Two q-congruences related to a congruence of Deines–Fuselier–Long–Swisher–Tu

Victor J. W. Guo

School of Mathematics, Hangzhou Normal University, Hangzhou 311121, People's Republic of China
jwguo@hzнu.edu.cn

Abstract. In 2016, Deines, Fuselier, Long, Swisher, and Tu proved that, for any integer $d \geqslant 3$ and prime $p \equiv 1 \pmod{d}$,

$$\sum_{k=0}^{p-1} (-1)^{dk} \binom{\frac{1-d}{d}}{k}^d \equiv -\Gamma_p(\tfrac{1}{d})^d \pmod{p^2},$$

where $\Gamma_p(x)$ denotes the p-adic Gamma function. They also conjectured that the above congruence holds modulo p^3. A q-analogue of the $d = 3$ case of this congruence modulo p^3 was recently given by Wei and Qin (Mediterr J Math 22:113, 2025) In this paper, we present two q-congruences related to this congruence modulo p^3 for $d = 4$ and $d = 5$, respectively. Our proofs employ Watson's ${}_8\phi_7$ transformation, the creative microscoping method developed in (Adv Math 346:329–358, 2019), and the Chinese remainder theorem for polynomials.

Keywords: congruences; q-congruences; p-adic Gamma function; cyclotomic polynomials; Watson's ${}_8\phi_7$ transformation; creative microscoping; Chinese remainder theorem for polynomials.

AMS Subject Classifications: 33D15; 11A07; 11B65

1. Introduction

Define the truncated hypergeometric series as follows:

$${}_{r+1}F_r \left[\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix} ; z \right]_m := \sum_{k=0}^{m} \frac{(a_1)_k (a_2)_k \cdots (a_{r+1})_k}{(b_1)_k \cdots (b_r)_k k!} z^k,$$

where $(x)_k = x(x+1)\cdots(x+k-1)$ is the *Pochhammer symbol*. In 1997, Van Hamme [14, (H.2)] proved the following congruence: for any odd prime p,

$${}_3F_2 \left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, \end{matrix} ; 1 \right]_{p-1} \equiv \begin{cases} -\Gamma_p(\frac{1}{4})^4 \pmod{p^2} & \text{if } p \equiv 1 \pmod{4}, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}, \end{cases} \tag{1.1}$$

where $\Gamma_p(x)$ denotes the *p-adic Gamma function*. In 2016, Long and Ramakrishna [11, Theorem 3] obtained the following generalization of (1.1):

$$
{}_3F_2\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ 1,1,\end{matrix};1\right]_{p-1} \equiv \begin{cases}-\Gamma_p(\frac{1}{4})^4 \pmod{p^3} & \text{if } p\equiv 1 \pmod 4,\\ -\frac{p^2}{16}\Gamma_p(\frac{1}{4})^4 \pmod{p^3} & \text{if } p\equiv 3 \pmod 4.\end{cases} \tag{1.2}
$$

Meanwhile, they also gave a similar congruence:

$$
{}_3F_2\left[\begin{matrix}\frac{1}{3},\frac{1}{3},\frac{1}{3}\\ 1,1,\end{matrix};1\right]_{p-1} \equiv \begin{cases}-\Gamma_p(\frac{1}{3})^6 \pmod{p^3} & \text{if } p\equiv 1 \pmod 6,\\ -\frac{p^2}{3}\Gamma_p(\frac{1}{3})^6 \pmod{p^3} & \text{if } p\equiv 5 \pmod 6.\end{cases} \tag{1.3}
$$

In the same year, Deines et al. [1, Theorem 6] proved the following congruence: for integers $d\geqslant 3$ and primes $p\equiv 1 \pmod d$,

$$
{}_dF_{d-1}\left[\begin{matrix}\frac{d-1}{d},\ldots,\frac{d-1}{d}\\ 1,\ldots,1\end{matrix};1\right]_{p-1} \equiv -\Gamma_p(\tfrac{1}{d})^d \pmod{p^2}. \tag{1.4}
$$

