Proof of a generalization of the (B.2) supercongruence of Van Hamme through a q-microscope

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Abstract. We prove some q-congruences for certain truncated basic hypergeometric series by using Andrews' multiseries generalization of Watson's transformation and the creative microscoping method, recently devised by the author and Wadim Zudilin. As a conclusion, we completely confirm Conjecture 1.1 in [Integral Transforms Spec. Funct. 28 (2017), 888–899] which is a generalization of the (B.2) supercongruence of Van Hamme, and partially confirm Conjecture 4.3 in the same paper. We also raise several related conjectures on q-congruences.

Keywords: basic hypergeometric series; Andrews' transformation; q-congruences; supercongruences; Euler number.

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1. Introduction

In 1859, Bauer [2] proved the following hypergeometric identity:

$$\sum_{k=0}^{\infty} \frac{4k+1}{(-64)^k} \binom{2k}{k}^3 = \frac{2}{\pi}. \tag{1.1}$$

One reason why such identities are interesting is that the fastest known algorithms for computing decimal digits of π are based on this kind of identities. See, for example, the monograph [3] by Borwein and Borwein. In 1997, Van Hamme [36, (B.2)] conjectured that the formula (1.1) possesses a nice p-adic analogue:

$$\sum_{k=0}^{(p-1)/2} \frac{4k+1}{(-64)^k} \binom{2k}{k}^3 \equiv p(-1)^{(p-1)/2} \pmod{p^3}, \tag{1.2}$$

where p is an odd prime. This supercongruence was first proved by Mortenson [30] using an idea of McCarthy and Osburn [29] to evaluate of a quotient of Gamma functions. It was reproved by Zudilin [40] using the Wilf–Zeilberger (WZ) method, and by Long [28] using hypergeometric series identities and evaluations. A refinement of (1.2) modulo p^4 was given by Sun [33] using the WZ method again together with some properties of the Euler numbers. Swisher [34, (B.3)] made an interesting conjecture on a further generalization of (1.2).

Motivated by Zudilin's work [40], the author [10] considered more WZ-pairs related to some generalizations of (1.2) and proved partial results on them. He also raised the following conjecture [10, Conjecture 1.1].

Conjecture 1.1. *For any positive odd integer s, there exists an integer a_s such that, for any odd prime p and positive integer r, there hold*

$$\sum_{k=0}^{(p^r-1)/2} \frac{(4k+1)^s}{(-64)^k} \binom{2k}{k}^3 \equiv a_s p^r (-1)^{\frac{(p-1)r}{2}} \pmod{p^{r+2}}, \tag{1.3}$$

$$\sum_{k=0}^{p^r-1} \frac{(4k+1)^s}{(-64)^k} \binom{2k}{k}^3 \equiv a_s p^r (-1)^{\frac{(p-1)r}{2}} \pmod{p^{r+2}}. \tag{1.4}$$

In particular, we have $a_1 = 1$, $a_3 = -3$, $a_5 = 41$, $a_7 = -1595$, $a_9 = 124689$ and $a_{11} = -16253107$.

As mentioned in [10], there are no 'Archimedean' analogues of (1.3) and (1.4) for $s \geqslant 3$, ie.,

$$\sum_{k=0}^{\infty} \frac{(4k+1)^s}{(-64)^k} \binom{2k}{k}^3 = \infty \quad \text{for } s \geqslant 3.$$

