q-Analogues of Dwork-type supercongruences

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Abstract. In 1997, Van Hamme conjectured 13 Ramanujan-type supercongruences. All of the 13 supercongruences have been confirmed by using a wide range of methods. In 2015, Swisher conjectured Dwork-type supercongruences related to the first 12 supercongruences of Van Hamme. Here we prove that the (C.3) and (J.3) supercongruences of Swisher are true modulo p^{3r} (the original modulus is p^{4r}) by establishing q-analogues of them. Our proof will use the "creative microscoping" method, recently introduced by the author in collaboration with Zudilin. We also raise conjectures on q-analogues of an equivalent form of the (M.2) supercongruence of Van Hamme, partially answering a question at the end of [Adv. Math. 346 (2019), 329–358].

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1. Introduction

In 1914, Ramanujan [25] mysteriously stated 17 hypergeometric series representations of $1/\pi$, including

$$\sum_{k=0}^{\infty}(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 4^k}=\frac{4}{\pi},$$

where $(a)_n=a(a+1)\cdots(a+n-1)$ denotes the Pochhammer symbol. In 1997, Van Hamme [29] listed 13 interesting p-adic analogues of Ramanujan's and Ramanujan-type formulas, such as

$$\sum_{k=0}^{(p-1)/2}(4k+1)\frac{(\frac{1}{2})_k^4}{k!^4}\equiv p \pmod{p^3}, \tag{1.1}$$

$$\sum_{k=0}^{(p-1)/2}(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 4^k}\equiv (-1)^{(p-1)/2}p \pmod{p^4}, \tag{1.2}$$

where $p>3$ is a prime. The supercongruence (1.1) was proved by Van Hamme [29] himself and Long [19] proved that it also holds modulo p^4 by using hypergeometric series identities and evaluations. The supercongruence (1.2) was also confirmed by Long [19]. It was not until 2016 that the last supercongruence of Van Hamme was proved by Osburn and Zudilin [23]. For some background on Ramanujan-type supercongruences, we refer the reader to Zudilin's paper [32].

In 2015, Swisher [27] proved and reproved several supercongruences of Van Hamme by utilizing Long's method. Furthermore, she proposed some conjectures on supercongruences that generalize Van Hamme's supercongruences (A.2)–(L.2). In particular, Swisher's conjectural (C.3) and (J.3) supercongruences, which are generalizations of (1.1) and (1.2) respectively, can be stated as follows: for any prime $p>3$,

$$\sum_{k=0}^{(p^r-1)/2}(4k+1)\frac{(\frac{1}{2})_k^4}{k!^4}\equiv p\sum_{k=0}^{(p^{r-1}-1)/2}(4k+1)\frac{(\frac{1}{2})_k^4}{k!^4}\pmod{p^{4r}},\tag{1.3}$$

$$\sum_{k=0}^{(p^r-1)/2}(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 4^k}\equiv (-1)^{(p-1)/2}p\sum_{k=0}^{(p^{r-1}-1)/2}(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 4^k}\pmod{p^{4r}}.\tag{1.4}$$

In recent years, q-analogues of congruences and supercongruences have been investigated by many authors (see, for example, [3–15, 17, 22, 26, 28, 30, 33]). In particular, using the q-WZ method [31] the author and Wang [13] gave a q-analogue of (1.1): for odd n,

$$\sum_{k=0}^{(n-1)/2}[4k+1]\frac{(q;q^2)_k^4}{(q^2;q^2)_k^4}\equiv q^{(1-n)/2}[n]+\frac{(n^2-1)(1-q)^2}{24}q^{(1-n)/2}[n]^3\pmod{[n]\Phi_n(q)^3}.\tag{1.5}$$

Moreover, the author [5] conjectured that, for odd n,

$$\sum_{k=0}^{(n-1)/2}q^{k^2}[6k+1]\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^4;q^4)_k^3}$$
$$\equiv(-q)^{(1-n)/2}[n]+\frac{(n^2-1)(1-q)^2}{24}(-q)^{(1-n)/2}[n]^3\pmod{[n]\Phi_n(q)^3}.\tag{1.6}$$

Here and in what follows we adopt the standard q-notation: $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ is the *q-shifted factorial*; $[n]=[n]_q=1+q+\cdots+q^{n-1}$ is the *q-integer*; and $\Phi_n(q)$ stands for the n-th *cyclotomic polynomial* in q:

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(n,k)=1}}(q-\zeta^k),$$

where ζ is an n-th primitive root of unity. The author [5] himself proved that the q-supercongruence (1.6) is true modulo $[n]\Phi_n(q)$. Shortly afterwards, the author and Zudilin [14] proved that (1.6) holds modulo $[n]\Phi_n(q)^2$ by a newly introduced method of creative microscoping.