The congruence (1.4) also holds for $n=2$, which was conjectured by Rodriguez-Villegas [13] and confirmed by Mortenson [12]. Deines et al. [1] numerically observed that (1.4) also holds modulo p^3, which was later confirmed by Wang and Pan [16], such as: for any prime $p\equiv 1 \pmod 3$,

$$
{}_3F_2\left[\begin{matrix}\frac{2}{3},\frac{2}{3},\frac{2}{3}\\ 1,1\end{matrix};1\right]_{p-1} \equiv -\Gamma_p(\tfrac{1}{3})^3 \pmod{p^3}, \tag{1.5}
$$

for any prime $p\equiv 1 \pmod 4$,

$$
{}_4F_3\left[\begin{matrix}\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4}\\ 1,1,1\end{matrix};1\right]_{p-1} \equiv -\Gamma_p(\tfrac{1}{4})^4 \pmod{p^3}, \tag{1.6}
$$

and for any prime $p\equiv 1 \pmod 5$,

$$
{}_5F_4\left[\begin{matrix}\frac{4}{5},\frac{4}{5},\frac{4}{5},\frac{4}{5},\frac{4}{5}\\ 1,1,1,1\end{matrix};1\right]_{p-1} \equiv -\Gamma_p(\tfrac{1}{5})^5 \pmod{p^3}. \tag{1.7}
$$

Let q be a complex number with $|q|<1$. Recall that the *q-shifted factorial* is defined by $(a;q)_0=1$ and $(a;q)_n=(1-a)(1-aq)\ldots(1-aq^{n-1})$ for $n\geqslant 1$ or $n=\infty$. For simplicity, we will also adopt the abbreviated notation $(a_1,\ldots,a_m;q)_n=(a_1;q)_n\ldots(a_m;q)_n$ for $n\geqslant 0$ or $n=\infty$. Let $\Phi_n(q)$ be the n-th *cyclotomic polynomial* in q, which is irreducible over the integers and can be factorized in the field of complex numbers as

$$
\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(n,k)=1}}(q-\zeta^k),
$$

where ζ is an n-th primitive root of unity. Moreover, let $[n]=(1-q^n)/(1-q)$ stand for the q-*integer*. It is well known that $\Phi_p(q)=[p]$ for any prime p.

In 2019, the author and Zudilin [9, Theorem 2] presented a q-analogue of (1.1) as follows: modulo $\Phi_n(q)^2$,

$$\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{2k}\equiv\begin{cases}\dfrac{(q^2;q^4)_{(n-1)/4}^2}{(q^4;q^4)_{(n-1)/4}^2}q^{(n-1)/2} & \text{if } n\equiv 1 \pmod 4,\\ 0 & \text{if } n\equiv 3 \pmod 4\end{cases}\tag{1.8}$$

(a special case has already been given in [7, Corollary 1.2]). Later, a generalization of (1.8) modulo $\Phi_n(q)^3$ for $n\equiv 3 \pmod 4$ was given by the author [5]. The corresponding generalizations for $n\equiv 1 \pmod 4$ were given by Wei [19] and Wang [15]. Moreover, Wei, Liu, and Wang [21] gave a q-analogue of (1.3). The author [4] formulated a q-analogue of Deines–Fuselier–Long–Swisher–Tu's congruence (1.4): for integers $d,n>1$ with $n\equiv 1 \pmod d$, modulo $\Phi_n(q)^2$,

$$\sum_{k=0}^{n-1}\frac{(q^{d-1};q^d)_k^d q^{dk}}{(q^d;q^d)_k^d}\equiv\frac{(q^d;q^d)_{(d-1)(n-1)/d}\,q^{(d-1)(n-1)(d+n-1)/(2d)}}{(q^d;q^d)_{(n-1)/d}^{d-1}(-1)^{(d-1)(n-1)/d}}.\tag{1.9}$$

Recently, Wei and Wang [22] presented the following q-analogue of (1.5): for $n\equiv 1 \pmod 3$, modulo $\Phi_n(q)^3$,

$$\sum_{k=0}^{n-1}\frac{(1+q^{3k+1})(q^2;q^3)_k^3}{(1+q)(q^3;q^3)_k^3}\equiv q^{(2-2n)/3}[n]_{q^2}\frac{(q^3;q^3)_{(2n-2)/3}}{(q^4;q^3)_{(2n-2)/3}}\left\{1-4[n]^2\sum_{j=1}^{(n-1)/3}\frac{q^{3j}}{[3j]^2}\right\}.$$

The first objective of this paper is to give the following q-congruence related to (1.6).

