Proof of a generalization of the (C.2) supercongruence of Van Hamme

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Abstract. We prove some q-supercongruences for certain truncated basic hypergeometric series by making use of Andrews' multiseries generalization of the Watson transformation, the creative microscoping method, and the Chinese remainder theorem for coprime polynomials. More precisely, we confirm Conjectures 5.2 and 5.3 in [Adv. Appl. Math. 116 (2020), Art. 102016]. As a conclusion, we also prove Conjecture 4.3 in [Integral Transforms Spec. Funct. 28 (2017), 888–899] which may be deemed a generalization of the (C.2) supercongruence of Van Hamme.

Keywords: basic hypergeometric series; Andrews' transformation; q-congruences; supercongruences.

AMS Subject Classifications: 11B65, 11A07, 33D15

1. Introduction

In 1859, Bauer [2] established the following representation for $1/\pi$:

$$\sum_{k=0}^{\infty} \frac{4k+1}{(-64)^k} \binom{2k}{k}^3 = \frac{2}{\pi},$$

which is now usually called a Ramanujan-type series of $1/\pi$, since Ramanujan later rather mysteriously recorded 17 rapidly convergent series of $1/\pi$ (see [4, p. 352]), including

$$\sum_{k=0}^{\infty} \frac{6k+1}{256^k} \binom{2k}{k}^3 = \frac{4}{\pi}.$$

Ramanujan's formulas gained unprecedented popularity in 1980's when they were found to provide fast algorithms for computing decimal digits of π. See, for example, the monograph [3].

In 1997, Van Hamme [30] observed that 13 Ramanujan's and Ramanujan-type formulas possess interesting p-adic analogues, such as

$$\sum_{k=0}^{(p-1)/2} \frac{4k+1}{(-64)^k} \binom{2k}{k}^3 \equiv p(-1)^{(p-1)/2} \pmod{p^3}, \tag{1.1}$$

$$\sum_{k=0}^{(p-1)/2} \frac{4k+1}{256^k} \binom{2k}{k}^4 \equiv p \pmod{p^3}, \tag{1.2}$$

$$\sum_{k=0}^{(p-1)/2} \frac{6k+1}{256^k} \binom{2k}{k}^3 \equiv p(-1)^{(p-1)/2} \pmod{p^4}, \quad p > 3,$$

where p is an odd prime. Van Hamme [30, (C.2)] himself proved (1.2) and another two supercongruences of his list. Long [24] proved that the supercongruence (1.2) is also true modulo p^4. The supercongruence (1.1) was first proved by Mortenson [25] and later reproved by Zudilin [36] and Long [24]. A generalization of (1.2) modulo p^4 was obtained by Sun [28]. For more Ramanujan-type supercongruences, the reader is referred to Zudilin's celebrated paper [36].

In 2017, inspired by Zudilin's WZ (Wilf–Zeilberger [35]) proof of (1.1), the author [8] investigated more WZ-pairs related to generalizations of (1.1) and proved some related supercongruences. He also proposed the following (corrected version of) conjecture on a generalization of (1.2) (see [8, Conjecture 4.3]).

Conjecture 1.1. *For any positive odd integer s, there exists an integer b_s such that, for any odd prime $p \geqslant (s+1)/2$ and positive integer r, there hold*

$$\sum_{k=0}^{(p^r-1)/2} \frac{(4k+1)^s}{256^k} \binom{2k}{k}^4 \equiv (-1)^{(s-1)/2} b_s p^r \pmod{p^{r+3}}, \tag{1.3}$$

$$\sum_{k=0}^{p^r-1} \frac{(4k+1)^s}{256^k} \binom{2k}{k}^4 \equiv (-1)^{(s-1)/2} b_s p^r \pmod{p^{r+3}}, \tag{1.4}$$

In particular, we have $b_1 = 1$, $b_3 = 1$, $b_5 = 3$, $b_7 = 23$, $b_9 = 371$ and $b_{11} = 10515$.

For $r = s = 1$, since $\binom{2k}{k} \equiv 0 \pmod{p}$ for $(p-1)/2 < k \leqslant p-1$, the supercongruence (1.3) is equivalent to (1.4), and the (C.2) supercongruence of Van Hamme [30] is the special case of them modulo p^3. Long [24] first proved the $r = s = 1$ case, and she also observed the supercongruence (1.3) is true for $s = 1$ and all positive integers r. Many more special cases of (1.3) were confirmed by Wang [31], Liu [22], the author and Wang [17], Hou, Mu, and Zeilberger [20], and the author [9, 11, 12]. In particular, the author [12] has proved that (1.3) and (1.4) are true modulo p^{r+2} for any odd prime p and arbitrary r using the method of 'creative microscoping' introduced by the author and Zudilin [18].

A purpose of this paper is prove the following q-analogue of (1.3) and (1.4), which was originally conjectured by the author [12, Conjecture 5.2].

Theorem 1.2. *Let n and s be positive odd integers with $n \geqslant (s+1)/2$. Then, modulo $[n]_{q^2}\Phi_n(q^2)^3$,*

$$\sum_{k=0}^{M}[4k+1]_{q^2}[4k+1]^{s-1}\frac{(q^2;q^4)_k^4}{(q^4;q^4)_k^4}q^{(2-2s)k}$$
$$\equiv q^{1-n}[n]_{q^2}B_s(q)+\frac{(n^2-1)(1-q^2)^2}{24}q^{1-n}[n]_{q^2}^3B_s(q), \tag{1.5}$$

where $M=(n-1)/2$ or $n-1$, and $B_s(q)$ is a rational function of q given by

$$B_s(q)=\sum_{j_1,\dots,j_{m-1}=0}^{1}(-1)^{j_1+\cdots+j_{m-1}}q^{-4(j_{m-2}+\cdots+(m-2)j_1)}$$
$$\times\frac{(q^5;q^4)_{j_1}^2\cdots(q^5;q^4)_{j_1+\cdots+j_{m-2}}^2(q^2;q^4)_{j_1+\cdots+j_{m-1}}^3}{(q;q^4)_{j_1}^2\cdots(q;q^4)_{j_1+\cdots+j_{m-1}}^2(q^4;q^4)_{j_1+\cdots+j_{m-1}}}\quad \textit{with } m=\frac{s+1}{2}.$$

Here and in what follows, $(a;q)_k=(1-a)(1-aq)\cdots(1-aq^{k-1})$ denotes the q-*shifted factorial*. For simplicity, we frequently use the compact notation $(a_1,\dots,a_m;q)_k=(a_1;q)_k\cdots(a_m;q)_k$. Moreover, $[n]=[n]_q=1+q+\cdots+q^{n-1}$ is the q-*integer*, and $\Phi_n(q)$ stands for the n-th *cyclotomic polynomial* in q, which may be given by

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(n,k)=1}}(q-\zeta^k),$$

where ζ is an n-th primitive root of unity.

