A q-Clausen-Orr type formula and its applications

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Abstract. We show that certain terminating ${}_6\phi_5$ series can be factorized into a product of two ${}_3\phi_2$ series. As applications we prove a summation formula for a product of two q-Delannoy numbers along with some congruences for sums involving q-Delannoy numbers. This confirms three recent conjectures of the second author.

Keywords: q-Delannoy numbers; q-analogue of Clausen's formula; q-binomial theorem; q-Chu-Vandermonde; q-Pfaff-Saalschütz.

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1. Introduction

Clausen's formula

$$\left\{ {}_2F_1\left[\begin{matrix} a, b \\ a+b+1/2 \end{matrix}; z\right]\right\}^2 = {}_3F_2\left[\begin{matrix} 2a, 2b, a+b \\ a+b+1/2, 2a+2b \end{matrix}; z\right]$$

plays a central role in Ramanujan's derivation for various series for $1/\pi$. See [1, 3] for some recent developments of this formula. More general formulas connecting products of two hypergeometric series as a single series were obtained by Orr in 1899 (see [15, p. 75]). This paper was motivated by a recent paper of the second author [7], where he proved some congruences of sums involving even powers of Delannoy numbers and raised some problems of finding the q-analogues. Recall that the Delannoy numbers $D(m,n)$ count lattice paths from $(0,0)$ to (m,n) consisting of horizontal (1,0), vertical (0,1), and diagonal (1,1) steps, and have the following explicit formulas in terms of binomial coefficients:

$$D(m,n) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+m-k}{n} = \sum_{k=0}^{n} \binom{n}{k} \binom{m}{k} 2^k. \tag{1.1}$$

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The reader is referred to Dziemiańczuk [4] and the references therein for how to generalize Delannoy numbers via counting weighted lattice paths. Recall that the *basic hypergeometric series* ${}_r\phi_s$ is defined as

$$ {}_r\phi_s\left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, z\right] = \sum_{k=0}^{\infty} \frac{(a_1;q)_k (a_2;q)_k \cdots (a_r;q)_k}{(q;q)_k (b_1;q)_k \cdots (b_s;q)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r} z^k, $$

where $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n = 1, 2, \dots$, and $(a;q)_0 = 1$. Aiming to answer the q-problems in [7], we are led to prove the following q-Clausen-Orr type formula.

Theorem 1.1. *Let n be a non-negative integer. Then*

$$ {}_3\phi_2\left[\begin{matrix} q^{-n}, a, x \\ c,\ 0 \end{matrix} ; q, q\right] {}_3\phi_2\left[\begin{matrix} q^{-n}, a, c/x \\ c,\ 0 \end{matrix} ; q, q\right] = a^n\, {}_6\phi_5\left[\begin{matrix} q^{-n}, cq^n, a, c/a, x, c/x \\ c, \sqrt{c}, -\sqrt{c}, \sqrt{cq}, -\sqrt{cq} \end{matrix} ; q, q\right]. \tag{1.2} $$

It is interesting to compare (1.2) with Jackson's q-analogue of Clausen's formula [9,10]:

$$ {}_2\phi_1\left[\begin{matrix} a, b \\ abq^{\frac{1}{2}} \end{matrix} ; q, z\right] {}_2\phi_1\left[\begin{matrix} a, b \\ abq^{\frac{1}{2}} \end{matrix} ; q, zq^{\frac{1}{2}}\right] = {}_4\phi_3\left[\begin{matrix} a, b, a^{\frac{1}{2}}b^{\frac{1}{2}}, -a^{\frac{1}{2}}b^{\frac{1}{2}} \\ ab, a^{\frac{1}{2}}b^{\frac{1}{2}}q^{\frac{1}{4}}, -a^{\frac{1}{2}}b^{\frac{1}{2}}q^{\frac{1}{4}} \end{matrix} ; q^{\frac{1}{2}}, z\right], \ |z| < 1, $$

and the following q-analogue of Clausen's formula [5, Appendix (III.22)]:

$$ \left\{ {}_4\phi_3\left[\begin{matrix} a, b, abz, ab/z \\ abq^{\frac{1}{2}}, -abq^{\frac{1}{2}}, -ab \end{matrix} ; q, q\right]\right\}^2 = {}_5\phi_4\left[\begin{matrix} a^2, b^2, ab, abz, ab/z \\ abq^{\frac{1}{2}}, -abq^{\frac{1}{2}}, -ab, a^2b^2 \end{matrix} ; q, q\right], \tag{1.3} $$

where both series are supposed to be terminated. Indeed, letting $c = x^2$, the identity (1.2) reduces to the following formula, which seems to be new.

Corollary 1.2. *Let n be a non-negative integer. Then*

$$ \left\{ {}_3\phi_2\left[\begin{matrix} q^{-n}, a, x \\ x^2,\ 0 \end{matrix} ; q, q\right]\right\}^2 = a^n\, {}_5\phi_4\left[\begin{matrix} q^{-n}, x^2q^n, a, x^2/a, x \\ x^2, -x, xq^{\frac{1}{2}}, -xq^{\frac{1}{2}} \end{matrix} ; q, q\right]. \tag{1.4} $$

In particular, the right-hand side of (1.4) is non-negative for real a, x, and q. Furthermore, if n is even, then by (1.3), the right-hand side of (1.4) may be written as

$$ a^n \left\{ {}_4\phi_3\left[\begin{matrix} q^{-\frac{n}{2}}, xq^{\frac{n}{2}}, a, x^2/a \\ xq^{\frac{1}{2}}, -xq^{\frac{1}{2}}, -x \end{matrix} ; q, q\right]\right\}^2. $$

Writing $n = 2m$ in (1.4) and taking the square root we obtain an identity between two polynomials in a of degree $2m$, where the sign is determined by comparing the coefficients of a^{2m}. We record the resulting formula as the second corollary.

