A NEW FAMILY OF q-SUPERCONGRUENCES FROM JACKSON'S ${}_6\phi_5$ SUMMATION

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Abstract. We obtain a new family of q-congruences modulo the fourth power of a cyclotomic polynomial. The key ingredients of our proof are the creative microscoping method, Jackson's ${}_6\phi_5$ summation, and the Chinese remainder theorem for polynomials.

1. Introduction

More than one century ago, Ramanujan mystically wrote down a number of infinite series for $1/\pi$ (see [2, p. 352]), which were later published in [13], such as

$$\sum_{k=0}^{\infty}(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 4^k}=\frac{4}{\pi}, \tag{1.1}$$

where $(a)_0=1$, $(a)_n=a(a+1)\cdots(a+n-1)$ $(n\geqslant 1)$ denotes the rising factorial. It was Van Hamme [17] who first observed that Ramanujan-type formulas have remarkable p-adic analogues. For example, Van Hamme [17, (C.2)] proved that, for any odd prime p,

$$\sum_{k=0}^{(p-1)/2}(4k+1)\frac{(\frac{1}{2})_k^4}{k!^4}\equiv p \pmod{p^3}. \tag{1.2}$$

Long [12] further proved that (1.2) holds modulo p^4 for $p>3$, and that

$$\sum_{k=0}^{(p-1)/2}(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 4^k}\equiv (-1)^{(p-1)/2}p \pmod{p^4}$$

for odd primes $p\geqslant 5$, which, corresponds to (1.1), was originally conjectured by Van Hamme [17, (J.2)].

The author and Wang [8, Theorem 1.2] established the following q-analogue of (1.2): for any positive odd integer n,

$$\sum_{k=0}^{(n-1)/2}[4k+1]\frac{(q;q^2)_k^4}{(q^2;q^2)_k^4}\equiv q^{(1-n)/2}[n]+\frac{(n^2-1)(1-q)^2}{24}q^{(1-n)/2}[n]^3 \pmod{[n]\Phi_n(q)^3}. \tag{1.3}$$

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As a conclusion, they obtained the following supercongruence: for any prime $p>3$ and positive integer r,

$$\sum_{k=0}^{(p^r-1)/2}(4k+1)\frac{(\frac{1}{2})_k^4}{k!^4}\equiv p^r \pmod{p^{r+3}},$$

confirming a result previously observed by Long [12]. Moreover, the author and Zudilin [9] introduced a method called 'creative microscoping' to prove q-supercongruences more or less systematically. For instance, they [9, Theorem 4.2] gave a parametric generalization of (1.3) modulo $[n]\Phi_n(q)^2$ as follows: for odd n, modulo $[n](1-aq^n)(a-q^n)$,

$$\sum_{k=0}^{N}[4k+1]\frac{(aq,q/a,q/b,q;q^2)_k}{(aq^2,q^2/a,bq^2,q^2;q^2)_k}b^k\equiv\frac{(b/q)^{(n-1)/2}(q^2/b;q^2)_{(n-1)/2}}{(bq^2;q^2)_{(n-1)/2}}[n], \tag{1.4}$$

where $N=(n-1)/2$ or $(n-1)$. Here it is proper to familiarize with the standard q-hypergeometric notation: $[n]=[n]_q=(1-q^n)/(1-q)$ is the *q-integer*; $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ denotes the *q-shifted factorial*, with the compact notation $(a_1,\dots,a_m;q)_n=(a_1;q)_n\cdots(a_m;q)_n$ for products of some q-shifted factorials. Moreover, let $\Phi_n(q)$ be the n-th *cyclotomic polynomial* in q, i.e.,

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}}(q-\zeta^k),$$

where ζ stands for an n-th primitive root of unity.