It is worth mentioning that (1.3) and (1.4) have the following companions: for any prime $p>3$,

$$\sum_{k=0}^{p^r-1}(4k+1)\frac{(\frac{1}{2})_k^4}{k!^4} \equiv p\sum_{k=0}^{p^{r-1}-1}(4k+1)\frac{(\frac{1}{2})_k^4}{k!^4} \pmod{p^{4r}}, \tag{1.7}$$

$$\sum_{k=0}^{p^r-1}(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 4^k} \equiv (-1)^{(p-1)/2}p\sum_{k=0}^{p^{r-1}-1}(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 4^k} \pmod{p^{4r}}, \tag{1.8}$$

which were conjectured in [13] and [5], respectively.

Given a prime p, a power series $f(z)=\sum_{k=0}^{\infty}A_k z^k$ is said to satisfy the Dwork congruence [1, 21] if

$$\frac{f_{r+1}(z)}{f_r(z^p)} \equiv \frac{f_r(z)}{f_{r-1}(z^p)} \pmod{p^r\mathbb{Z}_p[[z]]} \quad \text{for } r=1,2,\ldots, \tag{1.9}$$

where

$$f_r(z)=\sum_{k=0}^{p^r-1}A_k z^k$$

denotes the truncation of $f(z)$. Moreover, if the modulus in (1.9) can be replaced by $p^s\mathbb{Z}_p[[z]]$ for $s=s_r>r$, then we say that it satisfies a Dwork supercongruence. Formally, we need the condition $f_1(z^p)=\sum_{k=0}^{p-1}A_k z^{pk}\not\equiv 0 \pmod{p\mathbb{Z}_p[[z]]}$ satisfied to make sense of (1.9). However this may be relaxed to $f_1(z^p)\not\equiv 0 \pmod{p^m\mathbb{Z}_p[[z]]}$ if the congruences (1.9) hold modulo $p^{mr}\mathbb{Z}_p[[z]]$ for some $m>1$. This allows one to view Swisher's conjectures from [27] as particular instances of Dwork-type supercongruences.

The aim of this paper is to establish (partial) q-analogues of (1.3), (1.4), (1.7) and (1.8). We divide them into two theorems. It is reasonable to call the q-congruences in the theorems *q-analogues of Dwork-type supercongruences*.

Theorem 1.1. *Let $n>1$ be an odd integer and let $r\geqslant 1$. Then, modulo $[n^r]\prod_{j=1}^{r}\Phi_{n^j}(q)^2$,*

$$\sum_{k=0}^{(n^r-1)/2}[4k+1]\frac{(q;q^2)_k^4}{(q^2;q^2)_k^4} \equiv q^{(1-n)/2}[n]\sum_{k=0}^{(n^{r-1}-1)/2}[4k+1]_{q^n}\frac{(q^n;q^{2n})_k^4}{(q^{2n};q^{2n})_k^4}, \tag{1.10}$$

$$\sum_{k=0}^{n^r-1}[4k+1]\frac{(q;q^2)_k^4}{(q^2;q^2)_k^4} \equiv q^{(1-n)/2}[n]\sum_{k=0}^{n^{r-1}-1}[4k+1]_{q^n}\frac{(q^n;q^{2n})_k^4}{(q^{2n};q^{2n})_k^4}. \tag{1.11}$$

It is easy to see that, when $n=p$ is a prime and $q\to 1$, the q-supercongruences (1.10) and (1.11) reduce to (1.3) and (1.7) modulo p^{3r}.

