A NEW q-VARIATION OF THE (C.2) SUPERCONGRUENCE OF VAN HAMME

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Abstract. Long proved that Van Hamme's (C.2) supercongruence is also true modulo p^4 for any prime $p>3$. By making use of the q-WZ method, the author and Wang gave a q-analogue of Long's supercongruence. In this paper, employing the method of 'creative microscoping', introduced by the author and Zudilin in 2019, we obtain a generalization of this q-supercongruence. A limiting case of our result implies that, for $0\leqslant t\leqslant s\leqslant 10$ and any odd prime $p\geqslant 4s+1$ and integer $r\geqslant 1$,

$$\sum_{k=s}^{(p^r-1)/2+s}\frac{4k+1}{256^k}\binom{2k-2s}{k-s}\binom{2k+2s}{k+s}\binom{2k-2t}{k-t}\binom{2k+2t}{k+t}\equiv p^r \pmod{p^{r+3}}.$$

1. Introduction

In 1997, Van Hamme [12, (C.2)] proved that, for any prime $p\geqslant 3$,

$$\sum_{k=0}^{(p-1)/2}\frac{4k+1}{256^k}\binom{2k}{k}^4\equiv p \pmod{p^3}. \tag{1.1}$$

In 2011, Long [13, Theorem 1.1] further showed that (1.1) holds modulo p^4 for primes $p\geqslant 5$. Applying the q-WZ method, the author and Wang [7] gave a q-analogue of Long's result as follows: for any positive odd integer n,

$$\sum_{k=0}^{(n-1)/2}[4k+1]\frac{(q;q^2)_k^4}{(q^2;q^2)_k^4}\equiv [n]q^{(1-n)/2}\left(1+\frac{(n^2-1)(1-q)^2}{24}[n]^2\right) \pmod{[n]\Phi_n(q)^3}. \tag{1.2}$$

Here and in what follows, we adopt the standard q-notation: $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ denotes the *q-shifted factorial*, $[n]=(1-q^n)/(1-q)$ denotes the *q-integer*, and $\Phi_n(q)$ stands for the n-th *cyclotomic polynomial*, which can be written as

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}}(q-\zeta^k),$$

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where ζ is an n-th primitive root of unity. Moreover, the q-congruence $A_1(q)/A_2(q) \equiv 0 \pmod{P(q)}$ for integer coefficient polynomials $A_1(q), A_2(q), P(q)$ is meant that $P(q)$ divides the numerator of the reduced form of $A_1(q)/A_2(q)$. For two rational functions $A(q)$ and $B(q)$, the q-congruence $A(q) \equiv B(q) \pmod{P(q)}$ means $A(q) - B(q) \equiv 0 \pmod{P(q)}$.

It follows easily from (1.2) that, for any prime $p \geqslant 5$ and integer $r \geqslant 1$,

$$\sum_{k=0}^{(p^r-1)/2} \frac{4k+1}{256^k} \binom{2k}{k}^4 \equiv p^r \pmod{p^{r+3}}, \tag{1.3}$$

which was originally observed by Long [13]. Recently, Wang and Hu [15] proved the following generalization of (1.3):

$$\sum_{k=0}^{(p^r-1)/2} \frac{4k+1}{256^k} \binom{2k}{k}^4 \equiv p^r + \frac{7}{6} p^{r+3} B_{p-3} \pmod{p^{r+4}},$$

where B_{p-3} is the $(p-3)$-th Bernoulli number, confirming a previous conjecture of the author [4, Conjecture 6.2].

In 2019, the author and Zudilin [8] introduced a new method (called 'creative microscoping') to prove q-supercongruences systematically. Shortly afterwards, the author [4] provided a new proof of (1.2) by employing the method of 'creative microscoping' together with the Chinese remainder theorem for coprime polynomials. Using the same method but with more complicated calculation, Tang [11] gave a variation of (1.2) as follows: for any odd integer $n \geqslant 5$,

$$\begin{aligned}\sum_{k=1}^{(n+1)/2} [4k+1] \frac{(q;q^2)_{k-1}(q;q^2)_{k+1}(q;q^2)_k^2}{(q^2;q^2)_{k-1}(q^2;q^2)_{k+1}(q^2;q^2)_k^2} \\ \equiv [n]q^{(1-n)/2}\left(1+\frac{(n^2-1)(1-q)^2}{24}[n]^2\right) \pmod{[n]\Phi_n(q)^3}.\end{aligned} \tag{1.4}$$

It should be pointed out that many other authors have investigated q-supercongruences in recent years. See, for example, [1, 5, 6, 9, 10, 14, 16–18].

In this paper, we shall establish the following common generalization of (1.2) and (1.4).

Theorem 1.1. *Let s and t be non-negative integers with $s \geqslant t$, and let $n \geqslant 4s+1$ be an odd integer. Then*

$$\begin{aligned}\sum_{k=s}^{(n-1)/2+s} [4k+1] \frac{(q;q^2)_{k-s}(q;q^2)_{k+s}(q;q^2)_{k-t}(q;q^2)_{k+t}}{(q^2;q^2)_{k-s}(q^2;q^2)_{k+s}(q^2;q^2)_{k-t}(q^2;q^2)_{k+t}} \\ \equiv [n]q^{(1-n)/2}\left(1+\frac{(n^2-1)(1-q)^2}{24}[n]^2\right) \pmod{\Phi_n(q)^4}.\end{aligned} \tag{1.5}$$

Furthermore, if $s \leqslant 10$, then (1.5) also holds modulo $[n]\Phi_n(q)^3$.