On the basis of the q-congruence (1.4), by employing the Chinese remainder theorem for polynomials, the author [5, Theorem 1.1] gave a new proof of (1.3). On the other hand, the author and Schlosser [6] obtained a related q-supercongruence: for any odd integer $n>1$,

$$\sum_{k=0}^{(n+1)/2}[4k-1]\frac{(q^{-1};q^2)_k^4}{(q^2;q^2)_k^4}q^{4k}\equiv-(1+3q+q^2)[n]^4 \pmod{[n]^4\Phi_n(q)}. \tag{1.5}$$

Shortly afterwards, they [7] gave a generalization of (1.3) and the modulus $[n]\Phi_n(q)^3$ case of (1.5) as follows: Let d,n,r be integers satisfying $d\geqslant 2$, $r\leqslant d-2$, and $n\geqslant d-r$, such that d and r are coprime, and $n\equiv -r \pmod{d}$. Then

$$\sum_{k=0}^{M}[2dk+r]\frac{(q^r;q^d)_k^4}{(q^d;q^d)_k^4}q^{(d-2r)k}$$
$$\equiv\begin{cases}0 \pmod{[n]\Phi_n(q)^3} & \text{if } d=2,\\[2mm] q^{r(n+r-dn)/d}\dfrac{(q^{2r};q^d)_{(dn-n-r)/d}}{(q^d;q^d)_{(dn-n-r)/d}}[dn-n] \pmod{[n]\Phi_n(q)^3} & \text{if } d\geqslant 3,\end{cases} \tag{1.6}$$

where $M = (dn - n - r)/d$ or $n - 1$. We refer the reader to [4, 10, 16, 18, 19] for some other interesting work on q-congruences.

The aim of this paper is to build the following family of q-supercongruences modulo $[n]\Phi_n(q)^3$, which can also be considered as a new generalization of (1.3) and (1.5) in the modulus $[n]\Phi_n(q)^3$ case.

Theorem 1.1. *Let d, n, r be integers satisfying $d \geqslant 2$ and $0 \leqslant n - r \leqslant dn - d$ (in particular, r may be negative), such that d and r are coprime, and $n \equiv r \pmod{d}$. Then*

$$\sum_{k=0}^{M} [2dk + r] \frac{(q^r; q^d)_k^4}{(q^d; q^d)_k^4} q^{(d-2r)k}$$

$$\equiv [n] q^{r(r-n)/d} \frac{(q^{2r}; q^d)_{(n-r)/d}}{(q^d; q^d)_{(n-r)/d}} \left(1 - [n]^2 \sum_{j=1}^{(n-r)/d} \frac{q^{dj}}{[dj]^2} \right) \pmod{[n]\Phi_n(q)^3}, \tag{1.7}$$

where $M = (n - r)/d$ or $n - 1$.

The proof of Theorem 1.1 will be given in Section 3.

For $d = 3$ and $r = \pm 1$, we obtain the following two corollaries.

Corollary 1.2. *Let $n \equiv 1 \pmod 3$ be a positive integer. Then, modulo $[n]\Phi_n(q)^3$,*

$$\sum_{k=0}^{M} [6k + 1] \frac{(q; q^3)_k^4}{(q^3; q^3)_k^4} q^k \equiv [n] q^{(1-n)/3} \frac{(q^2; q^3)_{(n-1)/3}}{(q^3; q^3)_{(n-1)/3}} \left(1 - [n]^2 \sum_{j=1}^{(n-1)/3} \frac{q^{3j}}{[3j]^2} \right),$$

where $M = (n - 1)/3$ or $n - 1$.

Corollary 1.3. *Let $n \equiv 2 \pmod 3$ be a positive integer. Then, modulo $[n]\Phi_n(q)^3$,*

$$\sum_{k=0}^{M} [6k - 1] \frac{(q^{-1}; q^3)_k^4}{(q^3; q^3)_k^4} q^{5k} \equiv [n] q^{(n+1)/3} \frac{(q^{-2}; q^3)_{(n+1)/3}}{(q^3; q^3)_{(n+1)/3}} \left(1 - [n]^2 \sum_{j=1}^{(n+1)/3} \frac{q^{3j}}{[3j]^2} \right),$$

where $M = (n + 1)/3$ or $n - 1$.