Corollary 1.3. *Let m be a non-negative integer. Then*

$$ {}_3\phi_2\left[\begin{matrix} q^{-2m}, a, x \\ x^2,\ 0 \end{matrix}; q, q\right] = a^m\, {}_4\phi_3\left[\begin{matrix} q^{-m}, xq^m, a, x^2/a \\ xq^{\frac{1}{2}}, -xq^{\frac{1}{2}}, -x \end{matrix}; q, q\right]. $$

For some other q-Clausen type formulas, the reader is referred to Gasper and Rahman [5, Exercise 8.17] and Schlosser [13]. On the other hand, in their study of some q-supercongruences for certain truncated basic hypergeometric series related to [16, 17], Guo and Zeng [8] stumbled on the following q-Clausen-Orr type formula:

$$ \left(\sum_{k=s}^{n} \frac{(q^{-2n};q^2)_k (x;q)_k q^k}{(q;q)_{k-s}(q;q)_{k+s}}\right)\left(\sum_{k=s}^{n} \frac{(q^{-2n};q^2)_k (q/x;q)_k q^k}{(q;q)_{k-s}(q;q)_{k+s}}\right) $$
$$ = \frac{(-1)^n (q^2;q^2)_n^2 q^{-n^2}}{(q^2;q^2)_{n-s}(q^2;q^2)_{n+s}} \sum_{k=s}^{n} \frac{(-1)^k (q^2;q^2)_{n+k} (x;q)_k (q/x;q)_k q^{k^2-2nk}}{(q^2;q^2)_{n-k}(q;q)_{k-s}(q;q)_{k+s}(q;q)_{2k}}. \tag{1.5} $$

Noticing that $(c;q)_{2k} = (\sqrt{c};q)_k(-\sqrt{c};q)_k(\sqrt{cq};q)_k(-\sqrt{cq};q)_k$, we can rewrite (1.2) as

$$ \left(\sum_{k=0}^{n} \frac{(q^{-n};q)_k (a;q)_k (x;q)_k q^k}{(q;q)_k (c;q)_k}\right)\left(\sum_{k=0}^{n} \frac{(q^{-n};q)_k (a;q)_k (c/x;q)_k q^k}{(q;q)_k (c;q)_k}\right) $$
$$ = a^n \sum_{k=0}^{n} \frac{(q^{-n};q)_k (cq^n;q)_k (a;q)_k (c/a;q)_k (x;q)_k (c/x;q)_k q^k}{(q;q)_k (c;q)_k (c;q)_{2k}}. \tag{1.6} $$

Clearly (1.6) is an extension of (1.5). In the same vein we shall establish the following result.

Theorem 1.4. *Let n be a non-negative integer. Then*

$$ \left(\sum_{k=0}^{n} \frac{(q^{-n};q)_k (x;q^2)_k q^k}{(q;q)_k (c;q)_k}\right)\left(\sum_{k=0}^{n} \frac{(q^{-n};q)_k (x;q^2)_k c^k q^{nk-\binom{k}{2}}}{(q;q)_k (c;q)_k x^k}\right) $$
$$ = \sum_{k=0}^{n} \frac{(q^{-n};q)_k (cq^n;q)_k (x;q^2)_k (c^2/x;q^2)_k q^k}{(q;q)_k (c;q)_k (c;q)_{2k}}. \tag{1.7} $$

In this paper we shall consider two q-analogues of $D(m,n)$. We first recall some standard q-notation (see [5]). The *q-binomial coefficients* are given by

$$ \begin{bmatrix} n \\ k \end{bmatrix} := \begin{cases} \dfrac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} & \text{if } n \geqslant k \geqslant 0, \\ 0 & \text{otherwise.} \end{cases} $$

The following two natural q-analogues of Delannoy numbers were introduced in [4, p. 30] and [12]:

$$ D_q(m,n) := \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n+m-k \\ n \end{bmatrix}, \tag{1.8} $$

$$ D_q^*(m,n) := \sum_{k=0}^{n} q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n+m-k \\ n \end{bmatrix}. \tag{1.9} $$

Note that $D_q^*(m,n) := q^{mn} D_{q^{-1}}(m,n)$. We first show that both $D_q(m,n)$ and $D_q^*(m,n)$ have a q-analogue of the second expression in (1.1) and provide a q-analogue of [7, (3.1)], which was asked in [7, Problem 5.2].

Theorem 1.5. *Let m and n be non-negative integers. Then*

$$D_q(m,n) = \sum_{k=0}^{m} q^{(m-k)(n-k)} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} (-1;q)_k, \tag{1.10}$$

$$D_q^*(m,n) = \sum_{k=0}^{m} q^{(m-k)(n-k)} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} (-q;q)_k. \tag{1.11}$$

Moreover,

$$D_q(m,n) D_q^*(m,n) = \sum_{k=0}^{n} q^{(m-k)(n-k)} \begin{bmatrix} n+k \\ 2k \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} m+k \\ k \end{bmatrix} (-1;q)_k (-q;q)_k. \tag{1.12}$$

Applying the formula (1.12), we shall prove the following two results originally conjectured by the second author [7, Conjectures 5.3 and 5.4].

