}\equiv p^r\frac{(\frac35)_{(2p^r-2)/5}(\frac45)^2_{(2p^r-2)/5}}{(1)^2_{(2p^r-2)/5}(\frac65)_{(2p^r-2)/5}}\left\{1+4p^{2r}\sum_{j=1}^{(2p^r-2)/5}\left(\frac{1}{(5j-1)^2}-\frac{1}{25j^2}\right)\right\}.\quad(1.7)$$

Combining the supercongruences (1.5) and (1.7) for $r=1$, we have at the following result.

Corollary 1.6. *Let $p\equiv 1 \pmod 5$ be a prime. Then, modulo p^4,*

$$p\frac{(\frac{3}{5})_{(2p-2)/5}(\frac{4}{5})^2_{(2p-2)/5}}{(1)^2_{(2p-2)/5}(\frac{6}{5})_{(2p-2)/5}}\left\{1+4p^2\sum_{j=1}^{(2p-2)/5}\left(\frac{1}{(5j-1)^2}-\frac{1}{25j^2}\right)\right\}\equiv -\Gamma_p(\tfrac{1}{5})^5\Gamma_p(\tfrac{2}{5})^5.$$

Pan, Tauraso, and Wang (see [16, Theorem 6.3]) proved that, for odd primes $p\equiv 2 \pmod 5$,

$$ {}_5F_4\left[\begin{matrix}\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5}\\ 1,1,1,1\end{matrix};1\right]_{p-1}\equiv \frac{p}{10}\Gamma_p(\tfrac{1}{5})^5\Gamma_p(\tfrac{2}{5})^5 \pmod{p^6}. \tag{1.8}$$

For some other interesting supercongruences on truncated hypergeometric series and truncated basic hypergeometric series, see [1,12,13] and [4,5,10,11,14,17,20–22], respectively.

The third purpose of this paper is to give the following q-analogue of (1.8) modulo p^5.

Theorem 1.7. *Let $n>2$ be an integer with $n\equiv 2 \pmod 5$. Then, modulo $\Phi_n(q)^5$,*

$$\begin{aligned}&\sum_{k=0}^{n-1}\frac{(1+q^{5k+1})(q^2;q^5)_k^5}{(1+q)(q^5;q^5)_k^5}q^{5k}\\ &\quad\equiv\frac{(q^3,q^4,q^4,q^7;q^5)_{(n-2)/5}}{(q^2,q^5,q^5,q^6;q^5)_{(n-2)/5}}\left\{1+[n]^2q^{-n}\sum_{j=1}^{(n-2)/5}\left(\frac{q^{5j-1}}{[5j-1]^2}-\frac{q^{5j}}{[5j]^2}\right)\right\}.\end{aligned}\tag{1.9}$$

Similarly as before, we can obtain the following two corollaries from (1.8) and (1.9).

Corollary 1.8. *Let p be an odd prime and r a positive integer with $p^r\equiv 2 \pmod 5$. Then, modulo p^5,*

$$\sum_{k=0}^{p^r-1}\frac{(\frac{2}{5})_k^5}{k!^5}\equiv\frac{p^r(\frac{3}{5})_{(p^r-2)/5}(\frac{4}{5})^2_{(p^r-2)/5}}{2(1)^2_{(p^r-2)/5}(\frac{6}{5})_{(p^r-2)/5}}\left\{1+p^{2r}\sum_{j=1}^{(p^r-2)/5}\left(\frac{1}{(5j-1)^2}-\frac{1}{25j^2}\right)\right\}.$$

Corollary 1.9. *Let $p\equiv 2 \pmod 5$ be an odd prime. Then, modulo p^4,*

$$\frac{(\frac{3}{5})_{(p-2)/5}(\frac{4}{5})^2_{(p-2)/5}}{(1)^2_{(p-2)/5}(\frac{6}{5})_{(p-2)/5}}\left\{1+p^2\sum_{j=1}^{(p-2)/5}\left(\frac{1}{(5j-1)^2}-\frac{1}{25j^2}\right)\right\}\equiv\frac{1}{5}\Gamma_p(\tfrac{1}{5})^5\Gamma_p(\tfrac{2}{5})^5.$$

The paper is arranged as follows. We shall prove Theorems 1.1, 1.4, and 1.7 in Sections 2–4, respectively. Our proof relies on the creative microscoping method introduced by the author and Zudilin [9], and the Chinese remainder theorem for coprime polynomials. It should be mentioned that Jackson's ${}_8\phi_7$ summation will also play important part in this paper. We shall also present three more similar q-supercongruences in Section 5. Finally, in Section 6, we propose two open problems for further study.

2. Proof of Theorem 1.1

Recall that the basic hypergeometric ${}_{r+1}\phi_r$ series (see [3]) is defined by

$$ {}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix}; q, z\right] := \sum_{k=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_k}{(q, b_1, \dots, b_r; q)_k} z^k. $$

Then Jackson's ${}_8\phi_7$ summation [3, Appendix (II.22)] can be stated as follows:

$$ {}_8\phi_7\left[\begin{matrix} a, & qa^{\frac12}, & -qa^{\frac12}, & b, & c, & d, & e, & q^{-n} \\ & a^{\frac12}, & -a^{\frac12}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1} \end{matrix}; q, q\right] $$
$$ = \frac{(aq, aq/bc, aq/bd, aq/cd; q)_n}{(aq/b, aq/c, aq/d, aq/bcd; q)_n}, \tag{2.1} $$

where $a^2q = bcdeq^{-n}$.