Letting $t=0$ or $t=s$ in (1.5), we obtain the following results: for any non-negative integer s and odd integer $n\geqslant 4s+1$,

$$\sum_{k=s}^{(n-1)/2+s}[4k+1]\frac{(q;q^2)_{k-s}(q;q^2)_{k+s}(q;q^2)_k^2}{(q^2;q^2)_{k-s}(q^2;q^2)_{k+s}(q^2;q^2)_k^2}$$
$$\equiv [n]q^{(1-n)/2}\left(1+\frac{(n^2-1)(1-q)^2}{24}[n]^2\right) \pmod{\Phi_n(q)^4}, \tag{1.6}$$
$$\sum_{k=s}^{(n-1)/2+s}[4k+1]\frac{(q;q^2)_{k-s}^2(q;q^2)_{k+s}^2}{(q^2;q^2)_{k-s}^2(q^2;q^2)_{k+s}^2}$$
$$\equiv [n]q^{(1-n)/2}\left(1+\frac{(n^2-1)(1-q)^2}{24}[n]^2\right) \pmod{\Phi_n(q)^4}.$$

On the other hand, letting $n=p^r$ be a prime power, taking $q\to 1$ in (1.5), and noticing

$$\lim_{q\to 1}\frac{(q;q^2)_k}{(q^2;q^2)_k}=\frac{1}{4^k}\binom{2k}{k},$$

we arrive at the following supercongruence: for $s\geqslant t\geqslant 0$ and any odd prime $p\geqslant 4s+1$ and integer $r\geqslant 1$,

$$\sum_{k=s}^{(p^r-1)/2+s}\frac{4k+1}{256^k}\binom{2k-2s}{k-s}\binom{2k+2s}{k+s}\binom{2k-2t}{k-t}\binom{2k+2t}{k+t}\equiv p^r \pmod{p^4}. \tag{1.7}$$

Moreover, if $s\leqslant 10$, then (1.7) is also true modulo p^{r+3}.

The paper is organized as follows. In the next section, we give five lemmas on q-congruences. Three of them are deduced from Jackson's ${}_6\phi_5$ summation. In Section 3, we use these lemmas and the Chinese remainder theorem for coprime polynomials to deduce a parametric generalization of Theorem 1.1. Then we prove the q-supercongruence (1.5) from this parametric version by L'Hôpital's rule, and prove the modulus $[n]$ case for $s\leqslant 10$ by the asymptotics at roots of unity. Finally, in Section 4 we propose two related open problems for further study.

2. Some lemmas

In order to prove Theorem 1.1, we require five lemmas on q-congruences. The first one can be stated as follows.

Lemma 2.1. *Let s and t be non-negative integers with $s\geqslant t$, and let $n\geqslant 4s+1$ be an odd integer. Then*

$$\sum_{k=s}^{(n-1)/2+s}[4k+1]\frac{(q/b;q^2)_{k-s}(q;q^2)_{k+s}(aq;q^2)_{k-t}(q/a;q^2)_{k+t}}{(q^2;q^2)_{k-s}(bq^2;q^2)_{k+s}(aq^2;q^2)_{k-t}(q^2/a;q^2)_{k+t}}b^k\equiv 0 \pmod{\Phi_n(q)} \tag{2.1}$$

Proof. The author and Schlosser [6, Lemma 3.1] observed the simple q-congruence: for $0\leqslant k\leqslant (n-1)/2$,

$$\frac{(aq;q^2)_{(n-1)/2-k}}{(q^2/a;q^2)_{(n-1)/2-k}} \equiv (-a)^{(n-1)/2-2k}\frac{(aq;q^2)_k}{(q^2/a;q^2)_k}q^{(n-1)^2/4+k} \pmod{\Phi_n(q)}. \tag{2.2}$$

It follows that, for $s\leqslant k\leqslant (n-1)/2-s$,

$$\begin{aligned}
&\frac{(aq;q^2)_{(n-1)/2-k-s}}{(q^2/a;q^2)_{(n-1)/2-k+s}}\\
&\quad=\frac{(aq;q^2)_{(n-1)/2-k-s}/(q^2/a;q^2)_{(n-1)/2-k-s}}{(1-q^{n+1-2k-2s}/a)(1-q^{n+3-2k-2s}/a)\cdots(1-q^{n+2s-1-2k}/a)}\\
&\quad\equiv\frac{(-a)^{(n-1)/2-2k-2s}(aq;q^2)_{k+s}q^{(n-1)^2/4+k+s}}{(q^2/a;q^2)_{k+s}(1-q^{1-2k-2s}/a)(1-q^{3-2k-2s}/a)\cdots(1-q^{2s-1-2k}/a)}\\
&\quad=(-a)^{(n-1)/2-2k}\frac{(aq;q^2)_{k-s}}{(q^2/a;q^2)_{k+s}}q^{(n-1)^2/4+4ks+k+s} \pmod{\Phi_n(q)},
\end{aligned} \tag{2.3}$$

and similarly, modulo $\Phi_n(q)$,

$$\frac{(aq;q^2)_{(n-1)/2-k+s}}{(q^2/a;q^2)_{(n-1)/2-k-s}} \equiv (-a)^{(n-1)/2-2k}\frac{(aq;q^2)_{k+s}}{(q^2/a;q^2)_{k-s}}q^{(n-1)^2/4-4ks+k-s}. \tag{2.4}$$