Theorem 1.6. *Let p be an odd prime and m a positive integer. Then*

$$\begin{aligned} &\sum_{k=0}^{p-1} \frac{1-q^{2k+1}}{1-q} D_q(m,k) D_{q^{-1}}(m,k) q^{-k} \\ &\quad \equiv \begin{cases} \dfrac{1-q^{-2m}}{1-q^2} q \pmod{[p]^2} & \text{if } m \equiv 0 \pmod{p}, \\ \dfrac{1-q^{2m+2}}{1-q^2} q \pmod{[p]^2} & \text{if } m \equiv -1 \pmod{p}, \\ 0 \pmod{[p]^2} & \text{otherwise}, \end{cases} \end{aligned} \tag{1.13}$$

where $[p] = 1 + q + \cdots + q^{p-1}$, and the congruences are understood in the polynomial ring $\mathbb{Z}[q]$.

Theorem 1.7. *Let m, n, and r be positive integers. Then all of*

$$\sum_{k=0}^{n-1} \frac{(1-q^m)(1-q^{m+1})(1-q^{2k+1})}{(1-q^2)(1-q^n)^2} D_q(m,k) D_{q^{-1}}(m,k) q^{-k}, \tag{1.14}$$

$$\sum_{k=0}^{n-1} \frac{1-q^{2k+1}}{1-q^n} D_q(m,k)^r D_{q^{-1}}(m,k)^r q^{-k}, \tag{1.15}$$

$$\sum_{k=0}^{n-1} (-1)^{n-k-1} \frac{1-q^{2k+1}}{1-q^n} D_q(m,k)^r D_{q^{-1}}(m,k)^r q^{\binom{k}{2}} \tag{1.16}$$

are Laurent polynomials in q with non-negative integer coefficients.

Note that Theorem 1.7 is a q-analogue of [7, Theorem 1.1] for the first three polynomials. The rest of the paper is organized as follows. We shall give three lemmas in Section 2 and prove Theorem 1.1 in Section 3. In Sections 4 and 5 we prove Theorems 1.4 and 1.5. In Sections 6 and 7, by using Theorem 1.5 we give proofs of Theorems 1.6 and 1.7 respectively.

2. Three lemmas

The following three lemmas are crucial ingredients of our proof of Theorem 1.1.

Lemma 2.1. *Let n and h be positive integers and let m be a non-negative integer with $h \leqslant n-m$. Then*

$$\sum_{j=0}^{m}\sum_{k=0}^{n}\frac{(q^{-n};q)_j(q^{-n};q)_k(x;q)_j(x;q)_k(q^{j-m-h+1};q)_{h-1}(q^{k-m-h+1};q)_{h-1}(1-q^{k-j})q^{2j+k}}{(q;q)_j(q;q)_k(c;q)_j(c;q)_k}$$

$$=\frac{(q;q)_n(q;q)_{h-1}(x;q)_{m+h}(c/x;q)_{n-h}x^{n-h}q^{\frac{m^2+3m}{2}-mn-mh-h^2+h}}{(-1)^{m-1}(q;q)_m(c;q)_m(c;q)_n(q;q)_{n-m-h}}. \tag{2.1}$$

Proof. Note that both sides of (2.1) are polynomials in x of degree $m+n$ with the same leading coefficient. Therefore, to prove (2.1), it suffices to prove that both sides have the same roots as polynomials in x. Denote the left-hand side of (2.1) by $L_{m,n}(x)$. We first assert that

$$L_{m,n}(x)=\sum_{j=0}^{m}\sum_{k=m+1}^{n}\frac{(q^{-n};q)_j(q^{-n};q)_k(x;q)_j(x;q)_k}{(q;q)_j(q;q)_k(c;q)_j(c;q)_k}$$
$$\times(q^{j-m-h+1};q)_{h-1}(q^{k-m-h+1};q)_{h-1}(1-q^{k-j})q^{2j+k}. \tag{2.2}$$

In fact, since $(1-q^{j-k})q^{2k+j}=-(1-q^{k-j})q^{k+2j}$, we have $\sum_{j=0}^{m}\sum_{k=0}^{m}=0$ for the summands in $L_{m,n}(x)$. We now consider the following two cases.

- For $x=q^{-r}$ with $0\leqslant r\leqslant m+h-1$, we have

$$L_{m,n}(q^{-r})=\sum_{j=0}^{m}\sum_{k=0}^{n}\frac{(q^{-n};q)_j(q^{-n};q)_k(q^{-r};q)_j(q^{-r};q)_k}{(q;q)_j(q;q)_k(c;q)_j(c;q)_k}$$
$$\times(q^{j-m-h+1};q)_{h-1}(q^{k-m-h+1};q)_{h-1}(1-q^{k-j})q^{2j+k}.$$

  If $r\leqslant m$, then $L_{m,n}(q^{-r})=0$ by the antisymmetry of j and k in $L_{m,n}(q^{-r})$. If $r\geqslant m+1$, then $h\geqslant r-m+1$, i.e., $r-m-h+1\leqslant 0$, and so $(q^{k-m-h+1};q)_{h-1}=0$ for $m+1\leqslant k\leqslant r$. Hence, by (2.2), we again get $L_{m,n}(q^{-r})=0$.