We first give a parametric version of Theorem 1.1.

Lemma 2.1. *Let $n > 1$ be an integer with $n \equiv 1 \pmod 4$. Let a and b be indeterminates. Then, modulo $(1 - abq^{2n})(a - q^{2n})(b - q^{2n})$,*

$$ \sum_{k=0}^{n-1} \frac{(1 + q^{8k+1})(q^2/a, q^2/b, abq^2, q^2; q^8)_k}{(1 + q)(aq^8, bq^8, q^8/ab, q^8; q^8)_k} q^{8k} $$
$$ \equiv \frac{(b - q^{2n})(ab^2 - a^2b - 1 + abq^{2n})}{(a - b)(1 - ab^2)} \frac{(q^{10}, q^4, bq^3, bq^7; q^8)_{(n-1)/4}}{(q^9, q^5, bq^2, bq^8; q^8)_{(n-1)/4}} $$
$$ + \frac{(a - q^{2n})(1 - abq^{2n})}{(a - b)(1 - ab^2)} \frac{(q^{10}, q^4, aq^3, aq^7; q^8)_{(n-1)/4}}{(q^9, q^5, aq^2, aq^8; q^8)_{(n-1)/4}}. \tag{2.2} $$

Proof. For $a = q^{2n}$ or $a = q^{-2n}/b$, the left-hand side of (2.2) can be written as

$$ \sum_{k=0}^{n-1} [8k+1]_{q^2} \frac{(q^2, q^{2-2n}, q^2/b, bq^{2+2n}, q; q^8)_k}{(q^8, q^{8+2n}, bq^8, q^{8-2n}/b, q^9; q^8)_k} q^{8k} $$
$$ = {}_8\phi_7\left[\begin{matrix} q^2, & q^9, & -q^9, & q^2/b, & q, & q^5, & bq^{2+2n}, & q^{2-2n} \\ & q, & -q, & bq^8, & q^9, & q^5, & q^{8-2n}/b, & q^{8+2n} \end{matrix}; q^8, q^8\right]. \tag{2.3} $$

In view of Jackson's ${}_8\phi_7$ summation (2.1), the right-hand side of (2.3) is equal to

$$ \frac{(q^{10}, q^4, bq^3, bq^7; q^8)_{(n-1)/4}}{(q^9, q^5, bq^2, bq^8; q^8)_{(n-1)/4}}. $$

Since the polynomials $1 - abq^{2n}$ and $a - q^{2n}$ are coprime with each other, we immediately get the following q-congruence: modulo $(1 - abq^{2n})(a - q^{2n})$,

$$ \sum_{k=0}^{n-1} \frac{(1 + q^{8k+1})(q^2/a, q^2/b, abq^2, q^2; q^8)_k}{(1 + q)(aq^8, bq^8, q^8/ab, q^8; q^8)_k} q^{8k} \equiv \frac{(q^{10}, q^4, bq^3, bq^7; q^8)_{(n-1)/4}}{(q^9, q^5, bq^2, bq^8; q^8)_{(n-1)/4}}. $$

Interchanging the indeterminates a and b in the above q-congruence, we obtain the following result: modulo $b-q^{2n}$,

$$\sum_{k=0}^{n-1}\frac{(1+q^{8k+1})(q^2/a,q^2/b,abq^2,q^2;q^8)_k}{(1+q)(aq^8,bq^8,q^8/ab,q^8;q^8)_k}q^{8k}\equiv\frac{(q^{10},q^4,aq^3,aq^7;q^8)_{(n-1)/4}}{(q^9,q^5,aq^2,aq^8;q^8)_{(n-1)/4}}.$$

It is easy to verify that

$$\frac{(b-q^{2n})(ab^2-a^2b-1+abq^{2n})}{(a-b)(1-ab^2)}\equiv 1\pmod{(1-abq^{2n})(a-q^{2n})} \tag{2.4}$$

$$\frac{(a-q^{2n})(1-abq^{2n})}{(a-b)(1-ab^2)}\equiv 1\pmod{b-q^{2n}}. \tag{2.5}$$

Since $(a-q^{2n})(1-abq^{2n})$ and $b-q^{2n}$ are coprime polynomials, employing the Chinese remainder theorem for coprime polynomials and the relations (2.4) and (2.5), we obtain the q-congruence (2.2). □

Proof of Theorem 1.1. Note that $1-q^{2n}$ contains the factor $\Phi_n(q)$. Putting $b=1$ in (2.2), and using the following identity

$$(1-q^{2n})(1+a^2-a-aq^{2n})=(1-a)^2+(1-aq^{2n})(a-q^{2n}), \tag{2.6}$$

and the q-congruence

$$\begin{aligned}\frac{(aq^3;q^8)_{(n-1)/4}}{(aq^2;q^8)_{(n-1)/4}}&=q^{(n-1)/4}\frac{(q^{7-2n}/a;q^8)_{(n-1)/4}}{(q^{8-2n}/a;q^8)_{(n-1)/4}}\\&\equiv q^{(n-1)/4}\frac{(q^7/a;q^8)_{(n-1)/4}}{(q^8/a;q^8)_{(n-1)/4}}\pmod{\Phi_n(q)},\end{aligned} \tag{2.7}$$