Theorem 1.1. *Let $n>1$ be an integer with $n\equiv 1 \pmod 4$. Then, modulo $\Phi_n(q)^3$,*

$$\sum_{k=0}^{n-1}\frac{(1+q^{8k+3})(q^6;q^8)_k^4}{(1+q^3)(q^8;q^8)_k^4}\equiv q^{(9-9n)/2}\frac{(q^{12},q^{14};q^8)_{(3n-3)/4}}{(q^6,q^8;q^8)_{(3n-3)/4}}\left\{1-[6n]^2\sum_{j=1}^{(3n-3)/4}\frac{q^{8j}}{[8j]^2}\right\}$$
$$\times\sum_{k=0}^{(3n-3)/4}\frac{(q^4,q^6,q^6,q^6;q^8)_k}{(q^7,q^8,q^{11},q^{12};q^8)_k}q^{8k}.\tag{1.10}$$

Letting $n=p^r$ be a prime power and taking $q\to 1$ in (1.10), we obtain the following congruence.

Corollary 1.2. *Let p be an odd prime and r a positive integer with $p^r\equiv 1 \pmod 4$. Then*

$${}_4F_3\left[\begin{matrix}\frac34,\frac34,\frac34,\frac34\\ 1,1,1\end{matrix};1\right]_{p^r-1}\equiv\frac{(\frac32)_{(3p^r-3)/4}(\frac74)_{(3p^r-3)/4}}{(\frac34)_{(3p^r-3)/4}(1)_{(3p^r-3)/4}}\left\{1-\frac{9}{16}p^{2r}\sum_{j=1}^{(3p^r-3)/4}\frac{1}{j^2}\right\}$$
$$\times\sum_{k=1}^{(3p^r-3)/4}\frac{(\frac12)_k(\frac34)_k^3}{(\frac78)_k(1)_k(\frac{11}{8})_k(\frac32)_k}\pmod{p^3}.\tag{1.11}$$

Combining the congruence (1.5) and the $r=1$ case of (1.11), we get the following conclusion.

Corollary 1.3. *Let $p\equiv 1 \pmod 4$ be a prime. Then, modulo p^3,*

$$\frac{(\frac{3}{2})_{(3p-3)/4}(\frac{7}{4})_{(3p-3)/4}}{(\frac{3}{4})_{(3p-3)/4}(1)_{(3p-3)/4}}\left\{1-\frac{9}{16}p^2\sum_{j=1}^{(3p-3)/4}\frac{1}{j^2}\right\}\sum_{k=0}^{(3p-3)/4}\frac{(\frac{1}{2})_k(\frac{3}{4})_k^3}{(\frac{7}{8})_k(1)_k(\frac{11}{8})_k(\frac{3}{2})_k}\equiv -\Gamma_p(\tfrac{1}{4})^4.$$

The second objective of this paper is to present the following q-congruence related to (1.7).

Theorem 1.4. *Let $n>1$ be an integer with $n\equiv 1 \pmod 5$. Then, modulo $\Phi_n(q)^3$,*

$$\sum_{k=0}^{n-1}\frac{(1+q^{5k+2})(q^4;q^5)_k^5}{(1+q^2)(q^5;q^5)_k^5}\equiv q^{16(1-n)/5}\frac{(q^8,q^9;q^5)_{(4n-4)/5}}{(q^4,q^5;q^5)_{(4n-4)/5}}\left\{1-[4n]^2\sum_{j=1}^{(4n-4)/5}\frac{q^{5j}}{[5j]^2}\right\}$$
$$\times\sum_{k=0}^{(4n-4)/5}\frac{(q^3,q^4,q^4,q^4;q^5)_k}{(q^5,q^5,q^7,q^8;q^5)_k}q^{5k}. \qquad (1.12)$$

For any prime power $n=p^r$, letting $q\to 1$ in (1.10), we obtain the following result.