Theorem 1.2. *Let $n>1$ be an odd integer and let $r\geqslant 1$. Then, modulo $[n^r]\prod_{j=1}^r \Phi_{n^j}(q)^2$,*

$$\sum_{k=0}^{(n^r-1)/2} q^{k^2}[6k+1]\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^4;q^4)_k^3} \equiv (-q)^{(1-n)/2}[n]\sum_{k=0}^{(n^{r-1}-1)/2} q^{nk^2}[6k+1]_{q^n}\frac{(q^n;q^{2n})_k^2(q^{2n};q^{4n})_k}{(q^{4n};q^{4n})_k^3}, \tag{1.12}$$

$$\sum_{k=0}^{n^r-1} q^{k^2}[6k+1]\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^4;q^4)_k^3} \equiv (-q)^{(1-n)/2}[n]\sum_{k=0}^{n^{r-1}-1} q^{nk^2}[6k+1]_{q^n}\frac{(q^n;q^{2n})_k^2(q^{2n};q^{4n})_k}{(q^{4n};q^{4n})_k^3}. \tag{1.13}$$

Similarly, when $n=p$ is a prime and $q\to 1$, the q-supercongruences (1.12) and (1.13) reduce to (1.4) and (1.8) modulo p^{3r}.

We prove Theorems 1.1 and 1.2 in Sections 2 and 3, respectively. We shall accomplish this by using the creative microscoping method from [14]. More precisely, we shall give parametric generalizations of Theorems 1.1 and 1.2. In Section 4, we propose several conjectures on q-analogues of an equivalent form of the (M.2) supercongruence of Van Hamme [29], thus answering in part a suspicion of the author and Zudilin in [14].

2. Proof of Theorem 1.1

We first give the following result, which follows from the $c=1$ case of [14, Theorem 4.2].

Lemma 2.1. *Let n be a positive odd integer. Then*

$$\sum_{k=0}^{(n-1)/2}[4k+1]\frac{(aq;q^2)_k(q/a;q^2)_k(q;q^2)_k^2}{(aq^2;q^2)_k(q^2/a;q^2)_k(q^2;q^2)_k^2} \equiv 0 \pmod{[n]}.$$

We also need the following lemma, which is a also special case of [14, Theorem 4.2]. For the reader's convenience, we give a two-line proof here.

Lemma 2.2. *Let n be a positive odd integer. Then*

$$\sum_{k=0}^{(n-1)/2}[4k+1]\frac{(q^{1-n};q^2)_k(q^{1+n};q^2)_k(q;q^2)_k^2}{(q^{2-n};q^2)_k(q^{2+n};q^2)_k(q^2;q^2)_k^2} = q^{(1-n)/2}[n].$$

Proof. Recall that Jackson's ${}_6\phi_5$ summation formula can be written as

$$\sum_{k=0}^{N}\frac{(1-aq^{2k})(a;q)_k(b;q)_k(c;q)_k(q^{-N};q)_k}{(1-a)(q;q)_k(aq/b;q)_k(aq/c;q)_k(aq^{N+1};q)_k}\left(\frac{aq^{N+1}}{bc}\right)^k = \frac{(aq;q)_N(aq/bc;q)_N}{(aq/b;q)_N(aq/c;q)_N}$$

(see [2, Appendix (II.21)]). Performing the substitutions $q\mapsto q^2$, $a=b=q$, $c=q^{1+n}$ and $N=(n-1)/2$ in the above formula, we get the desired identity. $\square$

Like many theorems in [14], Theorem 1.1 has a parametric generalization.

Theorem 2.3. *Let $n>1$ be an odd integer and let $r\geqslant 1$. Then, modulo*

$$[n^r]\prod_{j=0}^{(n^{r-1}-1)/d}(1-aq^{(2j+1)n})(a-q^{(2j+1)n}),$$

we have

$$\begin{aligned}\sum_{k=0}^{(n^r-1)/d}&[4k+1]\frac{(aq;q^2)_k(q/a;q^2)_k(q;q^2)_k^2}{(aq^2;q^2)_k(q^2/a;q^2)_k(q^2;q^2)_k^2}\\ &\equiv q^{(1-n)/2}[n]\sum_{k=0}^{(n^{r-1}-1)/d}[4k+1]_{q^n}\frac{(aq^n;q^{2n})_k(q^n/a;q^{2n})_k(q^n;q^{2n})_k^2}{(aq^{2n};q^{2n})_k(q^{2n}/a;q^{2n})_k(q^{2n};q^{2n})_k^2},\end{aligned}\tag{2.1}$$

where $d=1,2$.