we are led to the following q-congruence: modulo $\Phi_n(q)(1-aq^{2n})(a-q^{2n})$,

$$\begin{aligned}&\sum_{k=0}^{n-1}\frac{(1+q^{8k+1})(q^2/a,q^2,aq^2,q^2;q^8)_k}{(1+q)(aq^8,q^8,q^8/a,q^8;q^8)_k}q^{8k}\\&\quad\equiv\frac{(q^{10},q^4,q^3,q^7;q^8)_{(n-1)/4}}{(q^9,q^5,q^2,q^8;q^8)_{(n-1)/4}}+\frac{(1-aq^{2n})(a-q^{2n})}{(1-a)^2}q^{(n-1)/4}\\&\quad\quad\times\left\{\frac{(q^{10},q^4,q^7,q^7;q^8)_{(n-1)/4}}{(q^9,q^5,q^8,q^8;q^8)_{(n-1)/4}}-\frac{(q^{10},q^4,q^7/a,aq^7;q^8)_{(n-1)/4}}{(q^9,q^5,q^8/a,aq^8;q^8)_{(n-1)/4}}\right\}.\end{aligned} \tag{2.8}$$

By the L'Hôpital rule, there holds

$$\begin{aligned}&\lim_{a\to 1}\frac{1}{(1-a)^2}\left\{\frac{(q^{10},q^4,q^7,q^7;q^8)_{(n-1)/4}}{(q^9,q^5,q^8,q^8;q^8)_{(n-1)/4}}-\frac{(q^{10},q^4,q^7/a,aq^7;q^8)_{(n-1)/4}}{(q^9,q^5,q^8/a,aq^8;q^8)_{(n-1)/4}}\right\}\\&\quad=\frac{1}{(1-q)^2}\frac{(q^{10},q^4,q^7,q^7;q^8)_{(n-1)/4}}{(q^9,q^5,q^8,q^8;q^8)_{(n-1)/4}}\sum_{j=1}^{(n-1)/4}\left(\frac{q^{8j-1}}{[8j-1]^2}-\frac{q^{8j}}{[8j]^2}\right).\end{aligned}$$

Hence, taking $a \to 1$ in (2.8) and applying the above limit and the $a = 1$ case of the q-congruence (2.7) again, we obtain the desired q-supercongruence (1.3). ☐

3. Proof of Theorem 1.4

Like before, we first establish a parametric version of Theorem 1.4.

Lemma 3.1. *Let $n > 1$ be an integer with $n \equiv 1 \pmod 5$. Let a and b be indeterminates. Then, modulo $(1-aq^{2n})(a-q^{2n})(1-bq^{2n})(b-q^{2n})$,*

$$\sum_{k=0}^{n-1} \frac{(1+q^{5k+1})(aq^2, q^2/a, bq^2, q^2/b, q^2; q^5)_k}{(1+q)(aq^5, q^5/a, bq^5, q^5/b, q^5; q^5)_k} q^{5k}$$
$$\equiv \frac{(1-bq^{2n})(b-q^{2n})(-1-a^2+aq^{2n})}{(a-b)(1-ab)} \frac{(bq^4, q^4/b, q^3, q^7; q^5)_{(2n-2)/5}}{(bq^5, q^5/b, q^2, q^6; q^5)_{(2n-2)/5}}$$
$$+ \frac{(1-aq^{2n})(a-q^{2n})(-1-b^2+bq^{2n})}{(b-a)(1-ba)} \frac{(aq^4, q^4/a, q^3, q^7; q^5)_{(2n-2)/5}}{(aq^5, q^5/a, q^2, q^6; q^5)_{(2n-2)/5}}. \tag{3.1}$$

Proof. For $a = q^{-2n}$ or $a = q^{2n}$, the left-hand side of (3.1) becomes

$$\sum_{k=0}^{(2n-2)/5} \frac{(1+q^{5k+1})(q^{2-2n}, q^{2+2n}, bq^2, q^2/b, q^2; q^5)_k}{(1+q)(q^{5+2n}, q^{5-2n}, bq^5, q^5/b, q^5; q^5)_k} q^{5k}$$
$$= {}_8\phi_7\left[\begin{matrix} q^2, & q^6, & -q^6, & q, & bq^2, & q^2/b, & q^{2+2n}, & q^{2-2n} \\ & q, & -q, & q^6, & q^5/b, & bq^5, & q^{5-2n}, & q^{5+2n} \end{matrix}; q^5, q^5\right], \tag{3.2}$$

where we have used $(q^{2-2n}; q^5)_k = 0$ for $k > (2n-2)/5$. By Jackson's summation (2.1), the right-hand side of (3.2) can be simplified as

$$\frac{(bq^4, q^4/b, q^3, q^7; q^5)_{(2n-2)/5}}{(bq^5, q^5/b, q^2, q^6; q^5)_{(2n-2)/5}}.$$

Noticing that the polynomial $1 - aq^{2n}$ is coprime with $a - q^{2n}$, we get the q-congruence: modulo $(1-aq^{2n})(a-q^{2n})$,

$$\sum_{k=0}^{n-1} \frac{(1+q^{5k+1})(aq^2, q^2/a, bq^2, q^2/b, q^2; q^5)_k}{(1+q)(aq^5, q^5/a, bq^5, q^5/b, q^5; q^5)_k} q^{5k} \equiv \frac{(bq^4, q^4/b, q^3, q^7; q^5)_{(2n-2)/5}}{(bq^5, q^5/b, q^2, q^6; q^5)_{(2n-2)/5}}. \tag{3.3}$$