Note that b_s may be defined as $(-1)^{(s-1)/2}B_s(1)$. For the reader's convenience, we list the first values of $B_s(q)$: $B_1(q)=1$, $B_3(q)=-2q/(q^2+1)$, and

$$B_5(q)=\frac{q^2(5q^4+4q^3+6q^2+4q+5)}{(q^4+1)(q^2+1)^2},$$
$$B_7(q)=-\frac{2q^3(7q^8+14q^7+23q^6+30q^5+36q^4+30q^3+23q^2+14q+7)}{(q^6+1)(q^4+1)(q^2+1)^2}.$$

It is easy to see that $B_s(1)$ is an integer (see [12]). Therefore, when $n=p^r$ and $q\to 1$, the congruence (1.5) reduces to (1.3) and (1.4). We should point out that (1.3) with $r=1$ also implies that [20, Conjecture 4.6] is true.

In a recent paper [14, Conjecture 5.2], the author and Liu made the following conjecture.

Conjecture 1.3. *For any odd positive integer s, there exists an integer d_s such that, for* *any odd prime p and positive integer r, there hold*

$$\sum_{k=0}^{(p^r+1)/2}(4k-1)^s\frac{(-\frac{1}{2})_k^4}{k!^4}\equiv d_sp^r \pmod{p^{r+3}}, \tag{1.6}$$

$$\sum_{k=0}^{p^r-1}(4k-1)^s\frac{(-\frac{1}{2})_k^4}{k!^4}\equiv d_sp^r \pmod{p^{r+3}}. \tag{1.7}$$

In particular, we have $d_1=d_3=0$, $d_5=16$, $d_7=80$, $d_9=192$, $d_{11}=640$, $d_{13}=-3472$, and $d_{15}=138480$.

The second purpose of this paper is to prove (1.6) and (1.7) by establishing the following q-analogue, which was previously conjectured in [12, Conjecture 5.3].

Theorem 1.4. *Let n and s be positive odd integers with $n\geqslant(s-1)/2$ and $n>1$. Then, modulo $[n]_{q^2}\Phi_n(q^2)^3$,*

$$\begin{aligned}\sum_{k=0}^{M}&[4k-1]_{q^2}[4k-1]^{s-1}\frac{(q^{-2};q^4)_k^4}{(q^4;q^4)_k^4}q^{(10-2s)k}\\ &\equiv q^{1-n}[n]_{q^2}D_s(q)+\frac{(n^2-1)(1-q^2)^2}{24}q^{1-n}[n]_{q^2}^3D_s(q),\end{aligned} \tag{1.8}$$

where $M=(n+1)/2$ or $n-1$, and $D_s(q)$ is a rational function of q given by

$$\begin{aligned}D_s(q)=-\frac{q^{3-s}}{(1-q^2)^2}&\sum_{j_1,\dots,j_{m-1}=0}^{1}(-1)^{j_1+\cdots+j_{m-1}}q^{-4(j_{m-2}+\cdots+(m-2)j_1)}\\ &\times\frac{(q^3;q^4)_{j_1}^2\cdots(q^3;q^4)_{j_1+\cdots+j_{m-2}}^2(q^{-2};q^4)_{j_1+\cdots+j_{m-1}}^3}{(q^{-1};q^4)_{j_1}^2\cdots(q^{-1};q^4)_{j_1+\cdots+j_{m-1}}^2(q^4;q^4)_{j_1+\cdots+j_{m-1}-2}}\end{aligned}$$

with $m=(s+1)/2$ and $1/(q^4;q^4)_k=0$ for any negative integer k.

Note that we may take $d_s=D_s(1)$. Here are the first values of $D_s(q)$: $D_1(q)=D_3(q)=0$, $D_5(q)=(q+1)^4/q^8$, and

$$D_7(q)=\frac{2(2q^2+q+2)(q+1)^4}{(q^2+1)q^{10}},$$

$$D_9(q)=\frac{(10q^8+8q^7+19q^6+4q^5+14q^4+4q^3+19q^2+8q+10)(q+1)^4}{(q^4+1)(q^2+1)^2q^{12}}.$$

It was shown in [12] that $D_s(1)$ is always an integer.

For more q-congruences in the literature, we refer the reader to [6, 7, 10, 15, 16, 19, 21, 23, 26, 27, 29, 32–34, 37].

We shall prove Theorems 1.2 and 1.4 in Sections 2 and 3, respectively. To this end we shall make use of the creative microscoping method [18], Andrews' multiseries generalization of Watson's transformation [1, Theorem 4], and the Chinese remainder theorem for coprime polynomials. In addition, a simple property of q-shifted factorials (Lemma 2.1) will play an important part in our proof.

2. Proof of Theorem 1.2

We require the following easily proved results (see [15, Lemma 3.1] and [13, Lemma 2.1]).