Proof. By Lemma 2.1 with $n\mapsto n^r$, we see that the left-hand side of (2.1) is congruent to 0 modulo $[n^r]$. On the other hand, letting $r\mapsto r-1$ and $q\mapsto q^n$ in Lemma 2.1, we conclude that the summation on the right-hand side of (2.1) is congruent to 0 modulo $[n^{r-1}]_{q^n}$. Furthermore, it is easy to see that, for odd n, the q-integer $[n]$ is relatively prime to $1+q^k$ for any positive integer k. Hence $[n]$ is also relatively prime to the denominators of the sum on the right-hand side of (2.1) because of the relation

$$\frac{(q^n;q^{2n})_k}{(q^{2n};q^{2n})_k}={2k\brack k}_{q^n}\frac{1}{(-q^n;q^n)_k^2},$$

where ${2k\brack k}_{q^n}=(q^n;q^n)_{2k}/(q^n;q^n)_k^2$ is the central q-binomial coefficient. This means that the right-hand side of (2.1) is congruent to 0 modulo $[n][n^{r-1}]_{q^n}=[n^r]$. Namely, the q-congruence (2.1) is true modulo $[n^r]$.

To prove that it is also true modulo

$$\prod_{j=0}^{(n^{r-1}-1)/d}(1-aq^{(2j+1)n})(a-q^{(2j+1)n}),\tag{2.2}$$

it suffices to prove that both sides of (2.1) are equal when $a=q^{-(2j+1)n}$ or $a=q^{(2j+1)n}$ for all $0\leqslant j\leqslant (n^{r-1}-1)/d$, i.e.,

$$\begin{aligned}\sum_{k=0}^{(n^r-1)/d}&[4k+1]\frac{(q^{1-(2j+1)n};q^2)_k(q^{1+(2j+1)n};q^2)_k(q;q^2)_k^2}{(q^{2-(2j+1)n};q^2)_k(q^{2+(2j+1)n};q^2)_k(q^2;q^2)_k^2}\\ &=q^{(1-n)/2}[n]\sum_{k=0}^{(n^{r-1}-1)/d}[4k+1]_{q^n}\frac{(q^{-2jn};q^{2n})_k(q^{(2j+2)n};q^{2n})_k(q^n;q^{2n})_k^2}{(q^{(1-2j)n};q^{2n})_k(q^{(2j+3)n};q^{2n})_k(q^{2n};q^{2n})_k^2}.\end{aligned}\tag{2.3}$$

It is clear that $(n^r-1)/d \geqslant ((2j+1)n-1)/2$ for $0 \leqslant j \leqslant (n^{r-1}-1)/d$, and $(q^{1-(2j+1)n}; q^2)_k = 0$ for $k > ((2j+1)n-1)/2$. By Lemma 2.2, the left-hand side of (2.3) is equal to $q^{(1-(2j+1)n)/2}[(2j+1)n]$. Similarly, the right-hand side of (2.3) is equal to

$$q^{(1-n)/2}[n] \cdot q^{-jn}[2j+1]_{q^n} = q^{(1-(2j+1)n)/2}[(2j+1)n].$$

This proves (2.3). Namely, the q-congruence (2.1) holds modulo (2.2). Since $\prod_{j=1}^r \Phi_{n^j}(q)$ and (2.2) are relatively prime polynomials, we complete the proof of (2.1). $\square$

Proof of Theorem 1.1. It is easy to see that the limit of (2.2) as $a \to 1$ has the factor

$$\begin{cases} \prod_{j=1}^r \Phi_{n^j}(q)^{2n^{r-j}}, & \text{if } d=1, \\ \prod_{j=1}^r \Phi_{n^j}(q)^{n^{r-j}+1}, & \text{if } d=2. \end{cases}$$