Applying the q-congruences (2.2)–(2.4), we can easily check that, for $N=(n-1)/2$ and $s\leqslant k\leqslant N-s$,

$$\begin{aligned}
&[4(N-k)+1]\frac{(q/b;q^2)_{N-k-s}(q;q^2)_{N-k+s}(aq;q^2)_{N-k-t}(q/a;q^2)_{N-k+t}}{(q^2;q^2)_{N-k-s}(bq^2;q^2)_{N-k+s}(aq^2;q^2)_{N-k-t}(q^2/a;q^2)_{N-k+t}}b^{N-k}\\
&\quad\equiv -[4k+1]\frac{(q/b;q^2)_{k-s}(q;q^2)_{k+s}(aq;q^2)_{k-t}(q/a;q^2)_{k+t}}{(q^2;q^2)_{k-s}(bq^2;q^2)_{k+s}(aq^2;q^2)_{k-t}(q^2/a;q^2)_{k+t}}b^k \pmod{\Phi_n(q)}.
\end{aligned}$$

This means that the partial sum of the left-hand side of (2.1) truncated at $k=(n-1)/2-s$ is congruent to 0 modulo $\Phi_n(q)$. Moreover, for k in the range $(n-1)/2-s<k\leqslant (n-1)/2+s$, we know that $(q;q^2)_{k+s}$ contains the factor $1-q^n$ and therefore each summand indexed by k on the left-hand side of (2.1) is congruent to 0 modulo $\Phi_n(q)$. This completes the proof of (2.1). ☐

Following Gasper and Rahman [2], the *basic hypergeometric series* ${}_{r+1}\phi_r$ is defined by (see [2])

$${}_{r+1}\phi_r\left[\begin{matrix}a_1,a_2,\dots,a_{r+1}\\ b_1,b_2,\dots,b_r\end{matrix};q,\,z\right]=\sum_{k=0}^{\infty}\frac{(a_1;q)_k(a_2;q)_k\cdots(a_{r+1};q)_k}{(q;q)_k(b_1;q)_k\cdots(b_r;q)_k}z^k.$$

Then a classical terminating ${}_6\phi_5$ summation of Jackson (see [2, Appendix (II.21)]) can be stated as follows:

$${}_6\phi_5\left[\begin{matrix}a,\,qa^{\frac12},\,-qa^{\frac12},\,b,\,c,\,q^{-n}\\ a^{\frac12},\,-a^{\frac12},\,aq/b,\,aq/c,\,aq^{n+1}\end{matrix};q,\,\frac{aq^{n+1}}{bc}\right]=\frac{(aq;q)_n(aq/bc;q)_n}{(aq/b;q)_n(aq/c;q)_n}. \tag{2.5}$$

We also need three q-congruences on the left-hand side of (2.1) modulo $1-aq^n$, $a-q^n$, and $b-q^n$, respectively. Note that all of the parametric q-congruences in [4, 8, 17] are symmetric in a and a^{-1}, and a q-congruence therein holds modulo $1-aq^n$ if and only if it also holds modulo $a-q^n$. However, this is not the case here, and we need to consider the q-congruences modulo $1-aq^n$ and $a-q^n$ individually.

Lemma 2.2. *Let s and t be non-negative integers with $s \geqslant t$, and let $n \geqslant 2s-2t+1$ be an odd integer. Then, modulo $1-aq^n$,*

$$\sum_{k=s}^{(n-1)/2+s} [4k+1] \frac{(q/b;q^2)_{k-s}(q;q^2)_{k+s}(aq;q^2)_{k-t}(q/a;q^2)_{k+t}}{(q^2;q^2)_{k-s}(bq^2;q^2)_{k+s}(aq^2;q^2)_{k-t}(q^2/a;q^2)_{k+t}} b^k$$
$$\equiv \frac{[n+2s+2t](aq;q^2)_{s-t}(q/a;q^2)_{s+t}(q^{n+2t-2s};q^2)_{2s}(q^2/b;q^2)_{(n+2t-2s-1)/2}b^s}{(q/b)^{(n+2t-2s-1)/2}(aq^2;q^2)_{s-t}(q^2/a;q^2)_{s+t}(bq^2;q^2)_{(n+2s+2t-1)/2}}. \quad (2.6)$$

Proof. For $a=q^{-n}$, the left-hand side of (2.6) is equal to

$$\sum_{k=s}^{(n-1)/2+s} [4k+1] \frac{(q^{1-n};q^2)_{k-t}(q^{1+n};q^2)_{k+t}(q/b;q^2)_{k-s}(q;q^2)_{k+s}}{(q^{2-n};q^2)_{k-t}(q^{2+n};q^2)_{k+t}(q^2;q^2)_{k-s}(bq^2;q^2)_{k+s}} b^k$$
$$= \sum_{k=0}^{(n-1)/2} [4k+4s+1] \frac{(q^{1-n};q^2)_{k+s-t}(q^{1+n};q^2)_{k+s+t}(q/b;q^2)_{k}(q;q^2)_{k+2s}}{(q^{2-n};q^2)_{k+s-t}(q^{2+n};q^2)_{k+s+t}(q^2;q^2)_{k}(bq^2;q^2)_{k+2s}} b^{k+s}$$
$$= [4s+1] \frac{(q^{1-n};q^2)_{s-t}(q^{1+n};q^2)_{s+t}(q;q^2)_{2s}}{(q^{2-n};q^2)_{s-t}(q^{2+n};q^2)_{s+t}(bq^2;q^2)_{2s}} b^s$$
$$\times {}_6\phi_5\left[\begin{matrix} q^{4s+1}, & q^{2s+\frac{5}{2}}, & -q^{2s+\frac{5}{2}}, & q/b, & q^{1+2s+2t+n}, & q^{1+2s-2t-n} \\ & q^{2s+\frac{1}{2}}, & -q^{2s+\frac{1}{2}}, & bq^{4s+2}, & q^{2+2s-2t-n}, & q^{2+2s+2t+n} \end{matrix} ; q^2, b\right]. \quad (2.7)$$