When $(d, r) = (2, 1)$, using the same technique given by Shi and Pan [15], we immediately get

$$\sum_{j=1}^{(n-1)/2} \frac{q^{2j}}{[2j]^2} \equiv \frac{1}{2} \sum_{j=1}^{n-1} \frac{q^j}{[j]^2} \equiv \frac{(1 - n^2)(1 - q)^2}{24} \pmod{\Phi_n(q)}, \tag{1.8}$$

and so the q-supercongruence (1.7) reduces to (1.3). Similarly, when $(d, r) = (2, 3)$, the q-supercongruence (1.7) reduces to the following result.

Corollary 1.4. *Let $n \geqslant 3$ be an odd integer. Then, modulo $[n]\Phi_n(q)^3$,*

$$\sum_{k=0}^{M} [4k + 3] \frac{(q^3; q^2)_k^4}{(q^2; q^2)_k^4} q^{-4k}$$

$$\equiv [n]q^{3(3-n)/2}\frac{(1-q^{n-1})(1-q^{n+1})}{(1-q^2)(1-q^4)}\left\{1+q[n]^2+[n]^2\frac{(n^2-1)(1-q)^2}{24}\right\},$$

where $M=(n-3)/2$ *or* $n-1$.

Letting $n=p^r$ be a prime power and taking $q\to 1$ in the above q-supercongruence, we get the following conclusion.

Corollary 1.5. *Let* $p>3$ *be a prime and let* r *be a positive integer. Then*

$$\sum_{k=0}^{m}(4k+3)\frac{(\frac{3}{2})_k^4}{k!^4}\equiv -\frac{p^r}{8} \pmod{p^{r+3}},$$

where $m=(p^r-3)/2$ *or* p^r-1.

Letting $d=4$ and $r=1$ in Theorem 1.1, we get the following result, which is equivalent to first q-supercongruence in [11, Theorem 1] in view of (1.8), and can be regarded as a further q-analogue of Van Hamme's (G.2) supercongruence [17].

Corollary 1.6. *Let* $n\equiv 1 \pmod 4$ *be a positive integer. Then, modulo* $[n]\Phi_n(q)^3$,

$$\sum_{k=0}^{M}[8k+1]\frac{(q;q^4)_k^4}{(q^4;q^4)_k^4}q^{2k}\equiv [n]q^{(1-n)/4}\frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}\left(1-[n]^2\sum_{j=1}^{(n-1)/4}\frac{q^{4j}}{[4j]^2}\right),$$

where $M=(n-1)/4$ *or* $n-1$.

When $d=2$ and $r<0$, we see that $(q^{2r};q^2)_{(n-r)/2}=0$ and the right-hand side of (1.7) vanishes, which coincides with the first part of (1.6).

Moreover, when $d>1$, letting $n=p^r$ be a prime power and taking the limits as $q\to 1$ in (1.7), we obtain the following result: for any prime p and positive integers d,r with $d>1$ and $p^r\equiv 1 \pmod d$,

$$\sum_{k=0}^{m}(2dk+1)\frac{(\frac{1}{d})_k^4}{k!^4}\equiv p^r\frac{(\frac{2}{d})_{(p^r-1)/d}}{(1)_{(p^r-1)/d}}\left(1-\frac{p^{2r}}{d^2}\sum_{j=1}^{(p^r-1)/d}\frac{1}{j^2}\right) \pmod{p^{r+3}}, \tag{1.9}$$

where $m=(p^r-1)/d$ or p^r-1. Note that Barman and Saikia [1, Theorem 1.2] proved that, for any prime $p\geqslant 5$ with $p\equiv 1 \pmod d$,

$$\sum_{k=0}^{p-1}(2dk+1)\frac{(\frac{1}{d})_k^4}{k!^4}\equiv (-1)^{1+(p-1)/d}p\Gamma_p(\tfrac{1}{d})^2\Gamma_p(\tfrac{d-2}{d}) \pmod{p^4}, \tag{1.10}$$

where $\Gamma_p(x)$ is the p-adic Gamma function (see [14]). Comparing (1.9) with (1.10), we have the following corollary.