Lemma 2.1. *Let n be a positive odd integer and let a be an indeterminate. Then, for $0 \leqslant k \leqslant (n-1)/2$, we have*

$$\frac{(aq;q^2)_{(n-1)/2-k}}{(q^2/a;q^2)_{(n-1)/2-k}} \equiv (-a)^{(n-1)/2-2k}\frac{(aq;q^2)_k}{(q^2/a;q^2)_k}q^{(n-1)^2/4+k} \pmod{\Phi_n(q)}.$$

Lemma 2.2. *Let n be a positive odd integer. Then*

$$(aq^2,q^2/a;q^2)_{(n-1)/2} \equiv (-1)^{(n-1)/2}\frac{(1-a^n)q^{-(n-1)^2/4}}{(1-a)a^{(n-1)/2}} \pmod{\Phi_n(q)},$$

$$(aq,q/a;q^2)_{(n-1)/2} \equiv (-1)^{(n-1)/2}\frac{(1-a^n)q^{(1-n^2)/4}}{(1-a)a^{(n-1)/2}} \pmod{\Phi_n(q)}.$$

We first establish the following q-congruence, which is a generalization of [12, Theorem 2.2] (the $b=1$ case) and may also be regarded as a generalization of [18, Theorem 4.2] (the $s=1$ case).

Theorem 2.3. *Let $n>1$ and $s>1$ be odd integers. Let a and b be indeterminates. Then, modulo $\Phi_n(q^2)(1-aq^{2n})(a-q^{2n})$,*

$$\begin{aligned}
&\sum_{k=0}^{(n-1)/2}[4k+1]_{q^2}[4k+1]^{s-1}\frac{(aq^2,q^2/a,q^2/b,q^2;q^4)_k}{(aq^4,q^4/a,bq^4,q^4;q^4)_k}b^kq^{(2-2s)k} \\
&\quad\equiv b^{(n-1)/2}q^{1-n}[n]_{q^2}\frac{(q^4/b;q^4)_{(n-1)/2}}{(bq^4;q^4)_{(n-1)/2}}\sum_{j_1,\dots,j_{m-1}=0}^{1}(-1)^{j_1+\cdots+j_{m-1}}q^{-4(j_{m-2}+\cdots+(m-2)j_1)} \\
&\quad\quad\times\frac{(q^5;q^4)_{j_1}^2\cdots(q^5;q^4)_{j_1+\cdots+j_{m-2}}^2(aq^2,q^2/a,q^2/b;q^4)_{j_1+\cdots+j_{m-1}}}{(q;q^4)_{j_1}^2\cdots(q;q^4)_{j_1+\cdots+j_{m-1}}^2(q^4/b;q^4)_{j_1+\cdots+j_{m-1}}},
\end{aligned}\tag{2.1}$$

where $m=(s+1)/2$

Proof. We need to use a transformation formula of Andrews [1, Theorem 4]:

$$\begin{aligned}
&\sum_{k\geqslant 0}\frac{(a,q\sqrt{a},-q\sqrt{a},b_1,c_1,\dots,b_m,c_m,q^{-N};q)_k}{(q,\sqrt{a},-\sqrt{a},aq/b_1,aq/c_1,\dots,aq/b_m,aq/c_m,aq^{N+1};q)_k}\left(\frac{a^mq^{m+N}}{b_1c_1\cdots b_mc_m}\right)^k \\
&\quad=\frac{(aq,aq/b_mc_m;q)_N}{(aq/b_m,aq/c_m;q)_N}\sum_{j_1,\dots,j_{m-1}\geqslant 0}\frac{(aq/b_1c_1;q)_{j_1}\cdots(aq/b_{m-1}c_{m-1};q)_{j_{m-1}}}{(q;q)_{j_1}\cdots(q;q)_{j_{m-1}}} \\
&\quad\quad\times\frac{(b_2,c_2;q)_{j_1}\dots(b_m,c_m;q)_{j_1+\cdots+j_{m-1}}}{(aq/b_1,aq/c_1;q)_{j_1}\dots(aq/b_{m-1},aq/c_{m-1};q)_{j_1+\cdots+j_{m-1}}} \\
&\quad\quad\times\frac{(q^{-N};q)_{j_1+\cdots+j_{m-1}}}{(b_mc_mq^{-N}/a;q)_{j_1+\cdots+j_{m-1}}}\frac{(aq)^{j_{m-2}+\cdots+(m-2)j_1}q^{j_1+\cdots+j_{m-1}}}{(b_2c_2)^{j_1}\cdots(b_{m-1}c_{m-1})^{j_1+\cdots+j_{m-2}}},
\end{aligned}\tag{2.2}$$

which is a multi-series generalization of the Watson transformation formula [5, Appendix (III.18)]:

$$
\begin{aligned}
&{}_8\phi_7\left[\begin{matrix} a, & qa^{\frac12}, & -qa^{\frac12}, & b, & c, & d, & e, & q^{-N} \\ & a^{\frac12}, & -a^{\frac12}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{N+1}\end{matrix} ; q,\ \frac{a^2q^{N+2}}{bcde}\right] \\
&\quad = \frac{(aq, aq/de; q)_N}{(aq/d, aq/e; q)_N}\,{}_4\phi_3\left[\begin{matrix} aq/bc,\ d,\ e,\ q^{-N} \\ aq/b,\ aq/c,\ deq^{-N}/a \end{matrix} ; q,\ q\right].
\end{aligned}
$$

For $a = q^{-2n}$ or $a = q^{2n}$, the left-hand side of (2.1) is equal to

$$
\begin{aligned}
&\sum_{k=0}^{(n-1)/2} [4k+1]_{q^2}[4k+1]^{s-1}\frac{(q^{2-2n}, q^{2+2n}, q^2/b, q^2; q^4)_k}{(q^{4-2n}, q^{4+2n}, bq^4, q^4; q^4)_k} b^k q^{(2-2s)k} \\
&\quad = \sum_{k=0}^{(n-1)/2} \frac{(q^2, q^5, -q^5, \overbrace{q^5, \dots, q^5}^{(s-1)\text{'s } q^5}, q^2/b, q^{2+2n}, q^{2-2n}; q^4)_k}{(q^4, q, -q, q, \dots, q, bq^4, q^{4-2n}, q^{4+2n}; q^4)_k} b^k q^{(2-2s)k}. \qquad (2.3)
\end{aligned}
$$