Since the left-hand side of (3.3) is symmetric in a and b, we conclude from (3.3) that, modulo $(1-bq^{2n})(b-q^{2n})$,

$$\sum_{k=0}^{n-1} \frac{(1+q^{5k+1})(aq^2, q^2/a, bq^2, q^2/b, q^2; q^5)_k}{(1+q)(aq^5, q^5/a, bq^5, q^5/b, q^5; q^5)_k} q^{5k} \equiv \frac{(aq^4, q^4/a, q^3, q^7; q^5)_{(2n-2)/5}}{(aq^5, q^5/a, q^2, q^6; q^5)_{(2n-2)/5}}. \tag{3.4}$$

Clearly, the polynomials $\Phi_n(q)$, $(1-aq^{2n})(a-q^{2n})$, and $(1-bq^{2n})(b-q^{2n})$ are pairwise coprime. Further, we have the following q-congruence:

$$\frac{(1-bq^{2n})(b-q^{2n})(-1-a^2+aq^{2n})}{(a-b)(1-ab)} \equiv 1 \pmod{(1-aq^{2n})(a-q^{2n})}, \tag{3.5}$$

which was first observed by Wei [20–22]. Using the Chinese remainder theorem for polynomials, from (3.5) and its equivalent form ($a \leftrightarrow b$), we arrive at the q-congruence (3.1). □

Proof of Theorem 1.4. Since $1-q^{2n}$ has the factor $\Phi_n(q)$, taking $b=1$ in (3.1), we obtain the following q-congruence: modulo $\Phi_n(q)^2(1-aq^{2n})(a-q^{2n})$,

$$\begin{aligned}
&\sum_{k=0}^{n-1} \frac{(1+q^{5k+1})(aq^2, q^2/a, q^2, q^2, q^2; q^5)_k}{(1+q)(aq^5, q^5/a, q^5, q^5, q^5; q^5)_k} q^{5k} \\
&\quad \equiv \frac{(1-q^{2n})^2(1+a^2-aq^{2n})}{(1-a)^2} \frac{(q^4, q^4, q^3, q^7; q^5)_{(2n-2)/5}}{(q^5, q^5, q^2, q^6; q^5)_{(2n-2)/5}} \\
&\qquad - \frac{(1-aq^{2n})(a-q^{2n})(2-q^{2n})}{(1-a)^2} \frac{(aq^4, q^4/a, q^3, q^7; q^5)_{(2n-2)/5}}{(aq^5, q^5/a, q^2, q^6; q^5)_{(2n-2)/5}} \\
&\quad = (1-q^{2n})^2 \frac{(q^4, q^4, q^3, q^7; q^5)_{(2n-2)/5}}{(q^5, q^5, q^2, q^6; q^5)_{(2n-2)/5}} \\
&\qquad + \frac{a(1-q^{2n})^2(2-q^{2n})}{(1-a)^2} \frac{(q^4, q^4, q^3, q^7; q^5)_{(2n-2)/5}}{(q^5, q^5, q^2, q^6; q^5)_{(2n-2)/5}} \\
&\qquad - \frac{(1-aq^{2n})(a-q^{2n})(2-q^{2n})}{(1-a)^2} \frac{(aq^4, q^4/a, q^3, q^7; q^5)_{(2n-2)/5}}{(aq^5, q^5/a, q^2, q^6; q^5)_{(2n-2)/5}}.
\end{aligned} \tag{3.6}$$

By the L'Hôpital rule, we have

$$\begin{aligned}
&\lim_{a\to 1} \left\{ \frac{a(1-q^{2n})^2}{(1-a)^2} \frac{(q^4; q^5)_{(2n-2)/5}^2}{(q^5; q^5)_{(2n-2)/5}^2} - \frac{(1-aq^{2n})(a-q^{2n})}{(1-a)^2} \frac{(aq^4, q^4/a; q^5)_{(2n-2)/5}}{(q^5/a, aq^5; q^5)_{(2n-2)/5}} \right\} \\
&\quad = \frac{(q^4; q^5)_{(2n-2)/5}^2}{(q^5; q^5)_{(2n-2)/5}^2} \left\{ q^{2n} + [2n]^2 \sum_{j=1}^{(2n-2)/5} \left(\frac{q^{5j-1}}{[5j-1]^2} - \frac{q^{5j}}{[5j]^2} \right) \right\}.
\end{aligned} \tag{3.7}$$

Hence, taking the limits as $a \to 1$ in (3.6), and applying (3.7) and the fact $[2n]^2(2-q^{2n}) \equiv [2n]^2 q^{-2n} \pmod{\Phi_n(q)^4}$, we get the q-supercongruence (1.6). □

4. Proof of Theorem 1.7

The proof is similar to that of Theorem 1.4. But we need an easily proved lemma, which first appears in [8, Lemma 2.1].

Lemma 4.1. *Let d, m and n be positive integers with $m \leqslant n-1$. Let r be an integer satisfying $dm \equiv -r \pmod{n}$. Then, for $0 \leqslant k \leqslant m$, we have*

$$\frac{(aq^r;q^d)_{m-k}}{(q^d/a;q^d)_{m-k}} \equiv (-a)^{m-2k}\frac{(aq^r;q^d)_k}{(q^d/a;q^d)_k} q^{m(dm-d+2r)/2+(d-r)k} \pmod{\Phi_n(q)}.$$

Using the above lemma, we can establish the following q-congruence.