Corollary 1.5. *Let p be an odd prime and r a positive integer with $p^r\equiv 1 \pmod 5$. Then*

$${}_5F_4\left[\begin{matrix}\frac{4}{5},\frac{4}{5},\frac{4}{5},\frac{4}{5},\frac{4}{5}\\ 1,1,1,1\end{matrix};1\right]_{p-1}\equiv\frac{(\frac{8}{5})_{(4p^r-4)/5}(\frac{9}{5})_{(4p^r-4)/5}}{(\frac{4}{5})_{(4p^r-4)/5}(1)_{(4p^r-4)/5}}\left\{1-\frac{16}{25}p^{2r}\sum_{j=1}^{(4p^r-4)/5}\frac{1}{j^2}\right\}$$
$$\times\sum_{k=0}^{(4p^r-4)/5}\frac{(\frac{3}{5})_k(\frac{4}{5})_k^3}{(1)_k^2(\frac{7}{5})_k(\frac{8}{5})_k}\pmod{p^3}. \qquad (1.13)$$

Comparing the congruences (1.7) with (1.13) for $r=1$ yields the following conclusion.

Corollary 1.6. *Let $p\equiv 1 \pmod 5$ be a prime. Then, modulo p^3,*

$$\frac{(\frac{8}{5})_{(4p-4)/5}(\frac{9}{5})_{(4p-4)/5}}{(\frac{4}{5})_{(4p-4)/5}(1)_{(4p-4)/5}}\left\{1-\frac{16}{25}p^2\sum_{j=1}^{(4p-4)/5}\frac{1}{j^2}\right\}\sum_{k=0}^{(4p-4)/5}\frac{(\frac{3}{5})_k(\frac{4}{5})_k^3}{(1)_k^2(\frac{7}{5})_k(\frac{8}{5})_k}\equiv -\Gamma_p(\tfrac{1}{5})^5.$$

We shall prove Theorems 1.1 and 1.4 in Sections 2 and 3, respectively, where we first establish the corresponding q-congruences modulo $(1-abq^{\alpha n})(a-q^{\alpha n})(b-q^{\alpha n})$. The proofs make use of the creative microscoping method introduced by the author and Zudilin [10], the Chinese remainder theorem for coprime polynomials [3], along with Watson's ${}_8\phi_7$ transformation. For more q-congruences relevant to Watson's transformation, see [6, 8, 18, 20].

2. Proof of Theorem 1.1

Recall that the basic hypergeometric ${}_{r+1}\phi_r$ series (see [2]) is defined by

$$ {}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix}; q, z\right] := \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_k}{(q, b_1, \ldots, b_r; q)_k} z^k. $$

Then Watson's ${}_8\phi_7$ transformation [2, Appendix (III.18)] can be stated as follows:

$$ {}_8\phi_7\left[\begin{matrix} a, & qa^{\frac12}, & -qa^{\frac12}, & b, & c, & d, & e, & q^{-n} \\ & a^{\frac12}, & -a^{\frac12}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1} \end{matrix}; q, \frac{a^2q^{n+2}}{bcde}\right] $$

$$ = \frac{(aq, aq/de; q)_n}{(aq/d, aq/e; q)_n} {}_4\phi_3\left[\begin{matrix} aq/bc, \ d, \ e, \ q^{-n} \\ aq/b, \ aq/c, \ deq^{-n}/a \end{matrix}; q, \ q\right]. \tag{2.1} $$

We first give a parametric version of Theorem 1.1.

Lemma 2.1. *Let $n > 1$ be an integer with $n \equiv 1 \pmod 4$. Let a and b be indeterminates. Then, modulo $(1 - abq^{6n})(a - q^{6n})(b - q^{6n})$,*

$$ \sum_{k=0}^{n-1} \frac{(1 + q^{8k+3})(q^6/a, q^6/b, abq^6, q^6; q^8)_k}{(1 + q^3)(aq^8, bq^8, q^8/ab, q^8; q^8)_k} $$

$$ \equiv \left\{ \frac{(b - q^{6n})(ab^2 - a^2b - 1 + abq^{6n})}{(a - b)(1 - ab^2)} \frac{(q^{14}, q^{2-6n}; q^8)_{(3n-3)/4}}{(bq^8, q^{8-6n}/b; q^8)_{(3n-3)/4}} \right. $$

$$ \left. + \frac{(a - q^{6n})(1 - abq^{6n})}{(a - b)(1 - ab^2)} \frac{(q^{14}, q^{2-6n}; q^8)_{(3n-3)/4}}{(aq^8, q^{8-6n}/a; q^8)_{(3n-3)/4}} \right\} $$

$$ \times \sum_{k=0}^{(3n-3)/4} \frac{(q^4, q^6/a, q^6/b, abq^6; q^8)_k}{(q^7, q^8, q^{11}, q^{12}; q^8)_k} q^{8k}. \tag{2.2} $$