By Andrews' transformation (2.2) with the parameter substitutions $m = (s+1)/2$, $q \mapsto q^4$, $a = q^2$, $b_1 = c_1 = \cdots = b_{m-1} = c_{m-1} = q^5$, $b_m = q^2/b$, $c_m = q^{2+2n}$ and $N = (n-1)/2$, the right-hand side of (2.3) may be written as

$$
\begin{aligned}
&\frac{(q^6, bq^{2-2n}; q^4)_{(n-1)/2}}{(bq^4, q^{4-2n}; q^4)_{(n-1)/2}} \sum_{j_1,\dots,j_{m-1}\geqslant 0} \frac{(q^{-4}; q^4)_{j_1}\cdots(q^{-4}; q^4)_{j_{m-1}}}{(q^4; q^4)_{j_1}\cdots(q^4; q^4)_{j_{m-1}}} q^{4(j_1+\cdots+j_{m-1})-4(j_{m-2}+\cdots+(m-2)j_1)} \\
&\quad\times \frac{(q^5; q^4)^2_{j_1}\cdots(q^5; q^4)^2_{j_1+\cdots+j_{m-2}}(q^2/b, q^{2+2n}, q^{2-2n}; q^4)_{j_1+\cdots+j_{m-1}}}{(q; q^4)^2_{j_1}\cdots(q; q^4)^2_{j_1+\cdots+j_{m-1}}(q^4/b; q^4)_{j_1+\cdots+j_{m-1}}} \qquad (2.4)
\end{aligned}
$$

Since

$$
\frac{(q^6, bq^{2-2n}; q^4)_{(n-1)/2}}{(bq^4, q^{4-2n}; q^4)_{(n-1)/2}} = b^{(n-1)/2}q^{1-n}[n]_{q^2}\frac{(q^4/b; q^4)_{(n-1)/2}}{(bq^4; q^4)_{(n-1)/2}},
$$

and

$$
\frac{(q^{-4}; q^4)_k}{(q^4; q^4)_k} = \begin{cases} (-1)^k q^{-4k}, & \text{if } k = 0, 1, \\ 0, & \text{otherwise,} \end{cases} \qquad (2.5)
$$

the expression (2.4) can be simplified as

$$
\begin{aligned}
&b^{(n-1)/2}q^{1-n}[n]_{q^2}\frac{(q^4/b; q^4)_{(n-1)/2}}{(bq^4; q^4)_{(n-1)/2}} \sum_{j_1,\dots,j_{m-1}=0}^{1} (-1)^{j_1+\cdots+j_{m-1}} q^{-4(j_{m-2}+\cdots+(m-2)j_1)} \\
&\quad\times \frac{(q^5; q^4)^2_{j_1}\cdots(q^5; q^4)^2_{j_1+\cdots+j_{m-2}}(q^2/b, q^{2-2n}, q^{2+2n}; q^4)_{j_1+\cdots+j_{m-1}}}{(q; q^4)^2_{j_1}\cdots(q; q^4)^2_{j_1+\cdots+j_{m-1}}(q^4/b; q^4)_{j_1+\cdots+j_{m-1}}}.
\end{aligned}
$$

This proves that the congruence (2.1) is true modulo $1-aq^{2n}$ and $a-q^{2n}$.

In addition, by Lemma 2.1, it is easy to check that the sum of the k-th and $((n-1)/2-k)$-th terms on the left-hand side of (2.1) is congruent to 0 modulo $\Phi_n(q^2)$ for any k in the range $0\leqslant k\leqslant (n-1)/2$. It follows that the left-hand side of (2.1) is congruent to 0 modulo $\Phi_n(q^2)$, and so (2.1) holds modulo $\Phi_n(q^2)$. Since $1-aq^{2n}$, $a-q^{2n}$ and $\Phi_n(q^2)$ are pairwise relatively polynomials, we complete the proof the theorem. □

We also need to establish the following simpler congruence.

Theorem 2.4. *Let $n>1$ and $s>1$ be odd integers. Let a and b be indeterminates. Then, modulo $b-q^{2n}$,*

$$\begin{aligned}
&\sum_{k=0}^{(n-1)/2}[4k+1]_{q^2}[4k+1]^{s-1}\frac{(aq^2,q^2/a,q^2/b,q^2;q^4)_k}{(aq^4,q^4/a,bq^4,q^4;q^4)_k}b^kq^{(2-2s)k}\\
&\quad\equiv\frac{[n]_{q^2}(q^2;q^4)_{(n-1)/2}^2}{(aq^4,q^4/a;q^4)_{(n-1)/2}}\sum_{j_1,\dots,j_{m-1}=0}^{1}(-1)^{j_1+\cdots+j_{m-1}}q^{-4(j_{m-2}+\cdots+(m-2)j_1)}\\
&\quad\times\frac{(q^5;q^4)_{j_1}^2\cdots(q^5;q^4)_{j_1+\cdots+j_{m-2}}^2(aq^2,q^2/a,q^2/b;q^4)_{j_1+\cdots+j_{m-1}}}{(q;q^4)_{j_1}^2\cdots(q;q^4)_{j_1+\cdots+j_{m-1}}^2(q^4/b;q^4)_{j_1+\cdots+j_{m-1}}},
\end{aligned}\tag{2.6}$$

where $m=(s+1)/2$

Proof. For $b=q^{2n}$, the left-hand side of (2.6) is equal to

$$\sum_{k=0}^{(n-1)/2}\frac{(q^2,q^5,-q^5,\overbrace{q^5,\dots,q^5}^{(s-1)\text{'s }q^5},aq^2,q^2/a,q^{2-2n};q^4)_k}{(q^4,q,-q,q,\dots,q,q^4/a,aq^4,q^{4+2n};q^4)_k}q^{(2n+2-2s)k},$$

which by Andrews' transformation (2.2) can be written as

$$\begin{aligned}
&\frac{(q^6,q^2;q^4)_{(n-1)/2}}{(aq^4,q^4/a;q^4)_{(n-1)/2}}\sum_{j_1,\dots,j_{m-1}\geqslant 0}\frac{(q^{-4};q^4)_{j_1}\cdots(q^{-4};q^4)_{j_{m-1}}}{(q^4;q^4)_{j_1}\cdots(q^4;q^4)_{j_{m-1}}}q^{4(j_1+\cdots+j_{m-1})-4(j_{m-2}+\cdots+(m-2)j_1)}\\
&\quad\times\frac{(q^5;q^4)_{j_1}^2\cdots(q^5;q^4)_{j_1+\cdots+j_{m-2}}^2(aq^2,q^2/a,q^{2-2n};q^4)_{j_1+\cdots+j_{m-1}}}{(q;q^4)_{j_1}^2\cdots(q;q^4)_{j_1+\cdots+j_{m-1}}^2(q^{4-2n};q^4)_{j_1+\cdots+j_{m-1}}}.
\end{aligned}$$