- For $x=cq^r$ with $0\leqslant r\leqslant n-h-1$, we shall prove that

$$\sum_{k=0}^{n}\frac{(q^{-n};q)_k(cq^r;q)_k(q^{k-m-h+1};q)_{h-1}(1-q^{k-j})q^{2j+k}}{(q;q)_k(c;q)_k}=0. \tag{2.3}$$

In fact, we can rewrite the left-hand side of (2.3) as

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{-nk+\binom{k}{2}} R_k, \tag{2.4}$$

where

$$R_k = \frac{(cq^r;q)_k (q^{k-m-h+1};q)_{h-1}(1-q^{k-j})q^{2j+k}}{(c;q)_k}.$$

Since

$$\frac{(cq^r;q)_k}{(c;q)_k} = \frac{(cq^k;q)_r}{(c;q)_r},$$

we see that R_k is a polynomial in q^k of degree $r+h-1+2 \leqslant n$ with no constant term. By the q-binomial theorem (see, for example, [2, Theorem 3.3])

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} x^k = (x;q)_n, \tag{2.5}$$

we have

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} q^{-ik} = 0 \quad \text{for } 0 \leqslant i \leqslant n-1.$$

It follows that the expression (2.4) is equal to 0. Namely, the identity (2.3) holds.

Hence, we see that all the $m+n$ roots of $L_{m,n}(x)$ are the same as those of the right-hand side of (2.1). □

Lemma 2.2. *Let n and h be positive integers and let m be a non-negative integer with $h \leqslant n-m$. Then*

$$\sum_{j=0}^{m}\sum_{k=m+h}^{n} \frac{(q^{-n};q)_j (q^{-n};q)_k (a;q)_j (a;q)_k (1-q^{k-j})q^{j+k+jh}}{(q;q)_j (q;q)_k (c;q)_j (c;q)_k} \begin{bmatrix} k-m-1 \\ h-1 \end{bmatrix} \begin{bmatrix} m+h-j-1 \\ h-1 \end{bmatrix}$$

$$= \frac{(q;q)_n (a;q)_{m+h} (c/a;q)_{n-h} a^{n-h} q^{\frac{m^2+m-h^2+h}{2}-mn}}{(-1)^{m-h}(q;q)_m (c;q)_m (c;q)_n (q;q)_{h-1}(q;q)_{n-m-h}}. \tag{2.6}$$

Proof. It is easy to see that $\begin{bmatrix} k-m-1 \\ h-1 \end{bmatrix} = 0$ for $m+1 \leqslant k < m+h$. Therefore, the left-hand side of (2.6) remains unchanged when we replace $\sum_{k=m+h}^{n}$ by $\sum_{k=m+1}^{n}$. Moreover,

$$\begin{bmatrix} k-m-1 \\ h-1 \end{bmatrix} \begin{bmatrix} m+h-j-1 \\ h-1 \end{bmatrix} = \frac{(q^{j-m-h+1};q)_{h-1}(q^{k-m-h+1};q)_{h-1} q^{(m-j)(h-1)+\binom{h}{2}}}{(-1)^{h-1}(q;q)_{h-1}^2}.$$

The proof then follows from (2.1) and (2.2) with $x=a$. □

The following result has been proved in [8, (3.5)].

Lemma 2.3 ([8]). *Let n be a positive integer. Then*

$$(x;q)_n+(a/x;q)_n=(x;q)_n(a/x;q)_n+(a;q)_n+\sum_{k=1}^{n-1}(x;q)_k(a/x;q)_kB_{n,k}(a), \tag{2.7}$$

where

$$B_{n,k}(a):=(1-q^n)\sum_{h=1}^{n-k}(-1)^h\begin{bmatrix}n-k-1\\ h-1\end{bmatrix}\begin{bmatrix}k+h-1\\ h-1\end{bmatrix}\frac{q^{\binom{h}{2}+kh}a^h}{1-q^h}.$$

3. Proof of Theorem 1.1

As

$$\left(\sum_{k=0}^{n}a_k\right)\left(\sum_{j=0}^{n}b_j\right)=\sum_{k=0}^{n}a_kb_k+\sum_{0\leqslant j<k\leqslant n}(a_kb_j+a_jb_k),$$

the left-hand side of (1.6) is equal to

$$\begin{aligned}&\sum_{k=0}^{n}\frac{(q^{-n};q)_k^2(a;q)_k^2q^{2k}}{(q;q)_k^2(c;q)_k^2}(x;q)_k(c/x;q)_k\\&\quad+\sum_{0\leqslant j<k\leqslant n}\frac{(q^{-n};q)_j(q^{-n};q)_k(a;q)_j(a;q)_kq^{j+k}\big((x;q)_j(c/x;q)_k+(x;q)_k(c/x;q)_j\big)}{(q;q)_j(q;q)_k(c;q)_j(c;q)_k}.\end{aligned} \tag{3.1}$$

For $0\leqslant j<k$, from (2.7) we deduce that

$$\begin{aligned}&(x;q)_j(c/x;q)_k+(x;q)_k(c/x;q)_j\\&\quad=(x;q)_j(c/x;q)_j\big((xq^j;q)_{k-j}+(cq^j/x;q)_{k-j}\big)\\&\quad=(x;q)_k(c/x;q)_k+(x;q)_j(c/x;q)_j(cq^{2j};q)_{k-j}+\sum_{i=1}^{k-j-1}(x;q)_{j+i}(c/x;q)_{j+i}B_{k-j,i}(cq^{2j})\\&\quad=(x;q)_k(c/x;q)_k+(x;q)_j(c/x;q)_j+\sum_{i=0}^{k-j-1}(x;q)_{j+i}(c/x;q)_{j+i}B_{k-j,i}(cq^{2j}),\end{aligned}$$

where we have used the q-binomial theorem (2.5) in the last step:

$$(cq^{2j};q)_{k-j}=1+\sum_{h=1}^{k-j}(-1)^h\begin{bmatrix}k-j\\ h\end{bmatrix}q^{\binom{h}{2}+2jh}c^h.$$