Performing the parameter substitutions $q \mapsto q^2$, $a = q^{4s+1}$, $b \mapsto q/b$, $c = q^{1+2s+2t+n}$, and $n \mapsto (n+2t-2s-1)/2$ in (2.5), one sees that the right-hand side of (2.7) can be written as

$$[4s+1] \frac{(q^{1-n};q^2)_{s-t}(q^{1+n};q^2)_{s+t}(q;q^2)_{2s}b^s}{(q^{2-n};q^2)_{s-t}(q^{2+n};q^2)_{s+t}(bq^2;q^2)_{2s}}$$
$$\times \frac{(q^{4s+3};q^2)_{(n+2t-2s-1)/2}(bq^{2s+1-2t-n};q^2)_{(n+2t-2s-1)/2}}{(bq^{4s+2};q^2)_{(n+2t-2s-1)/2}(q^{2s+2-2t-n};q^2)_{(n+2t-2s-1)/2}}$$
$$= [n+2s+2t] \frac{(q^{1-n};q^2)_{s-t}(q^{1+n};q^2)_{s+t}}{(q^{2-n};q^2)_{s-t}(q^{2+n};q^2)_{s+t}} b^s$$
$$\times \frac{(q;q^2)_{(n+2s+2t-1)/2}(bq^{2s+1-2t-n};q^2)_{(n+2t-2s-1)/2}}{(bq^2;q^2)_{(n+2s+2t-1)/2}(q^{2s+2-2t-n};q^2)_{(n+2t-2s-1)/2}}$$
$$= \frac{[n+2s+2t](q^{1-n};q^2)_{s-t}(q^{1+n};q^2)_{s+t}(q;q^2)_{(n+2s+2t-1)/2}(q^2/b;q^2)_{(n+2t-2s-1)/2}b^s}{(q/b)^{(n+2t-2s-1)/2}(q^{2-n};q^2)_{s-t}(q^{2+n};q^2)_{s+t}(bq^2;q^2)_{(n+2s+2t-1)/2}(q;q^2)_{(n+2t-2s-1)/2}},$$

which is the $a=q^{-n}$ case of the right-hand side of (2.6). Namely, the q-congruence (2.6) holds. □

Lemma 2.3. *Let s and t be non-negative integers with $s\geqslant t$, and let $n\geqslant 2s+2t+1$ be an odd integer. Then, modulo $a-q^n$,*

$$\sum_{k=s}^{(n-1)/2+s}[4k+1]\frac{(q/b;q^2)_{k-s}(q;q^2)_{k+s}(aq;q^2)_{k-t}(q/a;q^2)_{k+t}}{(q^2;q^2)_{k-s}(bq^2;q^2)_{k+s}(aq^2;q^2)_{k-t}(q^2/a;q^2)_{k+t}}b^k$$
$$\equiv\frac{[n+2s-2t](aq;q^2)_{s-t}(q/a;q^2)_{s+t}(q^{n-2s-2t};q^2)_{2s}(q^2/b;q^2)_{(n-2s-2t-1)/2}b^s}{(q/b)^{(n-2s-2t-1)/2}(aq^2;q^2)_{s-t}(q^2/a;q^2)_{s+t}(bq^2;q^2)_{(n+2s-2t-1)/2}}. \tag{2.8}$$

Proof. For $a=q^n$, the left-hand side of (2.6) is equal to

$$\sum_{k=s}^{(n-1)/2+s}[4k+1]\frac{(q^{1+n};q^2)_{k-t}(q^{1-n};q^2)_{k+t}(q/b;q^2)_{k-s}(q;q^2)_{k+s}}{(q^{2+n};q^2)_{k-t}(q^{2-n};q^2)_{k+t}(q^2;q^2)_{k-s}(bq^2;q^2)_{k+s}}b^k$$
$$=\sum_{k=0}^{(n-1)/2}[4k+4s+1]\frac{(q^{1+n};q^2)_{k+s-t}(q^{1-n};q^2)_{k+s+t}(q/b;q^2)_{k}(q;q^2)_{k+2s}}{(q^{2+n};q^2)_{k+s-t}(q^{2-n};q^2)_{k+s+t}(q^2;q^2)_{k}(bq^2;q^2)_{k+2s}}b^{k+s}$$
$$=[4s+1]\frac{(q^{1+n};q^2)_{s-t}(q^{1-n};q^2)_{s+t}(q;q^2)_{2s}}{(q^{2+n};q^2)_{s-t}(q^{2-n};q^2)_{s+t}(bq^2;q^2)_{2s}}b^s$$
$$\times{}_6\phi_5\left[\begin{matrix}q^{4s+1}, & q^{2s+\frac52}, & -q^{2s+\frac52}, & q/b, & q^{1+2s-2t+n}, & q^{1+2s+2t-n}\\ & q^{2s+\frac12}, & -q^{2s+\frac12}, & bq^{4s+2}, & q^{2+2s+2t-n}, & q^{2+2s-2t+n}\end{matrix};q^2,b\right]. \tag{2.9}$$

Making the parameter substitutions $q\mapsto q^2$, $a=q^{4s+1}$, $b\mapsto q/b$, $c=q^{1+2s-2t+n}$, and $n\mapsto(n-2s-2t-1)/2$ in (2.5), one sees that the right-hand side of (2.9) can be written as

$$[4s+1]\frac{(q^{1+n};q^2)_{s-t}(q^{1-n};q^2)_{s+t}(q;q^2)_{2s}b^s}{(q^{2+n};q^2)_{s-t}(q^{2-n};q^2)_{s+t}(bq^2;q^2)_{2s}}$$
$$\times\frac{(q^{4s+3};q^2)_{(n-2s-2t-1)/2}(bq^{2s+2t+1-n};q^2)_{(n-2s-2t-1)/2}}{(bq^{4s+2};q^2)_{(n-2s-2t-1)/2}(q^{2s+2t+2-n};q^2)_{(n-2s-2t-1)/2}}$$
$$=\frac{[n+2s-2t](q^{1+n};q^2)_{s-t}(q^{1-n};q^2)_{s+t}(q;q^2)_{(n+2s-2t-1)/2}(q^2/b;q^2)_{(n-2s-2t-1)/2}b^s}{(q/b)^{(n-2s-2t-1)/2}(q^{2+n};q^2)_{s-t}(q^{2-n};q^2)_{s+t}(bq^2;q^2)_{(n+2s-2t-1)/2}(q;q^2)_{(n-2s-2t-1)/2}},$$

which is the value of the right-hand side of (2.6) with $a=q^n$. That is, the q-congruence (2.8) holds. □