In view of (2.5), the above expression is just the $b=q^{2n}$ case of the right-hand side of (2.6). This proves that the two sides of (2.6) are equal for $b=q^{2n}$. Namely, the congruence (2.6) holds modulo $b-q^{2n}$. □

Note that the $s=1$ case of Theorem 1.2 was already proved by the author and Wang [17] (for $M=(n-1)/2$) and the author and Schlosser [15] (for $M=n-1$). On the basis of Theorems 2.3 and 2.4, we are now able to prove Theorem 1.2 for $s>1$. More concretely, we shall make use of the Chinese remainder theorem for coprime polynomials to give the following parametric generalization of Theorem 1.2 for $s>1$.

Theorem 2.5. *Let n and $s>1$ be odd integers with $n\geqslant (s+1)/2$. Then, modulo $\Phi_n(q^2)^2(1-aq^{2n})(a-q^{2n})$,*

$$\sum_{k=0}^{M}[4k+1]_{q^2}[4k+1]^{s-1}\frac{(aq^2,q^2/a,q^2,q^2;q^4)_k}{(aq^4,q^4/a,q^4,q^4;q^4)_k}q^{(2-2s)k}$$
$$\equiv q^{1-n}[n]_{q^2}\sum_{j_1,\ldots,j_{m-1}=0}^{1}(-1)^{j_1+\cdots+j_{m-1}}q^{-4(j_{m-2}+\cdots+(m-2)j_1)}$$
$$\times\frac{(q^5;q^4)_{j_1}^2\cdots(q^5;q^4)_{j_1+\cdots+j_{m-2}}^2(aq^2,q^2/a,q^2;q^4)_{j_1+\cdots+j_{m-1}}}{(q;q^4)_{j_1}^2\cdots(q;q^4)_{j_1+\cdots+j_{m-1}}^2(q^4;q^4)_{j_1+\cdots+j_{m-1}}}$$
$$\times\left\{1+\frac{(1-aq^{2n})(a-q^{2n})}{(1-a)^2}\left(1-\frac{n(1-a)a^{(n-1)/2}}{1-a^n}\right)\right\},\tag{2.7}$$

where $M=(n-1)/2$ or $n-1$, and $m=(s+1)/2$.

Proof. It is clear that the polynomials $\Phi_n(q^2)(1-aq^{2n})(a-q^{2n})$ and $b-q^{2n}$ are relatively prime. With the help of the Chinese reminder theorem for coprime polynomials, we may obtain the remainder of the left-hand side of (2.7) modulo $\Phi_n(q^2)(1-aq^{2n})(a-q^{2n})(b-q^{2n})$ from the previous congruences. More precisely, since

$$\frac{(b-q^{2n})(ab-1-a^2+aq^{2n})}{(a-b)(1-ab)}\equiv 1 \pmod{(1-aq^{2n})(a-q^{2n})},\tag{2.8}$$

$$\frac{(1-aq^{2n})(a-q^{2n})}{(a-b)(1-ab)}\equiv 1 \pmod{b-q^{2n}},\tag{2.9}$$

from (2.1) and (2.6) we deduce that

$$\sum_{k=0}^{(n-1)/2}[4k+1]_{q^2}[4k+1]^{s-1}\frac{(aq^2,q^2/a,q^2/b,q^2;q^4)_k}{(aq^4,q^4/a,bq^4,q^4;q^4)_k}b^kq^{(2-2s)k}$$
$$\equiv b^{(n-1)/2}q^{1-n}[n]_{q^2}\frac{(q^4/b;q^4)_{(n-1)/2}}{(bq^4;q^4)_{(n-1)/2}}B_s(a,b,q)\frac{(b-q^{2n})(ab-1-a^2+aq^{2n})}{(a-b)(1-ab)}$$
$$+\frac{[n]_{q^2}(q^2;q^4)_{(n-1)/2}^2}{(aq^4;q^4)_{(n-1)/2}(q^4/a;q^4)_{(n-1)/2}}B_s(a,b,q)\frac{(1-aq^{2n})(a-q^{2n})}{(a-b)(1-ab)}$$
$$\pmod{\Phi_n(q^2)(1-aq^{2n})(a-q^{2n})(b-q^{2n})},\tag{2.10}$$

where

$$B_s(a,b,q)=q^{-s-1}\sum_{j_1,\ldots,j_{m-1}=0}^{1}(-1)^{j_1+\cdots+j_{m-1}}q^{-4(j_{m-2}+\cdots+(m-2)j_1)}$$
$$\times\frac{(q^5;q^4)_{j_1}^2\cdots(q^5;q^4)_{j_1+\cdots+j_{m-2}}^2(aq^2,q^2/a,q^2/b;q^4)_{j_1+\cdots+j_{m-1}}}{(q;q^4)_{j_1}^2\cdots(q;q^4)_{j_1+\cdots+j_{m-1}}^2(q^4/b;q^4)_{j_1+\cdots+j_{m-1}}}.\tag{2.11}$$

In view of Lemma 2.2, we have

$$\frac{(q^2;q^4)_{(n-1)/2}^2}{(aq^4,q^4/a;q^4)_{(n-1)/2}} \equiv \frac{n(1-a)a^{(n-1)/2}}{(1-a^n)q^{n-1}} \pmod{\Phi_n(q^2)}. \tag{2.12}$$