On the other hand, the denominator of the right-hand side of (2.1) divides that of the left-hand side of (2.1). The factor of the latter related to a is $(aq^2;q^2)_{(n^r-1)/2}(q^2/a;q^2)_{(n^r-1)/2}$, the limit of which as $a \to 1$ only has the following factor

$$\begin{cases} \prod_{j=1}^r \Phi_{n^j}(q)^{2n^{r-j}-2}, & \text{if } d=1, \\ \prod_{j=1}^r \Phi_{n^j}(q)^{n^{r-j}-1}, & \text{if } d=2. \end{cases}$$

related to $\Phi_n(q), \Phi_{n^2}(q), \dots, \Phi_{n^r}(q)$. Thus, letting $a \to 1$ in (2.1), we see that (1.10) is true modulo $\prod_{j=1}^r \Phi_{n^j}(q)^3$, one product $\prod_{j=1}^r \Phi_{n^j}(q)$ of which comes from $[n^r]$.

Finally, by (1.5) and [14, Theorem 4.2], we see that

$$\sum_{k=0}^{(n-1)/2} [4k+1]\frac{(q;q^2)_k^4}{(q^2;q^2)_k^4} \equiv \sum_{k=0}^{n-1}[4k+1]\frac{(q;q^2)_k^4}{(q^2;q^2)_k^4} \equiv 0 \pmod{[n]}.$$

Replacing n by n^r in the above congruences, we see that the left-hand sides of (1.10) and (1.11) are congruent to 0 modulo $[n^r]$, while letting $q \mapsto q^n$ and $n \mapsto n^{r-1}$ in the above congruences, we conclude that the right-hand sides of them are congruent to 0 modulo $[n][n^{r-1}]_{q^n} = [n^r]$. It follows that (1.10) and (1.11) hold modulo $[n^r]$. The proof of the theorem then follows from the fact that the least common multiple of $\prod_{j=1}^r \Phi_{n^j}(q)^3$ and $[n^r]$ is just $[n^r]\prod_{j=1}^r \Phi_{n^j}(q)^2$. $\square$

3. Proof of Theorem 1.2

Similarly as before, we need the following lemma, which is a special case of [14, Theorem 4.3] and can be deduced from [24, eq. (4.6)].

Lemma 3.1. *Let n be a positive odd integer. Then*

$$\sum_{k=0}^{(n-1)/2} q^{k^2}[6k+1]\frac{(aq;q^2)_k(q/a;q^2)_k(q^2;q^4)_k}{(aq^4;q^4)_k(q^4/a;q^4)_k(q^4;q^4)_k} \equiv 0 \pmod{[n]}, \tag{3.1}$$

$$\sum_{k=0}^{(n-1)/2} q^{k^2}[6k+1]\frac{(q^{1-n};q^2)_k(q^{1+n};q^2)_k(q^2;q^4)_k}{(q^{4-n};q^4)_k(q^{4+n};q^4)_k(q^4;q^4)_k} = (-q)^{(1-n)/2}[n]. \tag{3.2}$$

We also need to establish the following parametric generalization of Theorem 1.2.

Theorem 3.2. *Let $n>1$ be an odd integer and let $r \geqslant 1$. Then, modulo*

$$[n^r]\prod_{j=0}^{(n^{r-1}-1)/d}(1-aq^{(2j+1)n})(a-q^{(2j+1)n}),$$

we have

$$\sum_{k=0}^{(n^r-1)/d} q^{k^2}[6k+1]\frac{(aq;q^2)_k(q/a;q^2)_k(q^2;q^4)_k}{(aq^4;q^4)_k(q^4/a;q^4)_k(q^4;q^4)_k}$$

$$\equiv (-q)^{(1-n)/2}[n]\sum_{k=0}^{(n^{r-1}-1)/d} q^{k^2}[6k+1]_{q^2}\frac{(aq^n;q^{2n})_k(q^n/a;q^{2n})_k(q^{2n};q^{4n})_k}{(aq^{4n};q^{4n})_k(q^{4n}/a;q^{4n})_k(q^{4n};q^{4n})_k}, \tag{3.3}$$

where $d=1,2$.