Proof. For $a = q^{6n}$ or $a = q^{-6n}/b$, the left-hand side of (2.2) can be written as

$$ \sum_{k=0}^{n-1} \frac{(1 + q^{8k+3})(q^{6-6n}, q^6/b, bq^{6+6n}, q^6; q^8)_k}{(1 + q^3)(q^{8+6n}, bq^8, q^{8-6n}/b, q^8; q^8)_k} $$

$$ = {}_8\phi_7\left[\begin{matrix} q^6, & q^{11}, & -q^{11}, & q^3, & q^7, & q^6/b, & bq^{6+6n}, & q^{6-6n} \\ & q^3, & -q^3, & q^{11}, & q^7, & bq^8, & q^{8-6n}/b, & q^{8+6n} \end{matrix}; q^8, 1\right]. \tag{2.3} $$

In view of Watson's ${}_8\phi_7$ transformation (2.1), the right-hand side of (2.3) is equal to

$$ \frac{(q^{14}, q^{2-6n}; q^8)_{(3n-3)/4}}{(bq^8, q^{8-6n}/b; q^8)_{(3n-3)/4}} \sum_{k=0}^{(3n-3)/4} \frac{(q^4, q^6/b, bq^{6+6n}, q^{6-6n}; q^8)_k}{(q^7, q^8, q^{11}, q^{12}; q^8)_k} q^{8k} $$

$$ = \frac{(q^{14}, q^{2-6n}; q^8)_{(3n-3)/4}}{(bq^8, q^{8-6n}/b; q^8)_{(3n-3)/4}} \sum_{k=0}^{(3n-3)/4} \frac{(q^4, q^6/a, q^6/b, abq^6; q^8)_k}{(q^7, q^8, q^{11}, q^{12}; q^8)_k} q^{8k} $$

Since the polynomials $1-abq^{6n}$ and $a-q^{6n}$ are coprime with each other, we immediately get the following q-congruence: modulo $(1-abq^{6n})(a-q^{6n})$,

$$\sum_{k=0}^{n-1}\frac{(1+q^{8k+3})(q^6/a,q^6/b,abq^6,q^6;q^8)_k}{(1+q^3)(aq^8,bq^8,q^8/ab,q^8;q^8)_k}$$
$$\equiv\frac{(q^{14},q^{2-6n};q^8)_{(3n-3)/4}}{(bq^8,q^{8-6n}/b;q^8)_{(3n-3)/4}}\sum_{k=0}^{(3n-3)/4}\frac{(q^4,q^6/a,q^6/b,abq^6;q^8)_k}{(q^7,q^8,q^{11},q^{12};q^8)_k}q^{8k}.$$

Interchanging the indeterminates a and b in the above q-congruence, we obtain the following result: modulo $b-q^{6n}$,

$$\sum_{k=0}^{n-1}\frac{(1+q^{8k+3})(q^6/a,q^6/b,abq^6,q^6;q^8)_k}{(1+q^3)(aq^8,bq^8,q^8/ab,q^8;q^8)_k}$$
$$\equiv\frac{(q^{14},q^{2-6n};q^8)_{(3n-3)/4}}{(aq^8,q^{8-6n}/a;q^8)_{(3n-3)/4}}\sum_{k=0}^{(3n-3)/4}\frac{(q^4,q^6/a,q^6/b,abq^6;q^8)_k}{(q^7,q^8,q^{11},q^{12};q^8)_k}q^{8k}.$$

Moreover, we can easily check that

$$\frac{(b-q^{6n})(ab^2-a^2b-1+abq^{6n})}{(a-b)(1-ab^2)}\equiv 1\pmod{(1-abq^{6n})(a-q^{6n})},\tag{2.4}$$
$$\frac{(a-q^{6n})(1-abq^{6n})}{(a-b)(1-ab^2)}\equiv 1\pmod{b-q^{6n}}.\tag{2.5}$$

Since $(a-q^{6n})(1-abq^{6n})$ and $b-q^{6n}$ are coprime polynomials in q, making use of the Chinese remainder theorem for coprime polynomials along with (2.4) and (2.5), we get the desired q-congruence (2.2). □