It follows that (3.1) can be written as $\sum_{m=0}^{n}\alpha_m(x;q)_m(c/x;q)_m$, where

$$\begin{aligned}\alpha_m=&\sum_{j=0}^{n}\frac{(q^{-n};q)_j(q^{-n};q)_m(a;q)_j(a;q)_mq^{j+m}}{(q;q)_j(q;q)_m(c;q)_j(c;q)_m}\\&+\sum_{j=0}^{m}\sum_{k=m+1}^{n}\frac{(q^{-n};q)_j(q^{-n};q)_k(a;q)_j(a;q)_kq^{j+k}}{(q;q)_j(q;q)_k(c;q)_j(c;q)_k}B_{k-j,m-j}(cq^{2j}).\end{aligned} \tag{3.2}$$

By the q-Chu-Vandermonde summation formula [5, Appendix (II.6)]:

$$ {}_2\phi_1\left[\begin{matrix} a, q^{-n} \\ c \end{matrix}; q, q\right] = \frac{(c/a;q)_n}{(c;q)_n} a^n, \tag{3.3} $$

we have

$$ \sum_{j=0}^{n} \frac{(q^{-n};q)_j (q^{-n};q)_m (a;q)_j (a;q)_m q^{j+m}}{(q;q)_j (q;q)_m (c;q)_j (c;q)_m} $$

$$ = (-1)^m \frac{(q;q)_n (a;q)_m (c/a;q)_n q^{\frac{m^2+m}{2}-mn}}{(q;q)_m (c;q)_m (c;q)_n (q;q)_{n-m}} a^n. \tag{3.4} $$

Substituting (3.4) and (2.6) into (3.2), we obtain

$$ \alpha_m = \frac{(-1)^m (q;q)_n q^{\frac{m^2+m}{2}-mn}}{(q;q)_m (c;q)_m (c;q)_n} \sum_{h=0}^{n-m} \frac{(a;q)_{m+h} (c/a;q)_{n-h} a^{n-h} q^{mh} c^h}{(q;q)_h (q;q)_{n-m-h}}. \tag{3.5} $$

The last sum can be summed again by the q-Chu-Vandermonde formula (3.3) and is equal to

$$ \frac{(a;q)_m (c/a;q)_m (cq^{2m};q)_{n-m} a^n}{(q;q)_{n-m}}. \tag{3.6} $$

It follows from (3.5) and (3.6) that α_m is just the coefficient of $(x;q)_m (c/x;q)_m$ on the right-hand side of (1.6).

Remark. Letting $c = q^{2s+1}$, $a = aq^s$, $x = xq^s$, and replacing n by $n-s$ in (1.6) ($0 \leqslant s \leqslant n$), we get the following result:

$$ \left(\sum_{k=s}^{n} \frac{(q^{-n};q)_k (a;q)_k (x;q)_k q^k}{(q;q)_{k-s} (q;q)_{k+s}}\right) \left(\sum_{k=s}^{n} \frac{(q^{-n};q)_k (a;q)_k (q/x;q)_k q^k}{(q;q)_{k-s} (q;q)_{k+s}}\right) $$

$$ = \frac{(q^{-n};q)_s (a;q)_s a^{n-s} q^{(n+1)s-s^2}}{(q^{n+1};q)_s (q/a;q)_s} \sum_{k=s}^{n} \frac{(q^{-n};q)_k (q^{n+1};q)_k (a;q)_k (q/a;q)_k (x;q)_k (q/x;q)_k q^k}{(q;q)_{k-s} (q;q)_{k+s} (q;q)_{2k}}. \tag{3.7} $$

It is clear that the $a = -q^{-n}$ case of (3.7) reduces to (1.5).

4. Proof of Theorem 1.4

We need a special case of Theorem 1.1. Letting $a = -x$ in (1.6), we are led to

$$ \left(\sum_{k=0}^{n} \frac{(q^{-n};q)_k (x^2;q^2)_k q^k}{(q;q)_k (c;q)_k}\right) \left(\sum_{k=0}^{n} \frac{(q^{-n};q)_k (-x;q)_k (c/x;q)_k q^k}{(q;q)_k (c;q)_k}\right) $$

$$ = (-x)^n \sum_{k=0}^{n} \frac{(q^{-n};q)_k (cq^n;q)_k (x^2;q^2)_k (c^2/x^2;q^2)_k q^k}{(q;q)_k (c;q)_k (c;q)_{2k}}. \tag{4.1} $$

We also need the following result.