Sketch of proof. By (3.1), we see that both sides of (3.3) are congruent to 0 modulo $[n^r]=[n][n^{r-1}]_{q^n}$. Thus, the congruence (3.3) holds modulo $[n^r]$. To prove that (3.3) also holds modulo (2.2), it suffices to show that both sides of (3.3) are identical when $a=q^{-(2j+1)n}$ or $a=q^{(2j+1)n}$ for all $0 \leqslant j \leqslant (n^{r-1}-1)/d$. We may accomplish this by applying (3.1) on both sides of (3.3) with $a=q^{-(2j+1)n}$. □

Proof of Theorem 1.2. The proof is similar to that of Theorem 1.1. Letting $a\to 1$ in (3.3), we conclude that (1.12) is true modulo $\prod_{j=1}^{r}\Phi_{n^j}(q)^3$. On the other hand, by [5, Theorem 1.3] or [14, Theorem 4.3], we have

$$\sum_{k=0}^{(n-1)/2} q^{k^2}[6k+1]\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^4;q^4)_k^3} \equiv \sum_{k=0}^{n-1} q^{k^2}[6k+1]\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^4;q^4)_k^3} \equiv 0 \pmod{[n]}.$$

Applying the above congruences, we immediately conclude that both sides of (1.12) and (1.13) are all congruent to 0 modulo $[n^r]$. This completes the proof. □

4. Concluding remarks

We have seen that supercongruences can be proved by establishing suitable q-analogues of them. As mentioned in [14], the creative microscoping method used to prove q-supercongruences cannot be transformed into a method used to prove usual supercongruences. Although we have only proved (1.3), (1.4), (1.7) and (1.8) modulo p^{3r} instead of expected p^{4r}, no proofs were known before for such congruences even modulo p^{2r}.

Note that a complete q-analogue of Swisher [27, (I.3)] has already been given by the author [6]; it has a somewhat different flavour. It is also possible to give q-analogues or partial q-analogues of some other supercongruences conjectured by Swisher in [27]. See the recent joint work with Zudilin [16].

In this context we also need to highlight that certain supercongruences are related to the coefficients of modular forms. One famous example is the supercongruence (see Van Hamme [29, (M.2)] and Kilbourn [18])

$$\sum_{k=0}^{p-1}\frac{(\frac{1}{2})_k^4}{k!^4}\equiv\sum_{k=0}^{(p-1)/2}\frac{(\frac{1}{2})_k^4}{k!^4}\equiv\gamma_p \pmod{p^3}$$

valid for an odd prime p, where γ_p stands for the p-th coefficient in the q-expansion

$$q(q^2;q^2)_\infty^4(q^4;q^4)_\infty^4=\sum_{n=1}^{\infty}\gamma_n q^n \quad \text{(of a modular form)}.$$

Although Swisher [27] did not give a generalization of the (M.2) supercongruence, Long et al. [20, Section 2.1] showed that the proof of the (M.2) supercongruence is equivalent to verifying that

$$\sum_{k=0}^{p^{r+1}-1}\frac{(\frac{1}{2})_k^4}{k!^4}\equiv\left(\sum_{k=0}^{p-1}\frac{(\frac{1}{2})_k^4}{k!^4}\right)\left(\sum_{k=0}^{p^r-1}\frac{(\frac{1}{2})_k^4}{k!^4}\right) \pmod{p^3} \tag{4.1}$$

holds for $r=1$ and 2 (hence for all $r\geqslant 1$). Recently, the author and Zudilin [14] suspected that a q-analogue of the supercongruences (4.1) should exist. Here we formulate such a q-analogue.

Conjecture 4.1. *Let $n>1$ be an odd integer and let $r\geqslant 1$. Then, modulo $\Phi_n(q)^3$,*

$$\sum_{k=0}^{n^{r+1}-1}\frac{(q;q^2)_k^4}{(q^2;q^2)_k^4}q^{2k}\equiv\left(\sum_{k=0}^{n-1}\frac{(q;q^2)_k^4}{(q^2;q^2)_k^4}q^{2k}\right)\left(\sum_{k=0}^{n^r-1}\frac{(q^{n^2};q^{2n^2})_k^4}{(q^{2n^2};q^{2n^2})_k^4}q^{2n^2k}\right). \tag{4.2}$$