Lemma 4.2. *Let $n>2$ be an integer with $n \equiv 2 \pmod 5$. Let a and b be indeterminates. Then*

$$\sum_{k=0}^{n-1} \frac{(1+q^{5k+1})(aq^2,q^2/a,bq^2,q^2/b,q^2;q^5)_k}{(1+q)(aq^5,q^5/a,bq^5,q^5/b,q^5;q^5)_k} q^{5k} \equiv 0 \pmod{\Phi_n(q)}. \tag{4.1}$$

Proof. In view of Lemma 4.1, for $0 \leqslant k \leqslant (n-2)/5$,

$$\frac{(aq^2;q^5)_{(n-2)/5-k}}{(q^5/a;q^5)_{(n-2)/5-k}} \equiv (-a)^{(n-2)/5-2k}\frac{(aq^2;q^5)_k}{(q^5/a;q^5)_k} q^{(n-2)(n-3)/10+3k} \pmod{\Phi_n(q)}.$$

By making use of the above q-congruence five times, and noticing $q^n \equiv 1 \pmod{\Phi_n(q)}$, we can verify that

$$\begin{aligned}
&\frac{(1+q^{5((n-2)/5-k)+1})(aq^2,q^2/a,bq^2,q^2/b,q^2;q^5)_{(n-2)/5-k}}{(1+q)(aq^5,q^5/a,bq^5,q^5/b,q^5;q^5)_{(n-2)/5-k}} q^{5(n-2)/5-5k} \\
&\quad \equiv -\frac{(1+q^{5k+1})(aq^2,q^2/a,bq^2,q^2/b,q^2;q^5)_k}{(1+q)(aq^5,q^5/a,bq^5,q^5/b,q^5;q^5)_k} q^{5k} \pmod{\Phi_n(q)}.
\end{aligned}$$

Namely, modulo $\Phi_n(q)$, the k-th and $((n-2)/5-k)$-th summands on the left-hand side of (4.1) cancel each other for $0 \leqslant k \leqslant (n-2)/5$. Therefore, the left-hand side of (4.1) truncated at $k=(n-2)/5$ is congruent to 0 modulo $\Phi_n(q)$. Further, for $(n-2)/5 < k \leqslant n-1$, the k-th summand on the left-hand side of (4.1) vanishes modulo $\Phi_n(q)$ because $(q^2;q^5)_k$ contains the factor $1-q^n$. This proves (4.1). ☐

We are now able to give a parametric version of Theorem 1.7.

Lemma 4.3. *Let $n>2$ be an integer with $n \equiv 2 \pmod 5$. Let a and b be indeterminates. Then, modulo $\Phi_n(q)(1-aq^n)(a-q^n)(1-bq^n)(b-q^n)$,*

$$\begin{aligned}
&\sum_{k=0}^{n-1} \frac{(1+q^{5k+1})(aq^2,q^2/a,bq^2,q^2/b,q^2;q^5)_k}{(1+q)(aq^5,q^5/a,bq^5,q^5/b,q^5;q^5)_k} q^{5k} \\
&\quad \equiv \frac{(1-bq^n)(b-q^n)(-1-a^2+aq^n)}{(a-b)(1-ab)} \frac{(bq^4,q^4/b,q^3,q^7;q^5)_{(n-2)/5}}{(bq^5,q^5/b,q^2,q^6;q^5)_{(n-2)/5}} \\
&\qquad + \frac{(1-aq^n)(a-q^n)(-1-b^2+bq^n)}{(b-a)(1-ba)} \frac{(aq^4,q^4/a,q^3,q^7;q^5)_{(n-2)/5}}{(aq^5,q^5/a,q^2,q^6;q^5)_{(n-2)/5}}.
\end{aligned} \tag{4.2}$$

Proof. Like the proof of Theorem 1.4, we can prove the truth of (4.2) modulo $(1-aq^n)(a-q^n)(1-bq^n)(b-q^n)$. This is because, for $a=q^n$ or $a=q^{-n}$, the left-hand side of (4.2) can be written as

$$\sum_{k=0}^{(n-2)/5}\frac{(1+q^{5k+1})(q^{2-n},q^{2+n},bq^2,q^2/b,q^2;q^5)_k}{(1+q)(q^{5+n},q^{5-n},bq^5,q^5/b,q^5;q^5)_k}q^{5k}$$

$$={}_8\phi_7\left[\begin{matrix} q^2, & q^6, & -q^6, & q, & bq^2, & q^2/b, & q^{2+n}, & q^{2-n} \\ & q, & -q, & q^6, & q^5/b, & bq^5, & q^{5-n}, & q^{5+n}\end{matrix};q^5,q^5\right], \tag{4.3}$$

where we have used $(q^{2-n};q^5)_k=0$ for $k>(n-2)/5$. In light of (2.1), the right-hand side of (4.3) is equal to

$$\frac{(bq^4,q^4/b,q^3,q^7;q^5)_{(n-2)/5}}{(bq^5,q^5/b,q^2,q^6;q^5)_{(n-2)/5}}.$$