Corollary 1.7. *Let $d>1$ be an integer and let $p\geqslant 5$ and $p\equiv 1 \pmod d$ be a prime. Then*

$$\frac{(\frac{2}{d})_{(p-1)/d}}{(1)_{(p-1)/d}}\left(1-\frac{p^2}{d^2}\sum_{j=1}^{(p-1)/d}\frac{1}{j^2}\right)\equiv(-1)^{1+(p-1)/d}\Gamma_p(\tfrac{1}{d})^2\Gamma_p(\tfrac{d-2}{d}) \pmod{p^3}.$$

2. Some lemmas

We require the following two result. For a proof of it, see [7, Lemma 2.2].

Lemma 2.1. *Let d,n be positive integers with $\gcd(d,n)=1$. Let r be an integer and let a,b be indeterminates. Then*

$$\sum_{k=0}^{m}[2dk+r]\frac{(aq^r,q^r/a,q^r/b,q^r;q^d)_k}{(aq^d,q^d/a,bq^d,q^d;q^d)_k}b^kq^{(d-2r)k}\equiv 0 \pmod{[n]}, \tag{2.1}$$

$$\sum_{k=0}^{n-1}[2dk+r]\frac{(aq^r,q^r/a,q^r/b,q^r;q^d)_k}{(aq^d,q^d/a,bq^d,q^d;q^d)_k}b^kq^{(d-2r)k}\equiv 0 \pmod{[n]}, \tag{2.2}$$

where $0\leqslant m\leqslant n-1$ and $dm\equiv -r \pmod n$.

We also need to establish the following q-congruence, which is a generalization of [9, Theorem 4.2].

Lemma 2.2. *Let d,n,r be integers satisfying $d\geqslant 2$ and $0\leqslant n-r\leqslant dn-d$, such that d and r are coprime, and $n\equiv r \pmod d$. Let a,b be indeterminates. Then, modulo $[n](1-aq^n)(a-q^n)$,*

$$\sum_{k=0}^{M}[2dk+r]\frac{(aq^r,q^r/a,q^r/b,q^r;q^d)_k}{(aq^d,q^d/a,bq^d,q^d;q^d)_k}b^kq^{(d-2r)k}$$

$$\equiv\frac{(q^{2r}/b;q^d)_{(n-r)/d}}{(bq^d;q^d)_{(n-r)/d}}[n]\left(\frac{b}{q^r}\right)^{(n-r)/d}, \tag{2.3}$$

where $M=(n-r)/d$ or $n-1$.

Proof. It is clear that $r\neq 0$ by the condition in the theorem. Recall that Jackson's ${}_6\phi_5$ summation can be stated as follows:

$$\sum_{k=0}^{N}\frac{(1-aq^{2k})(a,b,c,q^{-N};q)_k}{(1-a)(q,aq/b,aq/c,aq^{N+1};q)_k}\left(\frac{aq^{N+1}}{bc}\right)^k=\frac{(aq,aq/bc;q)_N}{(aq/b,aq/c;q)_N} \tag{2.4}$$

(see [3, Appendix (II.21)]). Making the parameter substitutions $q\mapsto q^d$, $a=q^r$, $b\mapsto q^r/b$, $c=q^{n+r}$ and $N=(n-r)/d$ in (2.4), we obtain

$$\sum_{k=0}^{M}\frac{[2dk+r](q^{r-n},q^{r+n},q^r/b,q^r;q^d)_k}{[r](q^{d-n},q^{d+n},bq^d,q^d;q^d)_k}b^kq^{(d-2r)k}$$

$$= \frac{(q^{d+r}, bq^{d-n-r}; q^d)_{(n-r)/d}}{(bq^d, q^{d-n}; q^d)_{(n-r)/d}}$$

$$= \frac{(q^{2r}/b; q^d)_{(n-r)/d}[n]}{(bq^d; q^d)_{(n-r)/d}[r]} \left(\frac{b}{q^r}\right)^{(n-r)/d}. \tag{2.5}$$

Namely, when $a = q^n$ or $a = q^{-n}$ both sides of (2.3) are equal. Thus, the q-congruence (2.3) is true modulo $(1-aq^n)(a-q^n)$.