Lemma 4.1. *Let n be a non-negative integer. Then*

$$\sum_{k=0}^{n}\frac{(q^{-n};q)_k(x;q)_k(y;q)_kq^k}{(q;q)_k(c;q)_k}=\sum_{k=0}^{n}(-1)^k\frac{(q^{-n};q)_k(x;q)_k(c/y;q)_kx^{n-k}y^kq^{nk-\binom{k}{2}}}{(q;q)_k(c;q)_k}. \tag{4.2}$$

Proof. This follows from combining Jackson's two transformations of terminating ${}_2\phi_1$ series [5, Appendix (III.7) and (III.8)]:

$${}_3\phi_2\left[\begin{matrix}q^{-n},b,bzq^{-n}/c\\ bq^{1-n}/c,0\end{matrix};q,q\right]=b^n{}_3\phi_1\left[\begin{matrix}q^{-n},b,q/z\\ bq^{1-n}/c\end{matrix};q,\frac{z}{c}\right]$$

with $b\to x,\ c\to xq^{1-n}/c$ and $z\to qy/c$. □

Replacing x and y by $-x$ and c/x respectively in (4.2), we are led to

$$\sum_{k=0}^{n}\frac{(q^{-n};q)_k(-x;q)_k(c/x;q)_kq^k}{(q;q)_k(c;q)_k}=(-x)^n\sum_{k=0}^{n}\frac{(q^{-n};q)_k(x^2;q^2)_kc^kq^{nk-\binom{k}{2}}}{(q;q)_k(c;q)_kx^{2k}}. \tag{4.3}$$

Combining (4.1) and (4.3) (also $x\to\sqrt{x}$), we obtain Theorem 1.4.

Remark. Letting $c=q^{2s+1}$, $x\to xq^{2s}$, and replacing n by $n-s$ $(0\leqslant s\leqslant n)$ in (1.7), we get the following identity

$$\begin{aligned}&\left(\sum_{k=s}^{n}\frac{(q^{-n};q)_k(x;q^2)_kq^k}{(q;q)_{k-s}(q;q)_{k+s}}\right)\left(\sum_{k=s}^{n}\frac{(q^{-n};q)_k(x;q^2)_kq^{(n+1)k-\binom{k}{2}}}{(q;q)_{k-s}(q;q)_{k+s}x^k}\right)\\&=\frac{(-1)^s(q;q)_n^2(x;q^2)_sq^s}{(q;q)_{n-s}(q;q)_{n+s}(q^2/x;q^2)_sx^s}\sum_{k=s}^{n}\frac{(q^{-n};q)_k(q^{n+1};q)_k(x;q^2)_k(q^2/x;q^2)_kq^k}{(q;q)_{k-s}(q;q)_{k+s}(q;q)_{2k}},\end{aligned}$$

which was originally conjectured in a preliminary version (arXiv:1408.0512v1) of [8].

5. Proof of Theorem 1.5

If we replace n by $n-i$ in the q-Chu-Vandermonde summation formula (3.3) with $a=q^{-m+i}$ and $c=q^{i+1}$ then

$$\sum_{k=i}^{m}q^{(m-k)(n-k)}\begin{bmatrix}m\\k\end{bmatrix}\begin{bmatrix}n-i\\k-i\end{bmatrix}=\begin{bmatrix}n+m-i\\n\end{bmatrix}.$$

Hence, by the q-binomial theorem (2.5) we have

$$\begin{aligned}\sum_{k=0}^{m}q^{(m-k)(n-k)}\begin{bmatrix}m\\k\end{bmatrix}\begin{bmatrix}n\\k\end{bmatrix}(x;q)_k&=\sum_{k=0}^{m}q^{(m-k)(n-k)}\begin{bmatrix}m\\k\end{bmatrix}\begin{bmatrix}n\\k\end{bmatrix}\sum_{i=0}^{k}(-1)^i\begin{bmatrix}k\\i\end{bmatrix}q^{\binom{i}{2}}x^i\\&=\sum_{i=0}^{m}(-1)^i\begin{bmatrix}n\\i\end{bmatrix}q^{\binom{i}{2}}x^i\sum_{k=i}^{m}q^{(m-k)(n-k)}\begin{bmatrix}m\\k\end{bmatrix}\begin{bmatrix}n-i\\k-i\end{bmatrix}\\&=\sum_{i=0}^{m}q^{\binom{i}{2}}\begin{bmatrix}n\\i\end{bmatrix}\begin{bmatrix}n+m-i\\n\end{bmatrix}(-x)^i.\end{aligned}$$

When $x=-1$ and $x=-q$, we obtain (1.10) and (1.11), respectively. Now, letting $c=q$ and $a=q^{-m}$ in (1.6), we get

$$\left(\sum_{k=0}^{m} q^{(m-k)(n-k)} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} (x;q)_k\right) \left(\sum_{k=0}^{m} q^{(m-k)(n-k)} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} (q/x;q)_k\right)$$
$$= \sum_{k=0}^{m} q^{(m-k)(n-k)} \begin{bmatrix} n+k \\ 2k \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} m+k \\ k \end{bmatrix} (x;q)_k (q/x;q)_k.$$

This entails the identity (1.12) by taking $x=-1$ or $-q$.

6. Proof of Theorem 1.6

The following identity can be easily proved by induction.

$$\sum_{k=j}^{n-1} (1-q^{2k+1}) \begin{bmatrix} k+j \\ 2j \end{bmatrix} q^{-(j+1)k} = \frac{(1-q^n)(1-q^{n-j})}{1-q^{j+1}} \begin{bmatrix} n+j \\ 2j \end{bmatrix} q^{-(j+1)(n-1)}. \tag{6.1}$$

By (1.12) and (6.1), the left-hand side of (1.13) is equal to

$$\sum_{k=0}^{p-1} \frac{1-q^{2k+1}}{1-q} \sum_{j=0}^{k} q^{j^2-mj-(j+1)k} \begin{bmatrix} k+j \\ 2j \end{bmatrix} \begin{bmatrix} m \\ j \end{bmatrix} \begin{bmatrix} m+j \\ j \end{bmatrix} (-1;q)_j (-q;q)_j$$
$$= \sum_{j=0}^{p-1} q^{j^2-mj-(j+1)(p-1)} \frac{(1-q^p)(1-q^{p-j})}{(1-q)(1-q^{j+1})} \begin{bmatrix} p+j \\ 2j \end{bmatrix} \begin{bmatrix} m \\ j \end{bmatrix} \begin{bmatrix} m+j \\ j \end{bmatrix} (-1;q)_j (-q;q)_j. \tag{6.2}$$