Note that, for $r = 1$, the supercongruence (1.3) is equivalent to (1.4), since $\binom{2k}{k} \equiv 0 \pmod{p}$ for $(p+1)/2 \leqslant k \leqslant p-1$. The author [10] himself proved that (1.3) holds modulo p^2 for $(r,s) = (1,3)$ and that it also holds modulo p^3 for $(r,s) = (1,3)$ and $p \not\equiv 3 \pmod{8}$. The author [11] later proved that (1.3) is true for $s = 1$ and all positive integers r. The author and Zudilin [24] proved that so is the supercongruence (1.4). Some other partial results of (1.3) were obtained by Liu [27], who showed that (1.3) is true for $r = 1$ and $s = 3, 5, 7, 9, 11$. Jana and Kalita [26] confirmed (1.3) for $s = 3$ and $r \geqslant 1$, and almost simultaneously the author [13] succeeded in proving (1.3) and (1.4) for $s = 3$ and $r \geqslant 1$. Recently, Gu and the author [8] proved (1.3) and (1.4) for $s = 5$ and $r \geqslant 1$. Moreover, Hou, Mu, and Zeilberger [25] further proved Conjecture 1.1 for $r = 1$ and all positive odd integers s. Until now, Conjecture 1.1 is still open for $s \geqslant 7$.

It is worth mentioning that q-analogues of supercongruences have been studied by many authors in recent years (see, for example, [6, 7, 9, 11–13, 15–22, 24, 31, 32, 35, 40]). In particular, the author and Zudilin [24] devised a method, called 'creative microscoping', to prove many q-supercongruences by introducing an extra parameter and considering asymptotic behavior of q-series at roots of unity. We believe that the creative microscoping method can be utilized to prove more supercongruences and q-supercongruences. In fact, the author [13] proved (1.3) and (1.4) for $s = 3$ by establishing their q-analogues in the spirit of [24].

We shall consider congruences in $\mathbb{Z}(a,q)$ (or in $\mathbb{Z}(q)$ when $a = 1$), a bivariate rational functional field. The congruence $A_1(a,q)/B_1(a,q) \equiv 0 \pmod{C(a,q)}$ for $A_1(a,q)$,

$B_1(a,q)$, $C(a,q)\in\mathbb{Z}[a,q]$ is meant that $A_1(a,q)$ is divisible by $C(a,q)$ in $\mathbb{Z}[a,q]$, while $B_1(a,q)$ is relatively prime to $C(a,q)$ in $\mathbb{Z}[a,q]$. More generally, $A(a,q)\equiv B(a,q) \pmod{C(a,q)}$ for rational functions $A(a,q)$, $B(a,q)\in\mathbb{Z}(a,q)$ is understood as $A(a,q)-B(a,q)\equiv 0 \pmod{C(a,q)}$.

The paper is a continuation of [13] and we shall confirm Conjecture 1.1 completely by establishing the following q-analogues of (1.3) and (1.4).

Theorem 1.2. *Let n and s be positive odd integers with $n>1$. Then, modulo $[n]_{q^2}\Phi_n(q^2)^2$,*

$$\sum_{k=0}^{(n-1)/2}(-1)^k[4k+1]_{q^2}[4k+1]^{s-1}\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3}q^{2k(k-s+1)}\equiv[n]_{q^2}(-1)^{(n-1)/2}q^{(n-1)^2/2}A_s(q), \quad (1.5)$$

$$\sum_{k=0}^{n-1}(-1)^k[4k+1]_{q^2}[4k+1]^{s-1}\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3}q^{2k(k-s+1)}\equiv[n]_{q^2}(-1)^{(n-1)/2}q^{(n-1)^2/2}A_s(q), \quad (1.6)$$

where $A_s(q)$ is a Laurent polynomial in q given by

$$A_s(q)=\sum_{l_1,\dots,l_{m-1}=0}^{1}(-1)^{l_1+\cdots+l_{m-1}}q^{-2(l_1+\cdots+l_{m-1})-4(l_{m-2}+\cdots+(m-2)l_1)}$$
$$\times\frac{(q^5;q^4)_{l_1}^2\cdots(q^5;q^4)_{l_1+\cdots+l_{m-2}}^2(q^2;q^4)_{l_1+\cdots+l_{m-1}}^2}{(q;q^4)_{l_1}^2\cdots(q;q^4)_{l_1+\cdots+l_{m-1}}^2}\quad \text{with } m=\frac{s+1}{2}. \quad (1.7)$$