By the Lucas theorem, one sees that $(\frac{1}{2})_k/k!=\binom{2k}{k}/4^k\equiv 0 \pmod{p}$ for k in the range $(p^s+1)/2\leqslant k\leqslant p^s-1$ where $s=1,2,\ldots$. Thus, the supercongruence (4.1) has the following equivalent form:

$$\sum_{k=0}^{(p^{r+1}-1)/2}\frac{(\frac{1}{2})_k^4}{k!^4}\equiv\left(\sum_{k=0}^{(p-1)/2}\frac{(\frac{1}{2})_k^4}{k!^4}\right)\left(\sum_{k=0}^{(p^r-1)/2}\frac{(\frac{1}{2})_k^4}{k!^4}\right) \pmod{p^3}. \tag{4.3}$$

Likewise, we have the following natural q-analogue of (4.3).

Conjecture 4.2. *Let $n>1$ be an odd integer and let $r\geqslant 1$. Then, modulo $\Phi_n(q)^3$,*

$$\sum_{k=0}^{(n^{r+1}-1)/2}\frac{(q;q^2)_k^4}{(q^2;q^2)_k^4}q^{2k}\equiv\left(\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k^4}{(q^2;q^2)_k^4}q^{2k}\right)\left(\sum_{k=0}^{(n^r-1)/2}\frac{(q^{n^2};q^{2n^2})_k^4}{(q^{2n^2};q^{2n^2})_k^4}q^{2n^2k}\right). \tag{4.4}$$

We should mention that the q-congruences (4.2) and (4.4) are not equivalent to each other. This is because the left-hand sides of (4.2) and (4.4) are not congruent to each other even modulo $\Phi_n(q)$. (Instead, they are congruent to each other modulo $\Phi_{n^{r+1}}(q)^4$.)

Finally, we propose the following partial q-analogues of (4.1) and (4.3).

Conjecture 4.3. *Let $n>1$ be an odd integer and let $r\geqslant 1$. Then, modulo $\Phi_n(q)^2$,*

$$\sum_{k=0}^{(n^{r+1}-1)/d}\frac{(q;q^2)_k^4}{(q^2;q^2)_k^4}q^{2k}\equiv\left(\sum_{k=0}^{(n-1)/d}\frac{(q;q^2)_k^4}{(q^2;q^2)_k^4}q^{2k}\right)\left(\sum_{k=0}^{(n^r-1)/d}\frac{(q^{n};q^{2n})_k^4}{(q^{2n};q^{2n})_k^4}q^{2nk}\right) \tag{4.5}$$

for $d=1,2$.

So far we did not find any parametric generalizations of (4.2), (4.4) and (4.5). This makes it difficult to use the creative microscoping method here. In any case, it is the first time when a q-version of (4.1) is given, though conjectural. We hope that an interested reader will make some progress on it.

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References

[1] B. Dwork, p-adic cycles, Publ. Math. Inst. Hautes Études Sci. 37 (1969), 27–115.

[2] G. Gasper and M. Rahman, Basic hypergeometric series, second edition, Encyclopedia of Mathematics and Its Applications 96, Cambridge University Press, Cambridge, 2004.

[3] O. Gorodetsky, q-Congruences, with applications to supercongruences and the cyclic sieving phenomenon, Int. J. Number Theory 15 (2019), 1919–1968.

[4] C.-Y. Gu and V.J.W. Guo, q-Analogues of two supercongruences of Z.-W. Sun, Czechoslovak Math. J., in press; `https://doi.org/10.21136/CMJ.2020.0516-18`

[5] V.J.W. Guo, A q-analogue of the (J.2) supercongruence of Van Hamme, J. Math. Anal. Appl. 466 (2018), 776–788.

[6] V.J.W. Guo, A q-analogue of the (I.2) supercongruence of Van Hamme, Int. J. Number Theory 15 (2019), 29–36.

[7] V.J.W. Guo, Common q-analogues of some different supercongruences, Results Math. 74 (2019), Art. 131.

[8] V.J.W. Guo, q-Analogues of two "divergent" Ramanujan-type supercongruences, Ramanujan J., in press; `https://doi.org/10.1007/s11139-019-00161-0`

[9] V.J.W. Guo, Proof of a generalization of the (B.2) supercongruence of Van Hamme through a q-microscope, Adv. Appl. Math. 116 (2020), Art. 102016.