By [7, Theorem 2.1], we know that

$$\frac{(1-q^m)(1-q^{m+1})(1-q^{p-j})}{(1-q)(1-q^p)(1-q^{j+1})} \begin{bmatrix} p+j \\ 2j \end{bmatrix} \begin{bmatrix} m \\ j \end{bmatrix} \begin{bmatrix} m+j \\ j \end{bmatrix}$$
$$= \frac{(1-q^{p-j})(1-q^{j+1})}{(1-q)(1-q^p)} \begin{bmatrix} p+j \\ 2j \end{bmatrix} \begin{bmatrix} m+1 \\ j+1 \end{bmatrix} \begin{bmatrix} m+j \\ j+1 \end{bmatrix}$$

is a polynomial in q with non-negative integer coefficients. Since $[p]=(1-q^p)/(1-q)$ is an irreducible polynomial in q for any prime p and $\gcd(q^m-1,q^n-1)=(q^{\gcd(m,n)}-1)$, we conclude that

$$\frac{1-q^{p-j}}{1-q^{j+1}} \begin{bmatrix} p+j \\ 2j \end{bmatrix} \begin{bmatrix} m \\ j \end{bmatrix} \begin{bmatrix} m+j \\ j \end{bmatrix} \equiv 0 \pmod{[p]} \quad \text{if } m \not\equiv 0,-1 \pmod{p},$$

and so the right-hand side of (6.2) is congruent to 0 modulo $[p]^2$ in this case.

On the other hand, if $m \equiv 0,-1 \pmod{p}$, then

$$\frac{1-q^{p-j}}{1-q^{j+1}} \begin{bmatrix} m \\ j \end{bmatrix} \begin{bmatrix} m+j \\ j \end{bmatrix} \equiv 0 \pmod{[p]} \quad \text{if } j=0,1,\ldots,p-2.$$

Therefore, if $m \equiv 0 \pmod{p}$, then the right-hand side of (6.2) is congruent to

$$\begin{aligned} q\begin{bmatrix}2p-1\\2p-2\end{bmatrix}\begin{bmatrix}m\\p-1\end{bmatrix}\begin{bmatrix}m+p-1\\p-1\end{bmatrix}(-1;q)_{p-1}(-q;q)_{p-1} &\equiv q\frac{1-q^{2p-1}}{1-q}\cdot\frac{1-q^{m}}{1-q^{p-1}}(-q^{-1})\cdot\frac{2q}{1+q}\\ &\equiv -\frac{2q(1-q^m)}{1-q^2}\\ &\equiv \frac{1-q^{-2m}}{1-q^2}q \pmod{[p]^2}, \end{aligned}$$

where we have used the congruence $(-q;q)_{p-1} \equiv 1 \pmod{[p]}$ (see [6, (1.6)] or [11]); while if $m \equiv -1 \pmod{p}$, then the right-hand side of (6.2) is congruent to

$$\begin{aligned} \frac{1-q^{2p-1}}{1-q}\cdot\frac{1-q^{m+1}}{1-q^{p-1}}\cdot\frac{2q}{1+q} &\equiv \frac{2q(1-q^{m+1})}{1-q^2}\\ &\equiv \frac{1-q^{2m+2}}{1-q^2}q \pmod{[p]^2}. \end{aligned}$$

7. Proof of Theorem 1.7

Similarly to (6.2), the left-hand side of (1.14) is equal to

$$\sum_{j=0}^{n-1} q^{j^2-mj-(j+1)(n-1)}\frac{(1-q^m)(1-q^{m+1})(1-q^{n-j})}{(1-q^2)(1-q^n)(1-q^{j+1})}\begin{bmatrix}n+j\\2j\end{bmatrix}\begin{bmatrix}m\\j\end{bmatrix}\begin{bmatrix}m+j\\j\end{bmatrix}(-1;q)_j(-q;q)_j.$$

It is easy to see that

$$\begin{aligned} &\frac{(1-q^m)(1-q^{m+1})(1-q^{n-j})}{(1-q^2)(1-q^n)(1-q^{j+1})}\begin{bmatrix}n+j\\2j\end{bmatrix}\begin{bmatrix}m\\j\end{bmatrix}\begin{bmatrix}m+j\\j\end{bmatrix}(-q;q)_j\\ &=\begin{cases}\dfrac{(1-q^m)(1-q^{m+1})}{(1-q^2)(1-q)} & \text{if } j=0,\\ \dfrac{(1-q^{n-j})(1-q^{j+1})}{(1-q)(1-q^n)}\begin{bmatrix}n+j\\2j\end{bmatrix}\begin{bmatrix}m+1\\j+1\end{bmatrix}\begin{bmatrix}m+j\\j+1\end{bmatrix}(-q^2;q)_{j-1} & \text{if } j\geqslant 1\end{cases} \end{aligned}$$

is a polynomial in q with non-negative integer coefficients by [7, Theorem 2.1]. We conclude that (1.14) is the desired Laurent polynomial in q.

Let

$$S_n(x_0,\dots,x_n;q)=\sum_{k=0}^{n}\begin{bmatrix}n+k\\2k\end{bmatrix}\begin{bmatrix}2k\\k\end{bmatrix}q^{-nk}x_k.$$

To prove that (1.15) and (1.16) also have the same properties, we first establish the following result.