Lemma 2.4. *Let s and t be non-negative integers with $s\geqslant t$, and let n be a positive odd integer. Then, modulo $b-q^n$,*

$$\sum_{k=s}^{(n-1)/2+s}[4k+1]\frac{(q/b;q^2)_{k-s}(q;q^2)_{k+s}(aq;q^2)_{k-t}(q/a;q^2)_{k+t}}{(q^2;q^2)_{k-s}(bq^2;q^2)_{k+s}(aq^2;q^2)_{k-t}(q^2/a;q^2)_{k+t}}b^k$$

$$\equiv \frac{[n](aq;q^2)_{s-t}(q/a;q^2)_{s+t}(q;q^2)_{(n-1)/2}^2 b^s}{(aq^2;q^2)_{(n+2s-2t-1)/2}(q^2/a;q^2)_{(n+2s+2t-1)/2}}. \tag{2.10}$$

Proof. For $b=q^n$, the left-hand side of (2.10) is equal to

$$\sum_{k=s}^{(n-1)/2+s}[4k+1]\frac{(q^{1-n};q^2)_{k-s}(q;q^2)_{k+s}(aq;q^2)_{k-t}(q/a;q^2)_{k+t}}{(q^2;q^2)_{k-s}(q^{2+n};q^2)_{k+s}(aq^2;q^2)_{k-t}(q^2/a;q^2)_{k+t}}q^{nk}$$
$$=\sum_{k=0}^{(n-1)/2}[4k+4s+1]\frac{(q^{1-n};q^2)_{k}(q;q^2)_{k+2s}(aq;q^2)_{k+s-t}(q/a;q^2)_{k+s+t}}{(q^2;q^2)_{k}(q^{2+n};q^2)_{k+2s}(aq^2;q^2)_{k+s-t}(q^2/a;q^2)_{k+s+t}}q^{nk+ns}$$
$$=[4s+1]\frac{(q;q^2)_{2s}(aq;q^2)_{s-t}(q/a;q^2)_{s+t}}{(q^{2+n};q^2)_{2s}(aq^2;q^2)_{s-t}(q^2/a;q^2)_{s+t}}q^{ns}$$
$$\times {}_6\phi_5\left[\begin{matrix} q^{4s+1}, & q^{2s+\frac52}, & -q^{2s+\frac52}, & aq^{2s-2t+1}, & q^{2s+2t+1}/a, & q^{1-n} \\ & q^{2s+\frac12}, & -q^{2s+\frac12}, & q^{2s+2t+2}/a, & aq^{2s-2t+2}, & q^{4s+2+n}\end{matrix};q^2,q^n\right]. \tag{2.11}$$

Letting $q\mapsto q^2$, $a=q^{4s+1}$, $b=aq^{2s-2t+1}$, $c=q^{2s+2t+1}/a$, and $n\mapsto(n-1)/2$ in (2.5), we see that the right-hand side of (2.7) can be simplified as

$$\frac{[4s+1](q;q^2)_{2s}(aq;q^2)_{s-t}(q/a;q^2)_{s+t}(q^{4s+3};q^2)_{(n-1)/2}(q;q^2)_{(n-1)/2}q^{ns}}{(q^{2+n};q^2)_{2s}(aq^2;q^2)_{s-t}(q^2/a;q^2)_{s+t}(aq^{2s-2t+2};q^2)_{(n-1)/2}(q^{2s+2t+2}/a;q^2)_{(n-1)/2}}$$
$$=\frac{[n](aq;q^2)_{s-t}(q/a;q^2)_{s+t}(q;q^2)_{(n-1)/2}^2 q^{ns}}{(aq^2;q^2)_{(n+2s-2t-1)/2}(q^2/a;q^2)_{(n+2s+2t-1)/2}}.$$

This proves the q-congruence (2.10). □

Finally, the following lemma was given in [4, Lemma 2.1] and will play an important part in our proof of Theorem 1.1.

Lemma 2.5. *Let n be a positive odd integer. Then*

$$(aq^2,q^2)_{(n-1)/2}(q^2/a,q^2)_{(n-1)/2}\equiv(-1)^{(n-1)/2}\frac{(1-a^n)q^{-(n-1)^2/4}}{(1-a)a^{(n-1)/2}} \pmod{\Phi_n(q)}, \tag{2.12}$$

$$(aq,q^2)_{(n-1)/2}(q/a,q^2)_{(n-1)/2}\equiv(-1)^{(n-1)/2}\frac{(1-a^n)q^{(1-n^2)/4}}{(1-a)a^{(n-1)/2}} \pmod{\Phi_n(q)}. \tag{2.13}$$

3. Proof of Theorem 1.1

On the basis of the previous lemmas in Section 2, we are now able to establish the following parametric version of Theorem 1.1. Note that the right-hand side of (3.1) has also appeared in [4,11]. However, our derivation of (3.1) here is more complicated.