Note that a_s may be defined by $A_s(1)$. We now already need to familiarize ourselves with the standard q-notation. The *q-shifted factorial* is defined as $(a;q)_0=1$ and $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n\geqslant 1$. For simplicity, we also compactly write $(a_1,a_2,\dots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n$. The *$q$-integer* is given by $[n]=[n]_q=1+q+\cdots+q^{n-1}$. Moreover, the n-th *cyclotomic polynomial*, denoted by $\Phi_n(q)$, is defined by

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(n,k)=1}}(q-\zeta^k),$$

where ζ is an n-th primitive root of unity. It is easy to see that $\Phi_n(q^2)=\Phi_n(q)\Phi_n(-q)$ for odd n.

Note that the indices $l_1,\dots,l_{m-1}$ in (1.7) take values 0 and 1, and so

$$\frac{(q^2;q^4)_{l_1+\cdots+l_{m-1}}}{(1-q)^{l_2+\cdots+l_{m-1}}(q;q^4)_{l_1}},\quad \frac{(1-q)^{l_2}(q^5;q^4)_{l_1}}{(q;q^4)_{l_1+l_2}},\dots,\frac{(1-q)^{l_{m-1}}(q^5;q^4)_{l_1+\cdots+l_{m-2}}}{(q;q^4)_{l_1+\cdots+l_{m-1}}} \quad (1.8)$$

are all polynomials in q. This means that the expression $A_s(q)$ given by (1.7) is indeed a Laurent polynomial in q. For the reader's convenience, we give the first values of $A_s(q)$

in Theorem 1.2 as follows: $A_1(q)=1$, $A_3(q)=-(2q+1)/q^2$, and

$$
\begin{aligned}
A_5(q)&=\frac{5q^6+8q^5+8q^4+8q^3+7q^2+4q+1}{q^8},\\
A_7(q)&=-\frac{(2q^5+2q^4+2q^3+2q^2+2q+1)}{q^{18}}\\
&\quad\times(7q^{10}+14q^9+18q^8+22q^7+23q^6+20q^5+16q^4+12q^3+8q^2+4q+1).
\end{aligned}
$$

It is clear that, for $k\geqslant 0$ and any prime power p^r, we have

$$
\lim_{q\to 1}\frac{(q^2;q^4)_k}{(q^4;q^4)_k}=\frac{1}{4^k}\binom{2k}{k}\quad\text{and}\quad \Phi_{p^r}(1)=p.
$$

Therefore, letting $n=p^r$ and $q\to 1$ in (1.5) and (1.6), and noticing that $(-1)^{(p^r-1)/2}=(-1)^{(p-1)r/2}$ for odd p, we are led to (1.3) and (1.4) immediately.

The second objective of this paper is to partially confirm the following (corrected version of) conjecture of the author (see [10, Conjecture 4.3]).

Conjecture 1.3. *For any positive odd integer s, there exists an integer b_s such that, for any odd prime $p\geqslant (s+1)/2$ and positive integer r, there hold*

$$
\sum_{k=0}^{(p^r-1)/2}\frac{(4k+1)^s}{256^k}\binom{2k}{k}^4\equiv(-1)^{(s-1)/2}b_s p^r \pmod{p^{r+3}}, \tag{1.9}
$$

$$
\sum_{k=0}^{p^r-1}\frac{(4k+1)^s}{256^k}\binom{2k}{k}^4\equiv(-1)^{(s-1)/2}b_s p^r \pmod{p^{r+3}}, \tag{1.10}
$$

In particular, we have $b_1=1$, $b_3=1$, $b_5=3$, $b_7=23$, $b_9=371$ and $b_{11}=10515$.

For $r=s=1$, the supercongruence (1.9) is equivalent to (1.10) and is a refinement of the (C.2) supercongruence of Van Hamme [36]. This case was first proved by Long [28, Theorem 1.1], who also observed the supercongruence (1.9) for $s=1$ and all positive integers r. Some other special cases of (1.9) were proved by Wang [37], Liu [27], the author and Wang [23], the author [13], and Hou, Mu, and Zeilberger [25]. Here we shall prove that (1.9) and (1.10) are true modulo p^{r+2} for any odd prime p and arbitrary r by establishing the following q-congruences.