On the other hand, in view of Lemma 2.1, the left-hand side of (2.3) is congruent to 0 modulo $[n]$. Since $(bq^d; q^d)_{(n-r)/d}$ is coprime with $[n]$, we conclude that (2.3) is also true modulo $[n]$. Noting that the polynomial $(1-aq^n)(a-q^n)$ is coprime with $[n]$, we complete the proof. □

The last result we need is a q-congruence modulo $b - q^n$.

Lemma 2.3. *Let d, n, r be integers satisfying $d \geqslant 2$ and $0 \leqslant n - r \leqslant dn - d$, such that d and r are coprime, and $n \equiv r \pmod{d}$. Let a and b be indeterminates. Then*

$$\sum_{k=0}^{M} [2dk+r] \frac{(aq^r, q^r/a, q^r/b, q^r; q^d)_k}{(aq^d, q^d/a, bq^d, q^d; q^d)_k} b^k q^{(d-2r)k}$$

$$\equiv \frac{(q^r, q^{d-r}; q^d)_{(n-r)/d}}{(aq^d, q^d/a; q^d)_{(n-r)/d}} [n] \pmod{b - q^n}, \tag{2.6}$$

where $M = (n-r)/d$ or $n-1$.

Proof. Performing the parameter substitutions $q \mapsto q^d$, $a = q^r$, $b = aq^r$, $c = q^r/a$ and $N = (n-r)/d$ in (2.4), we get

$$\sum_{k=0}^{(n-r)/d} \frac{[2dk+r](aq^r, q^r/a, q^{r-n}, q^r; q^d)_k}{[r](aq^d, q^d/a, q^{d+n}, q^d; q^d)_k} q^{(n+d-2r)k} = \frac{(q^{d+r}, q^{d-r}; q^d)_{(n-r)/d}}{(aq^d, q^d/a; q^d)_{(n-r)/d}}.$$

Namely, when $b = q^n$ the two sides of (2.6) are equal. This establishes the desired q-congruence (2.6). □

3. Proof of Theorem 1.1

It is obvious that the polynomials $[n](1-aq^n)(a-q^n)$ and $b-q^n$ are coprime. In light of the Chinese remainder theorem for polynomials, we can calculate the remainder of the left-hand side of (2.3) modulo $[n](1-aq^n)(a-q^n)(b-q^n)$ from the q-congruences (2.3) and (2.6). For this purpose, we need the following q-congruences:

$$\frac{(b-q^n)(ab-1-a^2+aq^n)}{(a-b)(1-ab)} \equiv 1 \pmod{(1-aq^n)(a-q^n)},$$

$$\frac{(1-aq^n)(a-q^n)}{(a-b)(1-ab)} \equiv 1 \pmod{b-q^n}.$$

Therefore, we deduce from (2.3) and (2.6) that, modulo $[n](1-aq^n)(a-q^n)(b-q^n)$,

$$\sum_{k=0}^{M}[2dk+r]\frac{(aq^r,q^r/a,q^r/b,q^r;q^d)_k}{(aq^d,q^d/a,bq^d,q^d;q^d)_k}b^kq^{(d-2r)k}$$
$$\equiv\frac{(q^{2r}/b;q^d)_{(n-r)/d}}{(bq^d;q^d)_{(n-r)/d}}\frac{(b-q^n)(ab-1-a^2+aq^n)}{(a-b)(1-ab)}[n]\left(\frac{b}{q^r}\right)^{(n-r)/d}$$
$$+\frac{(q^r,q^{d-r};q^d)_{(n-r)/d}}{(aq^d,q^d/a;q^d)_{(n-r)/d}}\frac{(1-aq^n)(a-q^n)}{(a-b)(1-ab)}[n]. \tag{3.1}$$