Since the above expression is congruent to 0 modulo $\Phi_n(q)$, by Lemma 4.2, we know that (4.2) is also true modulo $\Phi_n(q)$. As $(1-aq^n)(a-q^n)(1-bq^n)(b-q^n)$ is coprime with $\Phi_n(q)$, we complete the proof of the lemma. $\square$

Proof of Theorem 1.7. Letting $b=1$ in (4.2), we acquire the following q-congruence: modulo $\Phi_n(q)^3(1-aq^n)(a-q^n)$,

$$\sum_{k=0}^{n-1}\frac{(1+q^{5k+1})(aq^2,q^2/a,q^2,q^2,q^2;q^5)_k}{(1+q)(aq^5,q^5/a,q^5,q^5,q^5;q^5)_k}q^{5k}$$

$$\equiv(1-q^n)^2\frac{(q^4,q^4,q^3,q^7;q^5)_{(n-2)/5}}{(q^5,q^5,q^2,q^6;q^5)_{(n-2)/5}}+\frac{a(1-q^n)^2(2-q^n)}{(1-a)^2}\frac{(q^4,q^4,q^3,q^7;q^5)_{(n-2)/5}}{(q^5,q^5,q^2,q^6;q^5)_{(n-2)/5}}$$

$$-\frac{(1-aq^n)(a-q^n)(2-q^n)}{(1-a)^2}\frac{(aq^4,q^4/a,q^3,q^7;q^5)_{(n-2)/5}}{(aq^5,q^5/a,q^2,q^6;q^5)_{(n-2)/5}}. \tag{4.4}$$

By the L'Hôpital rule, there holds

$$\lim_{a\to1}\left\{\frac{a(1-q^n)^2}{(1-a)^2}\frac{(q^4;q^5)_{(n-2)/5}^2}{(q^5;q^5)_{(n-2)/5}^2}-\frac{(1-aq^n)(a-q^n)}{(1-a)^2}\frac{(aq^4,q^4/a;q^5)_{(n-2)/5}}{(q^5/a,aq^5;q^5)_{(n-2)/5}}\right\}$$

$$=\frac{(q^4;q^5)_{(n-2)/5}^2}{(q^5;q^5)_{(n-2)/5}^2}\left\{q^n+[n]^2\sum_{j=1}^{(n-2)/5}\left(\frac{q^{5j-1}}{[5j-1]^2}-\frac{q^{5j}}{[5j]^2}\right)\right\}. \tag{4.5}$$

Hence, taking $a\to1$ in (4.4), and applying (4.5) and the q-congruence $[n]^2(2-q^n)\equiv[n]^2q^{-n}\pmod{\Phi_n(q)^4}$, we are led to the q-supercongruence (1.9). $\square$

5. More q-supercongruences on truncated sums

We first give a result similar to Theorem 1.1.

Theorem 5.1. *Let n be a positive integer with $n\equiv 3 \pmod 4$. Then*

$$\sum_{k=0}^{n-1}\frac{(1+q^{8k+1})(q^2;q^8)_k^4}{(1+q)(q^8;q^8)_k^4}q^{8k}\equiv 0 \pmod{\Phi_n(q)^3}. \tag{5.1}$$

Sketch of proof. The proof is analogous to that of Theorem 1.1. This time, we have the following parametric version of (5.1): modulo $(1-abq^{6n})(a-q^{6n})(b-q^{6n})$,

$$\begin{aligned}\sum_{k=0}^{n-1}&\frac{(1+q^{8k+1})(q^2/a,q^2/b,abq^2,q^2;q^8)_k}{(1+q)(aq^8,bq^8,q^8/ab,q^8;q^8)_k}q^{8k}\\ &\equiv\frac{(b-q^{6n})(ab^2-a^2b-1+abq^{6n})}{(a-b)(1-ab^2)}\frac{(q^{10},q^4,bq^3,bq^7;q^8)_{(3n-1)/4}}{(q^9,q^5,bq^2,bq^8;q^8)_{(3n-1)/4}}\\ &\quad+\frac{(a-q^{6n})(1-abq^{6n})}{(a-b)(1-ab^2)}\frac{(q^{10},q^4,aq^3,aq^7;q^8)_{(3n-1)/4}}{(q^9,q^5,aq^2,aq^8;q^8)_{(3n-1)/4}}.\end{aligned} \tag{5.2}$$

Letting $b=1$, we conclude from (5.2) that, modulo $\Phi_n(q)(1-aq^{6n})(a-q^{6n})$,

$$\sum_{k=0}^{n-1}\frac{(1+q^{8k+1})(q^2/a,q^2,aq^2,q^2;q^8)_k}{(1+q)(aq^8,q^8,q^8/a,q^8;q^8)_k}q^{8k}\equiv\frac{(q^{10},q^4,q^3,q^7;q^8)_{(3n-1)/4}}{(q^9,q^5,q^2,q^8;q^8)_{(3n-1)/4}}.$$

Finally, taking $a=1$ in the above q-congruence, we complete the proof. □

It is easy to see that Theorem 5.1 has the following conclusion: for any prime $p\equiv 3 \pmod 4$,

$$ {}_4F_3\left[\begin{matrix}\frac14,\frac14,\frac14,\frac14\\ 1,1,1\end{matrix};1\right]_{p-1}\equiv 0 \pmod{p^3}, \tag{5.3}$$

of which a more general form was conjectured by the author (see [6, Conjecture 6.1]) and proved by Wang and Xia [19].