[10] V.J.W. Guo and J.-C. Liu, q-Analogues of two Ramanujan-type formulas for $1/\pi$, J. Difference Equ. Appl. 24 (2018), 1368–1373.

[11] V.J.W. Guo and M.J. Schlosser, Some new q-congruences for truncated basic hypergeometric series: even powers, Results Math. 75 (2020), Art. 1.

[12] V.J.W. Guo and M.J. Schlosser, A family of q-hypergeometric congruences modulo the fourth power of a cyclotomic polynomial, Israel J. Math., to appear.

[13] V.J.W. Guo and S.-D. Wang, Some congruences involving fourth powers of central q-binomial coefficients, Proc. Roy. Soc. Edinburgh Sect. A, `https://doi.org/10.1017/prm.2018.96`

[14] V.J.W. Guo and W. Zudilin, A q-microscope for supercongruences, Adv. Math. 346 (2019), 329–358.

[15] V.J.W. Guo and W. Zudilin, A common q-analogue of two supercongruences, Results Math. 75 (2020), Art. 46.

[16] V.J.W. Guo and W. Zudilin, Dwork-type supercongruences through a creative q-microscope, preprint, 2020, arXiv:2001.02311.

[17] J.-C. Liu and F. Petrov, Congruences on sums of q-binomial coefficients, Adv. Appl. Math. 116 (2020), Art. 102003.

[18] T. Kilbourn, An extension of the Apéry number supercongruence, Acta Arith. 123 (2006), 335–348.

[19] L. Long, Hypergeometric evaluation identities and supercongruences, Pacific J. Math. 249 (2011), 405–418.

[20] L. Long, F.-T. Tu, N. Yui, and W. Zudilin, Supercongruences for rigid hypergeometric Calabi–Yau threefolds, preprint, May 2017, arXiv:1705.01663; An announcement version in: 2017 MATRIX Annals, MATRIX Book Ser. 2 (2019), Springer, pp. 467–470.

[21] A. Mellit and M Vlasenko, Dworks congruences for the constant terms of powers of a Laurent polynomial, Int. J. Number Theory 12 (2016), 313–321.

[22] H.-X. Ni and H. Pan, Some symmetric q-congruences modulo the square of a cyclotomic polynomial, J. Math. Anal. Appl. 481 (2020), Art. 123372.

[23] R. Osburn and W. Zudilin, On the (K.2) supercongruence of Van Hamme, J. Math. Anal. Appl. 433 (2016), 706–711.

[24] M. Rahman, Some quadratic and cubic summation formulas for basic hypergeometric series, Canad. J. Math. 45 (1993), 394–411.

[25] S. Ramanujan, Modular equations and approximations to π, Quart. J. Math. Oxford Ser. (2) 45 (1914), 350–372.

[26] A. Straub, Supercongruences for polynomial analogs of the Apéry numbers, Proc. Amer. Math. Soc. 147 (2019), 1023–1036.

[27] H. Swisher, On the supercongruence conjectures of van Hamme, Res. Math. Sci. 2 (2015), Art. 18.

[28] R. Tauraso, Some q-analogs of congruences for central binomial sums, Colloq. Math. 133 (2013), 133–143.

[29] L. Van Hamme, Some conjectures concerning partial sums of generalized hypergeometric series, in: p-Adic functional analysis (Nijmegen, 1996), Lecture Notes in Pure and Appl. Math. 192, Dekker, New York, 1997, pp. 223–236.

[30] X. Wang and M. Yue, A q-analogue of the (A.2) supercongruence of Van Hamme for any prime $p \equiv 3 \pmod 4$, Int. J. Number Theory, in press; `https://doi.org/10.1142/S1793042120500694`

[31] H.S. Wilf and D. Zeilberger, An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities, Invent. Math. 108 (1992), 575–633.

[32] W. Zudilin, Ramanujan-type supercongruences, J. Number Theory 129 (2009), 1848–1857.

[33] W. Zudilin, Congruences for q-binomial coefficients, Ann. Combin. 23 (2019), 1123–1135.