Lemma 7.1. *Let n and r be positive integers. Then both*

$$\sum_{k=0}^{n-1}\frac{1-q^{2k+1}}{1-q^n}S_k(x_0,\dots,x_k)^r q^{-k} \text{ and } \sum_{k=0}^{n-1}(-1)^{n-k-1}\frac{1-q^{2k+1}}{1-q^n}S_k(x_0,\dots,x_k)^r q^{\binom{k}{2}}$$

are polynomials in $x_0,\dots,x_{n-1},q$ and q^{-1} with non-negative integer coefficients.

Proof. Recall the identity

$$\begin{bmatrix}k+i\\2i\end{bmatrix}\begin{bmatrix}2i\\i\end{bmatrix}\begin{bmatrix}k+j\\2j\end{bmatrix}\begin{bmatrix}2j\\j\end{bmatrix}=\sum_{s=i}^{i+j}\begin{bmatrix}i+j\\i\end{bmatrix}\begin{bmatrix}j\\s-i\end{bmatrix}\begin{bmatrix}s\\j\end{bmatrix}\begin{bmatrix}k+s\\2s\end{bmatrix}\begin{bmatrix}2s\\s\end{bmatrix}q^{(i+j-s)(k-s)}.$$

which can be proved using the q-Pfaff-Saalschütz identity (see [14, Lemma 2.1]). It follows that

$$\begin{aligned}S_k(x_0,\dots,x_k)^r &= \sum_{0\leqslant i_1,\dots,i_r\leqslant k}\prod_{j=1}^{r}\begin{bmatrix}k+i_j\\2i_j\end{bmatrix}\begin{bmatrix}2i_j\\i_j\end{bmatrix}q^{-ki_j}x_{i_j}\\ &= \sum_{0\leqslant i_1,\dots,i_r\leqslant k}x_{i_1}\cdots x_{i_r}\sum_{s=i_1}^{i_1+\cdots+i_r}P(i_1,\dots,i_r,s)\begin{bmatrix}k+s\\2s\end{bmatrix}\begin{bmatrix}2s\\s\end{bmatrix}q^{-ks},\end{aligned}\tag{7.1}$$

where $P(i_1,\dots,i_r,s)$ is a Laurent polynomial in q independent of k with non-negative integer coefficients. Therefore, by (6.1), we see that

$$\begin{aligned}&\sum_{k=0}^{n-1}\frac{1-q^{2k+1}}{1-q^n}S_k(x_0,\dots,x_k)^r q^{-k}\\ &= \sum_{0\leqslant i_1,\dots,i_r\leqslant n-1}x_{i_1}\cdots x_{i_r}\sum_{s=i_1}^{i_1+\cdots+i_r}P(i_1,\dots,i_r,s)\frac{1-q^{n-s}}{1-q^{s+1}}\begin{bmatrix}n+s\\2s\end{bmatrix}\begin{bmatrix}2s\\s\end{bmatrix}q^{-(s+1)(n-1)}\end{aligned}$$

is a polynomial in $x_0,\dots,x_{n-1},q$ and q^{-1} with non-negative integer coefficients since

$$\frac{1-q^{n-s}}{1-q^{s+1}}\begin{bmatrix}n+s\\2s\end{bmatrix}\begin{bmatrix}2s\\s\end{bmatrix}=\begin{bmatrix}n+s\\s\end{bmatrix}\begin{bmatrix}n\\s+1\end{bmatrix}.$$

Similarly, since

$$\sum_{k=s}^{n-1}(-1)^{n-k-1}\frac{1-q^{2k+1}}{1-q^n}\begin{bmatrix}k+s\\2s\end{bmatrix}\begin{bmatrix}2s\\s\end{bmatrix}q^{\binom{k}{2}-sk}=\begin{bmatrix}n-1\\s\end{bmatrix}\begin{bmatrix}n+s\\s\end{bmatrix}q^{\binom{n}{2}-sn},$$

we deduce from (7.1) that

$$\begin{aligned}&\sum_{k=0}^{n-1}(-1)^{n-k-1}\frac{1-q^{2k+1}}{1-q^n}S_k(x_0,\dots,x_k)^r q^{\binom{k}{2}}\\ &= \sum_{0\leqslant i_1,\dots,i_r\leqslant n-1}x_{i_1}\cdots x_{i_r}\sum_{s=i_1}^{i_1+\cdots+i_r}P(i_1,\dots,i_r,s)\begin{bmatrix}n-1\\s\end{bmatrix}\begin{bmatrix}n+s\\s\end{bmatrix}q^{\binom{n}{2}-sn}\end{aligned}$$

is a polynomial in $x_0, \ldots, x_{n-1}, q$ and q^{-1} with non-negative integer coefficients. □

For $k = 0, \ldots, n-1$, let

$$x_k = \begin{bmatrix} m+k \\ 2k \end{bmatrix} (-1;q)_k (-q;q)_k q^{k^2-mk}.$$

Then the identity (1.12) may be rewritten as

$$D_q(m,n) D_{q^{-1}}(m,n) = \sum_{k=0}^{n} \begin{bmatrix} n+k \\ 2k \end{bmatrix} \begin{bmatrix} 2k \\ k \end{bmatrix} q^{-nk} x_k.$$

It is clear that $x_0, \ldots, x_{n-1}$ are Laurent polynomials in q with non-negative integer coefficients. By Lemma 7.1, so are the expressions (1.15) and (1.16).

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