Theorem 1.4. *Let n and s be positive odd integers with $n>1$. Then, modulo $[n]_{q^2}\Phi_n(q^2)^2$,*

$$
\sum_{k=0}^{(n-1)/2}[4k+1]_{q^2}[4k+1]^{s-1}\frac{(q^2;q^4)_k^4}{(q^4;q^4)_k^4}q^{(2-2s)k}\equiv[n]_{q^2}q^{1-n}B_s(q)\pmod{[n]_{q^2}\Phi_n(q^2)^2}, \tag{1.11}
$$

$$
\sum_{k=0}^{n-1}[4k+1]_{q^2}[4k+1]^{s-1}\frac{(q^2;q^4)_k^4}{(q^4;q^4)_k^4}q^{(2-2s)k}\equiv[n]_{q^2}q^{1-n}B_s(q)\pmod{[n]_{q^2}\Phi_n(q^2)^2}, \tag{1.12}
$$

where $B_s(q)$ is a rational function of q given by

$$B_s(q) = \sum_{l_1,\dots,l_{m-1}=0}^{1} (-1)^{l_1+\cdots+l_{m-1}} q^{-4(l_{m-2}+\cdots+(m-2)l_1)}$$
$$\times \frac{(q^5;q^4)_{l_1}^2 \cdots (q^5;q^4)_{l_1+\cdots+l_{m-2}}^2 (q^2;q^4)_{l_1+\cdots+l_{m-1}}^3}{(q;q^4)_{l_1}^2 \cdots (q;q^4)_{l_1+\cdots+l_{m-1}}^2 (q^4;q^4)_{l_1+\cdots+l_{m-1}}} \quad \text{with } m = \frac{s+1}{2}. \tag{1.13}$$

Note that b_s can be defined as $(-1)^{(s-1)/2} B_s(1)$. The first values of $B_s(q)$ in Theorem 1.4 are listed as follows: $B_1(q) = 1$, $B_3(q) = -2q/(q^2+1)$, and

$$B_5(q) = \frac{q^2(5q^4+4q^3+6q^2+4q+5)}{(q^4+1)(q^2+1)^2},$$
$$B_7(q) = -\frac{2q^3(7q^8+14q^7+23q^6+30q^5+36q^4+30q^3+23q^2+14q+7)}{(q^6+1)(q^4+1)(q^2+1)^2}.$$

It is clear that, when $n = p^r$ and $q \to 1$, the congruences (1.11) and (1.12) reduce to (1.9) and (1.10) modulo p^{r+2} (for any odd prime p), respectively. To see that $B_s(1)$ is an integer, first notice that

$$\frac{(q^5;q^4)_{l_1}^2 \cdots (q^5;q^4)_{l_1+\cdots+l_{m-2}}^2 (q^2;q^4)_{l_1+\cdots+l_{m-1}}^3}{(q;q^4)_{l_1}^2 \cdots (q;q^4)_{l_1+\cdots+l_{m-1}}^2 (q^4;q^4)_{l_1+\cdots+l_{m-1}}}$$
$$= \frac{(q^5;q^4)_{l_1}^2 \cdots (q^5;q^4)_{l_1+\cdots+l_{m-2}}^2 (q^2;q^4)_{l_1+\cdots+l_{m-1}}^2}{(q;q^4)_{l_1}^2 \cdots (q;q^4)_{l_1+\cdots+l_{m-1}}^2} \cdot \frac{(q^2;q^4)_{l_1+\cdots+l_{m-1}}}{(q^4;q^4)_{l_1+\cdots+l_{m-1}}}.$$