Theorem 3.1. *Let s and t be non-negative integers with $s \geqslant t$, and let $n \geqslant 4s+1$ be an odd integer. Then, modulo $\Phi_n(q)^2(1-aq^n)(a-q^n)$,*

$$
\begin{aligned}
&\sum_{k=s}^{(n-1)/2+s}[4k+1]\frac{(q;q^2)_{k-s}(q;q^2)_{k+s}(aq;q^2)_{k-t}(q/a;q^2)_{k+t}}{(q^2;q^2)_{k-s}(q^2;q^2)_{k+s}(aq^2;q^2)_{k-t}(q^2/a;q^2)_{k+t}}\\
&\quad\equiv q^{(1-n)/2}[n]+q^{(1-n)/2}[n]\frac{(1-aq^n)(a-q^n)}{(1-a)^2}\left(1-\frac{n(1-a)a^{(n-1)/2}}{1-a^n}\right). \qquad (3.1)
\end{aligned}
$$

Proof. It is easy to see that $\Phi_n(q)$, $1-aq^n$, $a-q^n$, and $b-q^n$ are pairwise coprime polynomials. By the Chinese reminder theorem for coprime polynomials, we can determine the remainder of the left-hand side of (2.6) modulo $\Phi_n(q)(1-aq^n)(a-q^n)(b-q^n)$ from Lemmas 2.1–2.4. Note that the right-hand sides of (2.6)–(2.10) are all congruent to 0 modulo $\Phi_n(q)$. Thus, using the following q-congruences:

$$
\begin{aligned}
\frac{(a-q^n)(b-q^n)}{(1-a^2)(1-ab)}a^2 &\equiv 1 \pmod{1-aq^n},\\
\frac{(1-aq^n)(b-q^n)}{(1-a^2)(b-a)} &\equiv 1 \pmod{a-q^n},\\
\frac{(1-aq^n)(a-q^n)}{(a-b)(1-ab)} &\equiv 1 \pmod{b-q^n}.
\end{aligned}
$$

we conclude that, modulo $\Phi_n(q)(1-aq^n)(a-q^n)(b-q^n)$,

$$
\begin{aligned}
&\sum_{k=s}^{(n-1)/2+s}[4k+1]\frac{(q/b;q^2)_{k-s}(q;q^2)_{k+s}(aq;q^2)_{k-t}(q/a;q^2)_{k+t}}{(q^2;q^2)_{k-s}(bq^2;q^2)_{k+s}(aq^2;q^2)_{k-t}(q^2/a;q^2)_{k+t}}b^k\\
&\equiv \frac{[n+2s+2t](aq;q^2)_{s-t}(q/a;q^2)_{s+t}(q^{n+2t-2s};q^2)_{2s}(q^2/b;q^2)_{(n+2t-2s-1)/2}b^s}{(q/b)^{(n+2t-2s-1)/2}(aq^2;q^2)_{s-t}(q^2/a;q^2)_{s+t}(bq^2;q^2)_{(n+2s+2t-1)/2}}\\
&\qquad\times\frac{(a-q^n)(b-q^n)}{(1-a^2)(1-ab)}a^2\\
&\quad+\frac{[n+2s-2t](aq;q^2)_{s-t}(q/a;q^2)_{s+t}(q^{n-2s-2t};q^2)_{2s}(q^2/b;q^2)_{(n-2s-2t-1)/2}b^s}{(q/b)^{(n-2s-2t-1)/2}(aq^2;q^2)_{s-t}(q^2/a;q^2)_{s+t}(bq^2;q^2)_{(n+2s-2t-1)/2}}\\
&\qquad\times\frac{(1-aq^n)(b-q^n)}{(1-a^2)(b-a)}\\
&\quad+\frac{[n](aq;q^2)_{s-t}(q/a;q^2)_{s+t}(q;q^2)_{(n-1)/2}^2b^s}{(aq^2;q^2)_{(n+2s-2t-1)/2}(q^2/a;q^2)_{(n+2s+2t-1)/2}}\frac{(1-aq^n)(a-q^n)}{(a-b)(1-ab)}. \qquad (3.2)
\end{aligned}
$$

In what follows, we consider the $b=1$ case of (3.2). It is clear that $b-q^n=1-q^n$ contains the factor $\Phi_n(q)$, and the factor $(bq^2;q^2)_{(n-1)/2+2s}=(q^2;q^2)_{(n-1)/2+2s}$ in the denominator of the left-hand side of (3.2) is relatively prime to $\Phi_n(q)$ (since $n\geqslant 4s+1$). We can also easily check that

$$\frac{[n+2s+2t](aq;q^2)_{s-t}(q/a;q^2)_{s+t}(q^{n+2t-2s};q^2)_{2s}(q^2;q^2)_{(n+2t-2s-1)/2}}{q^{(n+2t-2s-1)/2}(aq^2;q^2)_{s-t}(q^2/a;q^2)_{s+t}(q^2;q^2)_{(n+2s+2t-1)/2}} \equiv q^{(1-n)/2}[n] \pmod{\Phi_n(q)(1-aq^n)} \tag{3.3}$$

(the modulus $\Phi_n(q)$ case is obvious and the modulus $1-aq^n$ case is equivalent to saying that both sides are equal for $a=q^{-n}$). Similarly, we have

$$\frac{[n+2s-2t](aq;q^2)_{s-t}(q/a;q^2)_{s+t}(q^{n-2s-2t};q^2)_{2s}(q^2;q^2)_{(n-2s-2t-1)/2}b^s}{q^{(n-2s-2t-1)/2}(aq^2;q^2)_{s-t}(q^2/a;q^2)_{s+t}(q^2;q^2)_{(n+2s-2t-1)/2}} \equiv q^{(1-n)/2}[n] \pmod{\Phi_n(q)(a-q^n)} \tag{3.4}$$