Proof of Theorem 1.1. It is easy to see that $1-q^{6n}$ contains the factor $\Phi_n(q)$. Putting $b=1$ in (2.2), and applying the following identity

$$(1-x)(1+a^2-a-x)=(1-a)^2+(1-ax)(a-x),\tag{2.6}$$

we arrive at the following q-congruence: modulo $\Phi_n(q)(1-aq^{6n})(a-q^{6n})$,

$$\sum_{k=0}^{n-1}\frac{(1+q^{8k+1})(q^2/a,q^2,aq^2,q^2;q^8)_k}{(1+q)(aq^8,q^8,q^8/a,q^8;q^8)_k}q^{8k}$$
$$\equiv\frac{(q^{14},q^{2-6n};q^8)_{(3n-3)/4}}{(q^8,q^{8-6n};q^8)_{(3n-3)/4}}\sum_{k=0}^{(3n-3)/4}\frac{(q^4,q^6/a,q^6,aq^6;q^8)_k}{(q^7,q^8,q^{11},q^{12};q^8)_k}q^{8k}$$
$$+\frac{(1-aq^{6n})(a-q^{6n})}{(1-a)^2}\sum_{k=0}^{(3n-3)/4}\frac{(q^4,q^6/a,q^6,aq^6;q^8)_k}{(q^7,q^8,q^{11},q^{12};q^8)_k}q^{8k}$$
$$\times\left\{\frac{(q^{14},q^{2-6n};q^8)_{(3n-3)/4}}{(q^8,q^{8-6n};q^8)_{(3n-3)/4}}-\frac{(q^{14},q^{2-6n};q^8)_{(3n-3)/4}}{(aq^8,q^{8-6n}/a;q^8)_{(3n-3)/4}}\right\}.\tag{2.7}$$

It is clear that

$$\frac{(q^{14},q^{2-6n};q^8)_{(3n-3)/4}}{(q^8,q^{8-6n};q^8)_{(3n-3)/4}}-\frac{(q^{14},q^{2-6n};q^8)_{(3n-3)/4}}{(aq^8,q^{8-6n}/a;q^8)_{(3n-3)/4}}$$
$$\equiv\frac{(q^{14},q^{2-6n};q^8)_{(3n-3)/4}}{(q^8,q^8;q^8)_{(3n-3)/4}}-\frac{(q^{14},q^{2-6n};q^8)_{(3n-3)/4}}{(aq^8,q^8/a;q^8)_{(3n-3)/4}}\pmod{\Phi_n(q)^3}. \tag{2.8}$$

By the L'Hôpital rule, there holds

$$\lim_{a\to 1}\frac{(1-aq^{6n})(a-q^{6n})}{(1-a)^2}\left\{\frac{(q^{14},q^{2-6n};q^8)_{(3n-3)/4}}{(q^8,q^8;q^8)_{(3n-3)/4}}-\frac{(q^{14},q^{2-6n};q^8)_{(3n-3)/4}}{(aq^8,q^8/a;q^8)_{(3n-3)/4}}\right\}$$
$$=-[6n]^2\frac{(q^{14},q^{2-6n};q^8)_{(3n-3)/4}}{(q^8,q^8;q^8)_{(3n-3)/4}}\sum_{j=1}^{(3n-3)/4}\frac{q^{8j}}{[8j]^2}.$$

Hence, substituting (2.8) into (2.7), taking the limits as $a\to 1$, and noticing the fact that

$$\frac{(q^{14},q^{2-6n};q^8)_{(3n-3)/4}}{(q^8,q^{8-6n};q^8)_{(3n-3)/4}}=q^{(9-9n)/2}\frac{(q^{12},q^{14};q^8)_{(3n-3)/4}}{(q^6,q^8;q^8)_{(3n-3)/4}},$$

we complete the proof of the theorem. □

3. Proof of Theorem 1.4

Similarly as before, we first establish a parametric version of Theorem 1.4.