It is easy to see that $1-q^{2n}$ contains the factor $\Phi_n(q^2)$. Meanwhile, when $b=1$, the factor $(bq^4;q^4)_{(n-1)/2}$ in the denominator of the left-hand side of (2.10) is coprime with $\Phi_n(q^2)$, so is the factor $(q^4/b;q^4)_{m-1}$ in the denominator of the right-hand side of (2.10) since $n \geqslant m$. Hence, putting $b=1$ in (2.10), and using the congruence (2.12) and the following relation

$$(1-q^{2n})(1+a^2-a-aq^{2n}) = (1-a)^2 + (1-aq^{2n})(a-q^{2n}), \tag{2.13}$$

we conclude that (2.7) is true for $M=(n-1)/2$. Finally, since the k-th summand on the left-hand side of (2.7) is congruent to 0 modulo $\Phi_n(q^2)^2(1-aq^{2n})(a-q^{2n})$ for $(n-1)/2 < k \leqslant n-1$, we see that (2.7) is also true for $M=n-1$. □

Proof of Theorem 1.2. Let $a \to 1$ in (2.7). By l'Hôpital's rule, there holds

$$\lim_{a\to 1} \frac{(1-aq^{2n})(a-q^{2n})}{(1-a)^2} \frac{(1-a^n-n(1-a)a^{(n-1)/2})}{(1-a^n)} = \frac{(n^2-1)(1-q^2)^2}{24}[n]_{q^2}^2,$$

and so the congruence (1.5) holds modulo $\Phi_n(q^2)^4$. Moreover, the author [12] has proved (1.5) also holds modulo $[n]_{q^2}$. Since the least common multiple of $\Phi_n(q^2)^4$ and $[n]_{q^2}$ is $[n]_{q^2}\Phi_n(q^2)^3$, we complete the proof of the theorem. □

3. Proof of Theorem 1.4

We need to establish the following q-congruence, of which the $b=1$ case reduces to [12, (4.5)].

Theorem 3.1. *Let $n>1$ and $s>1$ be odd integers. Let a and b be indeterminates. Then, modulo $\Phi_n(q^2)(1-aq^{2n})(a-q^{2n})$,*

$$\begin{aligned}
&\sum_{k=0}^{(n+1)/2} [4k-1]_{q^2}[4k-1]^{s-1} \frac{(aq^{-2},q^{-2}/a,q^{-2}/b,q^{-2};q^4)_k}{(aq^4,q^4/a,bq^4,q^4;q^4)_k} b^k q^{(10-2s)k} \\
&\quad \equiv -b^{(n-1)/2}q^{n-s-2}[n]_{q^2} \frac{(1/b;q^4)_{(n-1)/2}}{(bq^8;q^4)_{(n-1)/2}} \sum_{j_1,\dots,j_{m-1}=0}^{1} (-1)^{j_1+\cdots+j_{m-1}} q^{-4(j_{m-2}+\cdots+(m-2)j_1)} \\
&\qquad \times \frac{(q^3;q^4)_{j_1}^2 \cdots (q^3;q^4)_{j_1+\cdots+j_{m-2}}^2 (aq^{-2},q^{-2}/a,q^{-2}/b;q^4)_{j_1+\cdots+j_{m-1}}}{(q^{-1};q^4)_{j_1}^2 \cdots (q^{-1};q^4)_{j_1+\cdots+j_{m-1}}^2 (q^{-4}/b;q^4)_{j_1+\cdots+j_{m-1}}},
\end{aligned} \tag{3.1}$$

where $m=(s+1)/2$

Proof. For $a = q^{-2n}$ or $a = q^{2n}$, the left-hand side of (3.1) may be written as

$$\sum_{k=0}^{(n+1)/2} [4k-1]_{q^2}[4k-1]^{s-1} \frac{(q^{-2-2n}, q^{-2+2n}, q^{-2}/b, q^2; q^4)_k}{(q^{4-2n}, q^{4+2n}, bq^4, q^4; q^4)_k} b^k q^{(10-2s)k}$$

$$= -q^{-s-1} \sum_{k=0}^{(n+1)/2} \frac{(q^{-2}, q^3, -q^3, \overbrace{q^3, \ldots, q^3}^{(s-1)\text{'s } q^3}, q^{-2}/b, q^{-2+2n}, q^{-2-2n}; q^4)_k}{(q^4, q^{-1}, -q^{-1}, q^{-1}, \ldots, q^{-1}, bq^4, q^{4-2n}, q^{4+2n}; q^4)_k} b^k q^{(10-2s)k}. \quad (3.2)$$

By Andrews' transformation (2.2) with the parameter substitutions $m = (s+1)/2$, $q \mapsto q^4$, $a = q^{-2}$, $b_1 = c_1 = \cdots = c_{m-1} = d_{m-1} = q^3$, $b_m = q^{-2}/b$, $c_m = q^{-2+2n}$ and $N = (n+1)/2$. the right-hand side of (3.2) is equal to

$$\begin{aligned}
&- q^{-s-1} \frac{(q^2, bq^{6-2n}; q^4)_{(n+1)/2}}{(bq^4, q^{4-2n}; q^4)_{(n+1)/2}} \\
&\quad \times \sum_{j_1,\ldots,j_{m-1} \geqslant 0} \frac{(q^{-4}; q^4)_{j_1} \cdots (q^{-4}; q^4)_{j_{m-1}}}{(q^4; q^4)_{j_1} \cdots (q^4; q^4)_{j_{m-1}}} q^{4(j_1+\cdots+j_{m-1}) - 4(j_{m-2}+\cdots+(m-2)j_1)} \\
&\qquad \times \frac{(q^3; q^4)_{j_1}^2 \cdots (q^3; q^4)_{j_1+\cdots+j_{m-2}}^2 (q^{-2}/b, q^{-2+2n}, q^{-2-2n}; q^4)_{j_1+\cdots+j_{m-1}}}{(q^{-1}; q^4)_{j_1}^2 \cdots (q^{-1}; q^4)_{j_1+\cdots+j_{m-1}}^2 (q^{-4}/b; q^4)_{j_1+\cdots+j_{m-1}}}.
\end{aligned} \quad (3.3)$$

It is easy to see that

$$\frac{(q^2, bq^{6-2n}; q^4)_{(n+1)/2}}{(bq^4, q^{4-2n}; q^4)_{(n+1)/2}} = b^{(n-1)/2} q^{n-1} [n]_{q^2} \frac{(1/b; q^4)_{(n-1)/2}}{(bq^8; q^4)_{(n-1)/2}}.$$

Applying (2.5), we see that the expression (3.3) is just the $a = q^{-2n}$ or $a = q^{2n}$ case of the right-hand side of (3.1). This proves that (3.1) holds modulo $1 - aq^{2n}$ and $a - q^{2n}$.