We also have the following two results similar to Theorem 1.7.

Theorem 5.2. *Let n be a positive integer with $n\equiv 3 \pmod 5$. Then, modulo $\Phi_n(q)^4$,*

$$\sum_{k=0}^{n-1}\frac{(1+q^{5k+1})(q^2;q^5)_k^5}{(1+q)(q^5;q^5)_k^5}q^{5k}\equiv\frac{(q^3,q^4,q^4,q^7;q^5)_{(4n-2)/5}}{(q^2,q^5,q^5,q^6;q^5)_{(4n-2)/5}}\left\{1+\frac{[4n]^2q^{-n}}{[3n]^2}\right\}. \tag{5.4}$$

Sketch of proof. We have the following q-congruence: modulo $(1-aq^{4n})(a-q^{4n})(1-bq^{4n})(b-q^{4n})$,

$$\sum_{k=0}^{n-1}\frac{(1+q^{5k+1})(aq^2,q^2/a,bq^2,q^2/b,q^2;q^5)_k}{(1+q)(aq^5,q^5/a,bq^5,q^5/b,q^5;q^5)_k}q^{5k}$$

$$
\equiv \frac{(1-bq^{4n})(b-q^{4n})(-1-a^2+aq^{4n})}{(a-b)(1-ab)} \frac{(bq^4,q^4/b,q^3,q^7;q^5)_{(4n-2)/5}}{(bq^5,q^5/b,q^2,q^6;q^5)_{(4n-2)/5}}
$$

$$
+\frac{(1-aq^{4n})(a-q^{4n})(-1-b^2+bq^{4n})}{(b-a)(1-ba)} \frac{(aq^4,q^4/a,q^3,q^7;q^5)_{(4n-2)/5}}{(aq^5,q^5/a,q^2,q^6;q^5)_{(4n-2)/5}}. \tag{5.5}
$$

First letting $b=1$ and then taking $a\to 1$ in (5.5), we get the q-supercongruence: modulo $\Phi_n(q)^4$,

$$
\sum_{k=0}^{n-1}\frac{(1+q^{5k+1})(q^2;q^5)_k^5}{(1+q)(q^5;q^5)_k^5}q^{5k}
$$

$$
\equiv \frac{(q^3,q^4,q^4,q^7;q^5)_{(4n-2)/5}}{(q^2,q^5,q^5,q^6;q^5)_{(4n-2)/5}} \left\{1+[4n]^2q^{-4n}\sum_{j=1}^{(4n-2)/5}\left(\frac{q^{5j-1}}{[5j-1]^2}-\frac{q^{5j}}{[5j]^2}\right)\right\},
$$

which is equivalent to (5.4). □

From Theorem 5.2 we can deduce that, for any prime $p\equiv 3 \pmod 5$,

$$
{}_5F_4\left[\begin{matrix}\frac25,\frac25,\frac25,\frac25,\frac25\\ 1,1,1,1\end{matrix};1\right]_{p-1} \equiv \frac{50(\frac35)_{(4p-2)/5}(\frac45)_{(4p-2)/5}^2}{9(1)_{(4p-2)/5}^2(\frac65)_{(4p-2)/5}}p \pmod{p^4}.
$$

Theorem 5.3. *Let n be a positive integer with $n\equiv 4 \pmod 5$. Then, modulo $\Phi_n(q)^5$,*

$$
\sum_{k=0}^{n-1}\frac{(1+q^{5k+1})(q^2;q^5)_k^5}{(1+q)(q^5;q^5)_k^5}q^{5k} \equiv \frac{(q^3,q^4,q^4,q^7;q^5)_{(3n-2)/5}}{(q^2,q^5,q^5,q^6;q^5)_{(3n-2)/5}}\left\{1+\frac{[3n]^2q^{-2n}}{[n]^2}\right\}. \tag{5.6}
$$

Sketch of proof. Modulo $\Phi_n(q)(1-aq^{3n})(a-q^{3n})(1-bq^{3n})(b-q^{3n})$, there holds

$$
\sum_{k=0}^{n-1}\frac{(1+q^{5k+1})(aq^2,q^2/a,bq^2,q^2/b,q^2;q^5)_k}{(1+q)(aq^5,q^5/a,bq^5,q^5/b,q^5;q^5)_k}q^{5k}
$$

$$
\equiv \frac{(1-bq^{3n})(b-q^{3n})(-1-a^2+aq^{3n})}{(a-b)(1-ab)} \frac{(bq^4,q^4/b,q^3,q^7;q^5)_{(3n-2)/5}}{(bq^5,q^5/b,q^2,q^6;q^5)_{(3n-2)/5}}
$$

$$
+\frac{(1-aq^{3n})(a-q^{3n})(-1-b^2+bq^{3n})}{(b-a)(1-ba)} \frac{(aq^4,q^4/a,q^3,q^7;q^5)_{(3n-2)/5}}{(aq^5,q^5/a,q^2,q^6;q^5)_{(3n-2)/5}}. \tag{5.7}
$$

The rest is as before. □

It follows from Theorem 5.3 that, for primes $p\equiv 4 \pmod 5$,

$$
{}_5F_4\left[\begin{matrix}\frac25,\frac25,\frac25,\frac25,\frac25\\ 1,1,1,1\end{matrix};1\right]_{p-1} \equiv 15\frac{(\frac35)_{(3p-2)/5}(\frac45)_{(3p-2)/5}^2}{(1)_{(3p-2)/5}^2(\frac65)_{(3p-2)/5}}p \pmod{p^5}.
$$

6. Two open problems

It seems that the following generalization of (5.2) is true.