Lemma 3.1. *Let $n>1$ be an integer with $n\equiv 1\pmod 5$. Let a and b be indeterminates. Then, modulo $(1-abq^{4n})(a-q^{4n})(b-q^{4n})$,*

$$\sum_{k=0}^{n-1}\frac{(1+q^{5k+2})(q^4/a,q^4/b,abq^4,q^4,q^4;q^5)_k}{(1+q^2)(aq^5,bq^5,q^5/ab,q^5,q^5;q^5)_k}$$
$$\equiv\left\{\frac{(b-q^{4n})(ab^2-a^2b-1+abq^{4n})}{(a-b)(1-ab^2)}\frac{(q^9,q^{1-4n};q^5)_{(4n-4)/5}}{(bq^5,q^{5-4n}/b;q^5)_{(4n-4)/5}}\right.$$
$$\left.+\frac{(a-q^{4n})(1-abq^{4n})}{(a-b)(1-ab^2)}\frac{(q^9,q^{1-4n};q^5)_{(4n-4)/5}}{(aq^5,q^{5-4n}/a;q^5)_{(4n-4)/5}}\right\}$$
$$\times\sum_{k=0}^{(4n-4)/5}\frac{(q^3,q^4/a,q^4/b,abq^4;q^5)_k}{(q^5,q^5,q^7,q^8;q^5)_k}q^{5k}. \tag{3.1}$$

Proof. For $a=q^{4n}$ or $a=q^{-4n}/b$, the left-hand side of (3.1) becomes

$$\sum_{k=0}^{(4n-4)/5}\frac{(1+q^{5k+2})(q^{4-4n},q^4/b,bq^{4+4n},q^4,q^4;q^5)_k}{(1+q^2)(q^{5+4n},bq^5,q^{5-4n}/b,q^5,q^5;q^5)_k}$$
$$={}_8\phi_7\left[\begin{matrix}q^4, & q^7, & -q^7, & q^2, & q^4, & q^4/b, & bq^{4+4n}, & q^{4-4n}\\ & q^2, & -q^2, & q^7, & q^5, & bq^5, & q^{5-4n}/b, & q^{5+4n}\end{matrix};q^5,q^5\right], \tag{3.2}$$

where we have used $(q^{4-4n};q^5)_k=0$ for $k>(4n-4)/5$. By Watson's transformation (2.1), the right-hand side of (3.2) can be written as

$$
\begin{aligned}
&\frac{(q^9,q^{1-4n};q^5)_{(4n-4)/5}}{(bq^5,q^{5-4n}/b;q^5)_{(4n-4)/5}}\sum_{k=0}^{(4n-4)/5}\frac{(q^3,q^4/b,bq^{4+4n},q^{4-4n};q^5)_k}{(q^5,q^5,q^7,q^8;q^5)_k}q^{5k}\\
&=\frac{(q^9,q^{1-4n};q^5)_{(4n-4)/5}}{(bq^5,q^{5-4n}/b;q^5)_{(4n-4)/5}}\sum_{k=0}^{(4n-4)/5}\frac{(q^3,q^4/a,q^4/b,abq^4;q^5)_k}{(q^5,q^5,q^7,q^8;q^5)_k}q^{5k}.
\end{aligned}
$$

Observing that the polynomial $1-abq^{4n}$ is coprime with $a-q^{4n}$, we get the q-congruence: modulo $(1-abq^{4n})(a-q^{4n})$,

$$
\begin{aligned}
&\sum_{k=0}^{n-1}\frac{(1+q^{5k+2})(q^4/a,q^4/b,abq^4,q^4,q^4;q^5)_k}{(1+q^2)(aq^5,bq^5,q^5/ab,q^5,q^5;q^5)_k}\\
&\quad\equiv\frac{(q^9,q^{1-4n};q^5)_{(4n-4)/5}}{(bq^5,q^{5-4n}/b;q^5)_{(4n-4)/5}}\sum_{k=0}^{(4n-4)/5}\frac{(q^3,q^4/a,q^4/b,abq^4;q^5)_k}{(q^5,q^5,q^7,q^8;q^5)_k}q^{5k}. \qquad (3.3)
\end{aligned}
$$

Since the left-hand side of (3.3) is symmetric in a and b, we deduce from (3.3) that, modulo $b-q^{4n}$,

$$
\begin{aligned}
&\sum_{k=0}^{n-1}\frac{(1+q^{5k+2})(q^4/a,q^4/b,abq^4,q^4,q^4;q^5)_k}{(1+q^2)(aq^5,bq^5,q^5/ab,q^5,q^5;q^5)_k}\\
&\quad\equiv\frac{(q^9,q^{1-4n};q^5)_{(4n-4)/5}}{(aq^5,q^{5-4n}/a;q^5)_{(4n-4)/5}}\sum_{k=0}^{(4n-4)/5}\frac{(q^3,q^4/a,q^4/b,abq^4;q^5)_k}{(q^5,q^5,q^7,q^8;q^5)_k}q^{5k}.
\end{aligned}
$$