Note that $1-q^n$ contains the factor $\Phi_n(q)$. Thus, taking $b=1$ in (3.1) and observing that

$$(1-q^n)(1+a^2-a-aq^n)=(1-a)^2+(1-aq^n)(a-q^n),$$

we arrive at the following q-congruence: modulo $\Phi_n(q)^2(1-aq^n)(a-q^n)$,

$$\sum_{k=0}^{M}[2dk+r]\frac{(aq^r,q^r/a,q^r,q^r;q^d)_k}{(aq^d,q^d/a,q^d,q^d;q^d)_k}q^{(d-2r)k}$$
$$\equiv[n]q^{r(r-n)/d}\frac{(q^{2r};q^d)_{(n-r)/d}}{(q^d;q^d)_{(n-r)/d}}\left\{1+\frac{(1-aq^n)(a-q^n)}{(1-a)^2}\right\}$$
$$-\frac{(q^r,q^{d-r};q^d)_{(n-r)/d}}{(aq^d,q^d/a;q^d)_{(n-r)/d}}\frac{(1-aq^n)(a-q^n)}{(1-a)^2}[n]. \tag{3.2}$$

Since $q^n\equiv 1 \pmod{\Phi_n(q)}$, it is not difficult to see that

$$\begin{aligned}(q^r;q^d)_{(n-r)/d}&=(1-q^r)(1-q^{d+r})\cdots(1-q^{n-d})\\&\equiv(1-q^{r-n})(1-q^{d+r-n})\cdots(1-q^{-d})\\&=(-1)^{(n-r)/d}(q^d;q^d)_{(n-r)/d}q^{-(d+n-r)(n-r)/(2d)} \pmod{\Phi_n(q)},\end{aligned}$$

and similarly,

$$(q^{d-r};q^d)_{(n-r)/d}\equiv(-1)^{(n-r)/d}(q^{2r};q^d)_{(n-r)/d}q^{-(n+3r-d)(n-r)/(2d)} \pmod{\Phi_n(q)}.$$

It follows that

$$(q^r,q^{d-r};q^d)_{(n-r)/d}\equiv(q^{2r},q^d;q^d)_{(n-r)/d}q^{r(r-n)/d} \pmod{\Phi_n(q)},$$

and we may rewrite the q-congruence (3.2) as follows: modulo $\Phi_n(q)^2(1-aq^n)(a-q^n)$,

$$\sum_{k=0}^{M}[2dk+r]\frac{(aq^r,q^r/a,q^r,q^r;q^d)_k}{(aq^d,q^d/a,q^d,q^d;q^d)_k}q^{(d-2r)k}$$
$$\equiv[n]q^{r(r-n)/d}\frac{(q^{2r};q^d)_{(n-r)/d}}{(q^d;q^d)_{(n-r)/d}}$$

$$+ [n]q^{r(r-n)/d}\frac{(1-aq^n)(a-q^n)}{(1-a)^2}\left\{\frac{(q^{2r};q^d)_{(n-r)/d}}{(q^d;q^d)_{(n-r)/d}}-\frac{(q^{2r},q^d;q^d)_{(n-r)/d}}{(aq^d,q^d/a;q^d)_{(n-r)/d}}\right\}. \tag{3.3}$$

By L'Hôpital's rule, we get

$$\lim_{a\to 1}\frac{(1-aq^n)(a-q^n)}{(1-a)^2}\left\{\frac{(q^{2r};q^d)_{(n-r)/d}}{(q^d;q^d)_{(n-r)/d}}-\frac{(q^{2r},q^d;q^d)_{(n-r)/d}}{(aq^d,q^d/a;q^d)_{(n-r)/d}}\right\}$$
$$= -[n]^2\frac{(q^{2r};q^d)_{(n-r)/d}}{(q^d;q^d)_{(n-r)/d}}\sum_{j=1}^{(n-r)/d}\frac{q^{dj}}{[dj]^2}. \tag{3.4}$$

Taking the limits as $a\to 1$ in (3.3) and using the limit (3.4), we see that the q-congruence (1.7) holds modulo $\Phi_n(q)^4$. Note that the proof of [7, Lemma 2.2] also implies that the q-congruences (2.1) and (2.2) are true for $a=b=1$. Namely, the q-congruence (1.7) also holds modulo $[n]$ for $M=(n-r)/d$ or $n-1$. The proof then follows from the fact that the least common multiple of $\Phi_n(q)^4$ and $[n]$ is just $[n]\Phi_n(q)^3$.

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4. Declarations

Conflicts of interest: No potential conflict of interest was reported by the author.
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