Furthermore, in light of Lemma 2.1, we have

$$\begin{aligned}
\frac{(aq^{-1}; q^2)_{(n+1)/2-k}}{(q^2/a; q^2)_{(n+1)/2-k}} &= \frac{(1 - aq^{-1})(aq; q^2)_{(n-1)/2-k}}{(1 - q^{n+1-2k}/a)(q^2/a; q^2)_{(n-1)/2-k}} \\
&\equiv (-a)^{(n-1)/2-2k} \frac{(1 - aq^{-1})(aq; q^2)_k}{(1 - q^{1-2k}/a)(q^2/a; q^2)_k} q^{(n-1)^2/4+k} \\
&= (-a)^{(n+1)/2-2k} \frac{(aq^{-1}; q^2)_k}{(q^2/a; q^2)_k} q^{(n-1)^2/4+3k-1} \pmod{\Phi_n(q)}
\end{aligned}$$

for $0 \leqslant k \leqslant (n+1)/2$. Using the above congruence with $q \mapsto q^2$, we can easily check that the sum of the k-th and $((n+1)/2 - k)$-th terms on the left-hand side of (3.1) is congruent to 0 modulo $\Phi_n(q^2)$. Hence the congruence (3.1) holds modulo $\Phi_n(q^2)$. $\square$

As before, we also need the following congruence modulo $b - q^{2n}$.

Theorem 3.2. *Let $n>1$ and $s>1$ be odd integers. Let a and b be indeterminates. Then, modulo $b-q^{2n}$,*

$$\sum_{k=0}^{(n+1)/2}[4k-1]_{q^2}[4k-1]^{s-1}\frac{(aq^{-2},q^{-2}/a,q^{-2}/b,q^{-2};q^4)_k}{(aq^4,q^4/a,bq^4,q^4;q^4)_k}b^kq^{(10-2s)k}$$
$$\equiv -\frac{[n]_{q^2}(q^2;q^4)_{(n-1)/2}^2(1-b)(1-bq^4)}{q^{s+1}(aq^4,q^4/a;q^4)_{(n+1)/2}}\sum_{j_1,\dots,j_{m-1}=0}^{1}(-1)^{j_1+\cdots+j_{m-1}}q^{-4(j_{m-2}+\cdots+(m-2)j_1)}$$
$$\times\frac{(q^3;q^4)_{j_1}^2\cdots(q^3;q^4)_{j_1+\cdots+j_{m-2}}^2(aq^{-2},q^{-2}/a,q^{-2}/b;q^4)_{j_1+\cdots+j_{m-1}}}{(q^{-1};q^4)_{j_1}^2\cdots(q^{-1};q^4)_{j_1+\cdots+j_{m-1}}^2(q^{-4}/b;q^4)_{j_1+\cdots+j_{m-1}}}, \tag{3.4}$$

where $m=(s+1)/2$

Proof. For $b=q^{2n}$, by (2.2) and (2.5), the left-hand side of (3.4) is equal to

$$-q^{-s-1}\sum_{k=0}^{(n+1)/2}\frac{(q^{-2},q^3,-q^3,\overbrace{q^3,\dots,q^3}^{(s-1)\text{'s } q^3},aq^{-2},q^{-2}/a,q^{-2-2n};q^4)_k}{(q^4,q^{-1},-q^{-1},q^{-1},\dots,q^{-1},q^4/a,aq^4,q^{4+2n};q^4)_k}b^kq^{(10-2s)k}$$
$$=-q^{-s-1}\frac{(q^2,q^6;q^4)_{(n+1)/2}}{(aq^4,q^4/a;q^4)_{(n+1)/2}}\sum_{j_1,\dots,j_{m-1}=0}^{1}(-1)^{j_1+\cdots+j_{m-1}}q^{-4(j_{m-2}+\cdots+(m-2)j_1)}$$
$$\times\frac{(q^3;q^4)_{j_1}^2\cdots(q^3;q^4)_{j_1+\cdots+j_{m-2}}^2(aq^{-2},q^{-2}/a,q^{-2-2n};q^4)_{j_1+\cdots+j_{m-1}}}{(q^{-1};q^4)_{j_1}^2\cdots(q^{-1};q^4)_{j_1+\cdots+j_{m-1}}^2(q^{-4-2n};q^4)_{j_1+\cdots+j_{m-1}}},$$

which is just the right-hand side of (3.4). Namely, the congruence (3.4) holds. □

With the help of Theorems 3.1 and 3.2, we can prove the following parametric generalization of Theorem 1.4.

Theorem 3.3. *Let $n>1$ and $s>1$ be odd integers with $n\geqslant(s-1)/2$. Then, modulo $\Phi_n(q^2)^2(1-aq^{2n})(a-q^{2n})$,*

$$\sum_{k=0}^{M}[4k-1]_{q^2}[4k-1]^{s-1}\frac{(aq^{-2},q^{-2}/a,q^{-2},q^{-2};q^4)_k}{(aq^4,q^4/a,q^4,q^4;q^4)_k}q^{(10-2s)k}$$
$$\equiv -q^{4-n-s}[n]_{q^2}X_s(a,q)\left\{\frac{1}{(1-q^{2-2n})(1-q^{2+2n})}\right.$$
$$\left.+\frac{(1-aq^{2n})(a-q^{2n})}{(1-a)^2(1-q^2)^2}\left(1-\frac{n(1-a)a^{(n-1)/2}(1-q^2)^2}{(1-a^n)(1-aq^2)(1-q^2/a)}\right)\right\}, \tag{3.5}$$

where $M=(n+1)/2$ *or* $n-1$, *and*

$$X_s(a,q)=\sum_{j_1,\dots,j_{m-1}=0}^{1}(-1)^{j_1+\cdots+j_{m-1}}q^{-4(j_{m-2}+\cdots+(m-2)j_1)}$$
$$\times\frac{(q^3;q^4)_{j_1}^2\cdots(q^3;q^4)_{j_1+\cdots+j_{m-2}}^2(aq^{-2},q^{-2}/a,q^{-2};q^4)_{j_1+\cdots+j_{m-1}}}{(q^{-1};q^4)_{j_1}^2\cdots(q^{-1};q^4)_{j_1+\cdots+j_{m-1}}^2(q^4;q^4)_{j_1+\cdots+j_{m-1}-2}}. \tag{3.6}$$

with $m=(s+1)/2$.