Clearly, the polynomials $(1-abq^{4n})(a-q^{4n})$, and $b-q^{4n}$ are coprime polynomials in q. Using the relations (2.4) and (2.5) with $q^{6n}\mapsto q^{4n}$, we get the desired q-congruence (3.1). □

Proof of Theorem 1.4. Letting $b=1$ in (2.2), and then applying the identity (2.6), we are led to the following q-congruence: modulo $\Phi_n(q)(1-aq^{4n})(a-q^{4n})$,

$$
\begin{aligned}
&\sum_{k=0}^{n-1}\frac{(1+q^{5k+2})(q^4/a,q^4,aq^4,q^4,q^4;q^5)_k}{(1+q^2)(aq^5,q^5,q^5/a,q^5,q^5;q^5)_k}\\
&\quad\equiv\frac{(q^9,q^{1-4n};q^5)_{(4n-4)/5}}{(q^5,q^{5-4n};q^5)_{(4n-4)/5}}\sum_{k=0}^{(4n-4)/5}\frac{(q^3,q^4/a,q^4,aq^4;q^5)_k}{(q^5,q^5,q^7,q^8;q^5)_k}q^{5k}\\
&\qquad+\frac{(1-aq^{4n})(a-q^{4n})}{(1-a)^2}\sum_{k=0}^{(4n-4)/5}\frac{(q^3,q^4/a,q^4,aq^4;q^5)_k}{(q^5,q^5,q^7,q^8;q^5)_k}q^{5k}\\
&\qquad\times\left\{\frac{(q^9,q^{1-4n};q^5)_{(4n-4)/5}}{(q^5,q^{5-4n};q^5)_{(4n-4)/5}}-\frac{(q^9,q^{1-4n};q^5)_{(4n-4)/5}}{(aq^5,q^{5-4n}/a;q^5)_{(4n-4)/5}}\right\}. \qquad (3.4)
\end{aligned}
$$

It is easy to see that

$$\frac{(q^9,q^{1-4n};q^5)_{(4n-4)/5}}{(q^5,q^{5-4n};q^5)_{(4n-4)/5}}-\frac{(q^9,q^{1-4n};q^5)_{(4n-4)/5}}{(aq^5,q^{5-4n}/a;q^5)_{(4n-4)/5}}$$
$$\equiv\frac{(q^9,q^{1-4n};q^5)_{(4n-4)/5}}{(q^5,q^5;q^5)_{(4n-4)/5}}-\frac{(q^9,q^{1-4n};q^5)_{(4n-4)/5}}{(aq^5,q^5/a;q^5)_{(4n-4)/5}}\pmod{\Phi_n(q)^3}. \tag{3.5}$$

By the L'Hôpital rule, we have

$$\lim_{a\to1}\frac{(1-aq^{4n})(a-q^{4n})}{(1-a)^2}\left\{\frac{(q^9,q^{1-4n};q^5)_{(4n-4)/5}}{(q^5,q^5;q^5)_{(4n-4)/5}}-\frac{(q^9,q^{1-4n};q^5)_{(4n-4)/5}}{(aq^5,q^5/a;q^5)_{(4n-4)/5}}\right\}$$
$$=-[4n]^2\frac{(q^9,q^{1-4n};q^5)_{(4n-4)/5}}{(q^5,q^5;q^5)_{(4n-4)/5}}\sum_{j=1}^{(4n-4)/5}\frac{q^{5j}}{[5j]^2}.$$

Hence, substituting (3.5) into (3.4), taking the limits as $a\to1$, and noticing the fact that

$$\frac{(q^9,q^{1-4n};q^5)_{(4n-4)/5}}{(q^5,q^{5-4n};q^5)_{(4n-4)/5}}=q^{16(1-n)/5}\frac{(q^8,q^9;q^5)_{(4n-4)/5}}{(q^4,q^5;q^5)_{(4n-4)/5}},$$

we finish the proof of the theorem. □

Declarations

Data Availability. Data sharing not applicable to this article.
Conflict of interest. The author declares no conflict of interest.

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