Furthermore, since $q^n\equiv 1 \pmod{\Phi_n(q)}$, from (2.12) and (2.13) we deduce that

$$\begin{aligned}
&\frac{(aq;q^2)_{s-t}(q/a;q^2)_{s+t}(q;q^2)_{(n-1)/2}^2}{(aq^2;q^2)_{(n+2s-2t-1)/2}(q^2/a;q^2)_{(n+2s+2t-1)/2}}\\
&\quad=\frac{(aq;q^2)_{s-t}(q/a;q^2)_{s+t}(q;q^2)_{(n-1)/2}^2}{(aq^{n+1};q^2)_{s-t}(q^{n+1}/a;q^2)_{s+t}(aq^2,q^2)_{(n-1)/2}(q^2/a,q^2)_{(n-1)/2}}\\
&\quad\equiv\frac{n(1-a)a^{(n-1)/2}}{(1-a^n)q^{(n-1)/2}} \pmod{\Phi_n(q)}.
\end{aligned} \tag{3.5}$$

Thus, putting $b=1$ in (3.2) and applying the q-congruences (3.3)–(3.5), we conclude that the right-hand side of (3.2) modulo $\Phi_n(q)^2(1-aq^n)(a-q^n)$ reduces to

$$\begin{aligned}
&q^{(1-n)/2}[n]\frac{(a-q^n)(1-q^n)}{(1-a^2)(1-a)}a^2+q^{(1-n)/2}[n]\frac{(1-aq^n)(1-q^n)}{(1-a^2)(1-a)}\\
&\quad-[n]\frac{(1-aq^n)(a-q^n)}{(1-a)^2}\frac{n(1-a)a^{(n-1)/2}}{(1-a^n)q^{(n-1)/2}},
\end{aligned}$$

which is equal to the right-hand side of (3.1). This completes the proof. □

Proof of Theorem 1.1. By L'Hôpital's rule, there holds

$$\lim_{a\to 1}\frac{(1-aq^n)(a-q^n)}{(1-a)^2}\frac{(1-a^n-n(1-a)a^{(n-1)/2})}{(1-a^n)}=\frac{(n^2-1)(1-q)^2}{24}[n]^2.$$

Therefore, taking the limits of the two sides of (3.1) as $a\to 1$, we see that (1.5) is true modulo $\Phi_n(q)^4$.

We shall prove that (1.5) is also true modulo $[n]$ for $s\leqslant 10$. Namely,

$$\sum_{k=s}^{(n-1)/2+s}[4k+1]\frac{(q;q^2)_{k-s}(q;q^2)_{k+s}(q;q^2)_{k-t}(q;q^2)_{k+t}}{(q^2;q^2)_{k-s}(q^2;q^2)_{k+s}(q^2;q^2)_{k-t}(q^2;q^2)_{k+t}}\equiv 0 \pmod{[n]},$$

or, equivalently,

$$\sum_{k=0}^{(n-1)/2}[4k+4s+1]\frac{(q;q^2)_k(q;q^2)_{k+2s}(q;q^2)_{k+s-t}(q;q^2)_{k+s+t}}{(q^2;q^2)_k(q^2;q^2)_{k+2s}(q^2;q^2)_{k+s-t}(q^2;q^2)_{k+s+t}}\equiv 0 \pmod{[n]}. \quad (3.6)$$

The proof is similar to that of [6, Theorem 12.9] (or [8, Theorem 4.2]). For the reader's convenience, we include a detailed proof here.

Let $\zeta\neq 1$ stand for an n-th root of unity, possibly not primitive. That is to say, ζ is a primitive root of unity of degree d subject to $d\mid n$. Let $c_q(k)$ be the k-th summand on the left-hand side of (3.6), i.e.,

$$c_q(k)=[4k+4s+1]\frac{(q;q^2)_k(q;q^2)_{k+2s}(q;q^2)_{k+s-t}(q;q^2)_{k+s+t}}{(q^2;q^2)_k(q^2;q^2)_{k+2s}(q^2;q^2)_{k+s-t}(q^2;q^2)_{k+s+t}}.$$

Via the mathematical software `Maple`, we can check that (3.6) holds modulo $\Phi_n(q)$ for all non-negative integers $t\leqslant s\leqslant 10$ and positive odd integers $n\leqslant 4s-1$. This, together with (1.5), implies that the q-congruence (3.6) holds modulo $\Phi_n(q)$ for all $t\leqslant s\leqslant 10$ and odd $n>1$. This q-congruence is also true when the left-hand side is summing over k from 0 to $n-1$, because each summand is congruent to 0 modulo $\Phi_n(q)$ for k in the range $(n-1)/2<k\leqslant n-1$. Taking $n=d$ yields that

$$\sum_{k=0}^{(d-1)/2}c_\zeta(k)=\sum_{k=0}^{d-1}c_\zeta(k)=0.$$

Observing that

$$\frac{c_\zeta(\ell d+k)}{c_\zeta(\ell d)}=\lim_{q\to\zeta}\frac{c_q(\ell d+k)}{c_q(\ell d)}=\frac{c_\zeta(k)}{c_\zeta(0)},$$

we obtain

$$\begin{aligned}\sum_{k=0}^{(n-1)/2}c_\zeta(k)&=\sum_{\ell=0}^{(n/d-3)/2}\sum_{k=0}^{d-1}c_\zeta(\ell d+k)+\sum_{k=0}^{(d-1)/2}c_\zeta((n-d)/2+k)\\&=\frac{1}{c_\zeta(0)}\sum_{\ell=0}^{(n/d-3)/2}c_\zeta(\ell d)\sum_{k=0}^{d-1}c_\zeta(k)+\sum_{k=0}^{(d-1)/2}c_\zeta((n-d)/2+k)\\&=0.\end{aligned}$$