Conjecture 6.1. *The q-supercongruence* (5.4) *holds modulo* $\Phi_n(q)^5$.

To prove the above conjecture, it suffices to show that (5.5) holds modulo $\Phi_n(q)$. But the same technique in the proof of Lemma 4.2 does not work here.

Numerical computation implies that the following q-analogue of (1.2) modulo p should be true.

Conjecture 6.2. *Let $n > 1$ be an integer with $n \equiv 1 \pmod 4$. Then*

$$\sum_{k=0}^{n-1} \frac{(q;q^4)_k^4}{(q^4;q^4)_k^4} q^{4k} \equiv q^{(n-1)/4} \frac{(q;q^2)_{(n-1)/4}^2 (q^2;q^4)_{(n-1)/4}}{(q^2;q^2)_{(n-1)/4}^2 (q^4;q^4)_{(n-1)/4}} \pmod{\Phi_n(q)}. \tag{6.1}$$

Furthermore, can we extend the q-congruence (6.1) to the modulus $\Phi_n(q)^2$ (or higher powers of $\Phi_n(q)$) case? We leave this task to an interested reader.

Data Availability Statements. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Acknowledgment. The author would like to thank the anonymous referee for a careful reading of a previous version of this paper.

References

[1] R. Barman and N. Saikia, Supercongruences for truncated hypergeometric series and p-adic gamma function, Math. Proc. Camb. Phil. Soc. 168 (2020), 171–195.
[2] A. Deines, J. G. Fuselier, L. Long, H. Swisher, and F.-T. Tu, Hypergeometric series, truncated hypergeometric series, and Gaussian hypergeometric functions, in: Directions in number theory, Assoc. Women Math. Ser. 3, Springer, Cham, 2016, 125–159.
[3] G. Gasper and M. Rahman, Basic Hypergeometric Series, Second Edition, Encyclopedia of Mathematics and Its Applications 96, Cambridge University Press, Cambridge, 2004.
[4] G. Gu and X. Wang, Proof of some conjectures of Guo and of Tang, Forum Math. 37 (2025), 811–820.
[5] G. Gu and X. Wang, Proof of two conjectures of Guo and of Tang, J. Math. Anal. Appl. 541 (2025), Art. 128712.
[6] V.J.W. Guo, Factors of some truncated basic hypergeometric series, J. Math. Anal. Appl. 476 (2019), 851–859.
[7] V.J.W. Guo, Some q-supercongruences from Gasper's Karlsson–Minton type summation, Ramanujan J. 60 (2023), 825–835.
[8] V.J.W. Guo and M.J. Schlosser, A new family of q-supercongruences modulo the fourth power of a cyclotomic polynomial, Results Math. 75 (2020), Art. 155.
[9] V.J.W. Guo and W. Zudilin, A q-microscope for supercongruences, Adv. Math. 346 (2019), 329–358.
[10] Q. Hu, New q-analogues of Van Hamme's (F.2) supercongruence and of some related supercongruences, Math. Slovaca 74 (2024), 629–636.
[11] L. Li and S.-D. Wang, Proof of a q-supercongruence conjectured by Guo and Schlosser, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM 114 (2020), Art. 190.
[12] L. Long, Hypergeometric evaluation identities and supercongruences, Pacific J. Math. 249 (2011), 405–418.

[13] L. Long and R. Ramakrishna, Some supercongruences occurring in truncated hypergeometric series, Adv. Math. 290 (2016), 773–808.
[14] S. Maity and R. Barman, A q-supercongruence from finite q-binomial theorem, Res. Number Theory 11 (2025), Art. 34.
[15] E. Mortenson, A supercongruence conjecture of Rodriguez-Villegas for a certain truncated hypergeometric function, J. Number Theory 99 (2003), 139–147.
[16] H. Pan, R. Tauraso, and C. Wang, A local-global theorem for p-adic supercongruences, J. Reine Angew. Math. 790 (2022), 53–83.
[17] H. Song and C. Wang, Further generalizations of the (A.2) and (H.2) supercongruences of Van Hamme, Results Math. 79 (2024), Art. 147.
[18] C. Wang and H. Pan, Supercongruences concerning truncated hypergeometric series, Math. Z. 300 (2022), Art. 161177.
[19] C. Wang and W. Xia, Divisibility results concerning truncated hypergeometric series, J. Math. Anal. Appl. 491 (2020), Art. 124402.
[20] C. Wei, q-Supercongruences from Gasper and Rahman's summation formula, Adv. Appl. Math. 139 (2022), Art. 102376.
[21] C. Wei, A q-supercongruence from a q-analogue of Whipple's ${}_3F_2$ summation formula, J. Combin. Theory Ser. A 194 (2023), Art. 105705.
[22] C. Wei, q-supercongruences from Jackson's ${}_8\phi_7$ summation and Watson's ${}_8\phi_7$ transformation, J. Combin. Theory Ser. A 204 (2024), Art. 105853.

School of Mathematics, Hangzhou Normal University, Hangzhou 311121, People's Republic of China
E-mail address: jwguo@math.ecnu.edu.cn