Proof. Applying (2.8) and (2.9), from (3.1) and (3.4) we immediately deduce that

$$\sum_{k=0}^{(n+1)/2}[4k-1]_{q^2}[4k-1]^{s-1}\frac{(aq^{-2},q^{-2}/a,q^{-2}/b,q^{-2};q^4)_k}{(aq^4,q^4/a,bq^4,q^4;q^4)_k}b^kq^{(10-2s)k}$$
$$\equiv b^{(n-1)/2}q^{n-1}[n]_{q^2}\frac{(1/b;q^4)_{(n-1)/2}}{(bq^8;q^4)_{(n-1)/2}}D_s(a,b,q)\frac{(b-q^{2n})(ab-1-a^2+aq^{2n})}{(a-b)(1-ab)}$$
$$+\frac{[n]_{q^2}(q^2;q^4)_{(n-1)/2}^2(1-b)(1-bq^4)}{(aq^4;q^4)_{(n+1)/2}(q^4/a;q^4)_{(n+1)/2}}D_s(a,b,q)\frac{(1-aq^{2n})(a-q^{2n})}{(a-b)(1-ab)}$$
$$\pmod{\Phi_n(q^2)(1-aq^{2n})(a-q^{2n})(b-q^{2n})}, \tag{3.7}$$

where

$$D_s(a,b,q)=-q^{-s-1}\sum_{j_1,\dots,j_{m-1}=0}^{1}(-1)^{j_1+\cdots+j_{m-1}}q^{-4(j_{m-2}+\cdots+(m-2)j_1)}$$
$$\times\frac{(q^3;q^4)_{j_1}^2\cdots(q^3;q^4)_{j_1+\cdots+j_{m-2}}^2(aq^{-2},q^{-2}/a,q^{-2}/b;q^4)_{j_1+\cdots+j_{m-1}}}{(q^{-1};q^4)_{j_1}^2\cdots(q^{-1};q^4)_{j_1+\cdots+j_{m-1}}^2(q^{-4}/b;q^4)_{j_1+\cdots+j_{m-1}}}.$$

It is easy to see that

$$\lim_{b\to1}\frac{(1/b;q^4)_{(n-1)/2}}{(bq^8;q^4)_{(n-1)/2}}D_s(a,b,q)=-\frac{q^{5-2n-s}}{(1-q^{2-2n})(1-q^{2+2n})}X_s(a,q),$$
$$\lim_{b\to1}\frac{(1-b)(1-bq^4)}{(1-aq^{2n+2})(1-q^{2n+2}/a)}D_s(a,b,q)=-\frac{q^{3-s}}{(1-aq^{2n+2})(1-q^{2n+2}/a)}X_s(a,q),$$

where $X_s(a,q)$ is given by (3.6). Note that the condition $n\geqslant(s-1)/2$ in the theorem guarantees the denominator of $X_s(a,q)$ is always relatively prime to $\Phi_n(q^2)$. Let $b=1$ in (3.7) and use the above limits. Then we apply the congruence (2.12) and the relation

(2.13) to obtain

$$\sum_{k=0}^{(n+1)/2}[4k-1]_{q^2}[4k-1]^{s-1}\frac{(aq^{-2},q^{-2}/a,q^{-2},q^{-2};q^4)_k}{(aq^4,q^4/a,q^4,q^4;q^4)_k}q^{(10-2s)k}$$
$$\equiv -\frac{q^{4-n-s}[n]_{q^2}X_s(a,q)}{(1-q^{2-2n})(1-q^{2+2n})}\left(1+\frac{(1-aq^{2n})(a-q^{2n})}{(1-a)^2}\right)$$
$$+\frac{q^{4-n-s}[n]_{q^2}X_s(a,q)}{(1-aq^{2n+2})(1-q^{2n+2}/a)}\frac{(1-aq^{2n})(a-q^{2n})na^{(n-1)/2}}{(1-a)(1-a^n)}$$
$$\pmod{\Phi_n(q^2)^2(1-aq^{2n})(a-q^{2n})}.$$

In view of $q^{2n}\equiv 1 \pmod{\Phi_n(q^2)}$, the above congruence is equivalent to (3.5) for $M=(n+1)/2$. Finally, since the k-th summand on the left-hand side of (3.5) is congruent to 0 modulo $\Phi_n(q^2)^2(1-aq^{2n})(a-q^{2n})$ for $(n+1)/2<k\leqslant n-1$, we conclude that (3.5) also holds for $M=n-1$. □

Proof of Theorem 1.4. Let $a\to 1$ in (3.5). By l'Hôpital's rule, there holds

$$\lim_{a\to 1}\frac{(1-aq^{2n})(a-q^{2n})}{(1-a)^2}\left(1-\frac{n(1-a)a^{(n-1)/2}(1-q^2)^2}{(1-a^n)(1-aq^2)(1-q^2/a)}\right)$$
$$=\left(\frac{(n^2-1)(1-q^2)^2}{24}-q^2\right)[n]_{q^2}^2.$$

Employing the following easily checked congruence

$$\frac{1}{(1-q^{2-2n})(1-q^{2+2n})}-\frac{q^2[n]_{q^2}^2}{(1-q^2)^2}\equiv\frac{1}{(1-q^2)^2}\pmod{\Phi_n(q^2)^3},$$

we conclude that (1.8) holds modulo $\Phi_n(q^2)^4$. In addition, the author [12] has proved (1.8) also holds modulo $[n]_{q^2}$. This completes the proof. □

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