This means that the sum $\sum_{k=0}^{(n-1)/2}c_q(k)$ is congruent to 0 modulo $\Phi_d(q)$. Since each cyclotomic polynomial $\Phi_d(q)$ is irreducible in $\mathbb{Z}[q]$, we conclude that the left-hand side of (3.6) is congruent to 0 modulo

$$\prod_{d\mid n,\, d>1}\Phi_d(q)=[n].$$

Namely, the q-congruence (1.5) holds modulo $[n]$ for $s\leqslant 10$. Since the least common multiple of $\Phi_n(q)^4$ and $[n]$ is $[n]\Phi_n(q)^3$, we finish the proof. □

4. Two open problems

It is natural to conjecture that the condition $s \leqslant 10$ in Theorem 1.1 is not necessary. Namely, we believe that the following stronger version of Theorem 1.1 is true.

Conjecture 4.1. *The q-supercongruence* (1.5) *holds modulo $[n]\Phi_n(q)^3$. In particular, the supercongruence* (1.7) *holds modulo p^{r+3}.*

By the proof the second part of Theorem 1.1, in order to prove Conjecture 4.1, it suffices to establish the following result: for all non-negative integers $s \geqslant t$ and odd integers $n > 1$,

$$\sum_{k=s}^{(n-1)/2+s} [4k+1] \frac{(q;q^2)_{k-s}(q;q^2)_{k+s}(q;q^2)_{k-t}(q;q^2)_{k+t}}{(q^2;q^2)_{k-s}(q^2;q^2)_{k+s}(q^2;q^2)_{k-t}(q^2;q^2)_{k+t}} \equiv 0 \pmod{\Phi_n(q)},$$

which is left to an interested reader.

We also find that the following refinement of (1.7) for $s = (p^r-1)/6$ and $t = 0$ seems to be true.

Conjecture 4.2. *Let p be an odd prime and $r \geqslant 1$ with $p^r \equiv 1 \pmod 6$, and let $s = (p^r-1)/6$. Then*

$$\sum_{k=s}^{(p^r-1)/2+s} \frac{4k+1}{256^k} \binom{2k-2s}{k-s} \binom{2k+2s}{k+s} \binom{2k}{k}^2 \equiv p^r \pmod{p^{r+4}}.$$

It should be mentioned that the previous q-supercongruence (1.6) does not hold modulo $[n]\Phi_n(q)^4$ for $s = (n-1)/6$. For this reason, we think that Conjecture 4.2 is challenging even in the $r = 1$ case.

5. Declarations

Data Availability. Data sharing not applicable to this article.

Conflict of interest. The author declares no conflict of interest.

References

[1] M. El Bachraoui, On supercongruences for truncated sums of squares of basic hypergeometric series, Ramanujan J. 54 (2021), 415–426.

[2] G. Gasper and M. Rahman, Basic Hypergeometric Series, second edition, Encyclopedia of Mathematics and Its Applications 96, Cambridge University Press, Cambridge, 2004.

[3] V.J.W. Guo, A q-analogue of a Ramanujan-type supercongruence involving central binomial coefficients, J. Math. Anal. Appl. 458 (2018), 590–600.

[4] V.J.W. Guo, q-Supercongruences modulo the fourth power of a cyclotomic polynomial via creative microscoping, Adv. Appl. Math. 120 (2020), Art. 102078.

[5] V.J.W. Guo, A new extension of the (A.2) supercongruence of Van Hamme, Results Math. 77 (2022), Art. 96.

[6] V.J.W. Guo and M.J. Schlosser, Some q-supercongruences from transformation formulas for basic hypergeometric series, Constr. Approx. 53 (2021), 155–200.

[7] V.J.W. Guo and S.-D. Wang, Some congruences involving fourth powers of central q-binomial coefficients, Proc. Roy. Soc. Edinburgh Sect. A 150 (2020), 1127–1138.
[8] V.J.W. Guo and W. Zudilin, A q-microscope for supercongruences, Adv. Math. 346 (2019), 329–358.
[9] H. He and X. Wang, Some congruences that extend Van Hamme's (D.2) supercongruence, J. Math. Anal. Appl. 527 (2023), Art. 127344.
[10] J.-C. Liu, A variation of the q-Wolstenholme theorem, Ann. Mat. Pura Appl. (4) 201 (2022), 1993–2000.
[11] N. Tang, A new q-supercongruence modulo the fourth power of a cyclotomic polynomial, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. 117 (2023), Art. 101.
[12] L. Van Hamme, Some conjectures concerning partial sums of generalized hypergeometric series, in: p-Adic Functional Analysis (Nijmegen, 1996), Lecture Notes in Pure and Appl. Math. 192, Dekker, New York, 1997, pp. 223–236.
[13] L. Long, Hypergeometric evaluation identities and supercongruences, Pacific J. Math. 249 (2011), 405–418.
[14] C. Wang, A new q-extesion of the (H.2) congruence of Van Hamme for primes $p \equiv 1 \pmod 4$, Results Math. 76 (2021), Art. 205.
[15] C. Wang and D.-W. Hu, Proof of some supercongruences concerning truncated hypergeometric series, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. 117 (2023), Art. 99.
[16] X. Wang and C. Xu, q-Supercongruences on triple and quadruple sums, Results Math. 78 (2023), Art. 27.
[17] C. Wei, Some q-supercongruences modulo the fourth power of a cyclotomic polynomial, J. Combin. Theory, Ser. A 182 (2021), Art. 105469.
[18] C. Wei, A q-supercongruence from a q-analogue of Whipple's ${}_3F_2$ summation formula, J. Combin. Theory Ser. A 194 (2023), Art. 105705.

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