Moreover, for $l_1,\dots,l_{m-1} \in \{0,1\}$, the first fraction is a square of the product of the $m-1$ polynomials in (1.8), and the first one in (1.8) is clearly divisible by $(1+q)^{l_1+\cdots+l_{m-1}}$. Thus, the limit of the first fraction as $q \to 1$ is an integer divisible by $2^{2(l_1+\cdots+l_{m-1})}$. Finally, observe that

$$\lim_{q\to 1} \frac{(q^2;q^4)_{l_1+\cdots+l_{m-1}}}{(q^4;q^4)_{l_1+\cdots+l_{m-1}}} = 2^{-2(l_1+\cdots+l_{m-1})} \binom{2(l_1+\cdots+l_{m-1})}{l_1+\cdots+l_{m-1}}.$$

The rest of the paper is organized as follows. We shall prove Theorems 1.2 and 1.4 in Sections 2 and 3, respectively. To accomplish this we shall make use of not only the aforementioned creative microscoping method [24] but also Andrews' multiseries generalization of the Watson transformation [1, Theorem 4] (Andrews' was already utilized by Zudilin [39] to solve a problem of Schmidt, and was used in [14] to prove some q-analogues of Calkin's congruence [4]. It was also applied by the author and Schlosser [18, 20, 22] to prove some q-congruences for truncated basic hypergeometric series). Meanwhile, a simple property of fractions of q-shifted factorials (Lemma 2.1) plays an important role in our proof. We shall give some similar q-congruences in Section 4. Finally, in Section 5, we propose three related conjectures including a refinement of Theorem 1.4, which is also a complete q-analogue of Conjecture 1.3.

2. Proof of Theorem 1.2

We will need the following easily proved result (see [21, Lemma 3.1]).

Lemma 2.1. *Let n be a positive odd integer and let a be an indeterminate. Then, for $0 \leqslant k \leqslant (n-1)/2$, we have*

$$\frac{(aq;q^2)_{(n-1)/2-k}}{(q^2/a;q^2)_{(n-1)/2-k}} \equiv (-a)^{(n-1)/2-2k} \frac{(aq;q^2)_k}{(q^2/a;q^2)_k} q^{(n-1)^2/4+k} \pmod{\Phi_n(q)}.$$

We first establish the following parametric generalization of Theorem 1.2.

Theorem 2.2. *Let n and s be positive odd integers with $n>1$ and let a be an indeterminate. Then, modulo $\Phi_n(q^2)(1-aq^{2n})(a-q^{2n})$,*

$$\sum_{k=0}^{(n-1)/2} (-1)^k [4k+1]_{q^2} [4k+1]^{s-1} \frac{(q^2, aq^2, q^2/a; q^4)_k}{(q^4, aq^4, q^4/a; q^4)_k} q^{2k(k-s+1)}$$

$$\equiv [n]_{q^2} (-1)^{(n-1)/2} q^{(n-1)^2/2} \sum_{l_1,\dots,l_{m-1}=0}^{1} (-1)^{l_1+\cdots+l_{m-1}} q^{-2(l_1+\cdots+l_{m-1})-4(l_{m-2}+\cdots+(m-2)l_1)}$$

$$\times \frac{(q^5;q^4)_{l_1}^2 \cdots (q^5;q^4)_{l_1+\cdots+l_{m-2}}^2 (aq^2, q^2/a; q^4)_{l_1+\cdots+l_{m-1}}}{(q;q^4)_{l_1}^2 \cdots (q;q^4)_{l_1+\cdots+l_{m-1}}^2}, \tag{2.1}$$

where $m=(s+1)/2$.

Proof. The $s=1$ case is just [24, Theorem 4.1] with $q \mapsto q^2$. Now suppose that $s \geqslant 3$. We need to use a complicated transformation formula due to Andrews [1, Theorem 4]:

$$\sum_{k\geqslant 0} \frac{(a, q\sqrt{a}, -q\sqrt{a}, b_1, c_1, \dots, b_m, c_m, q^{-N}; q)_k}{(q, \sqrt{a}, -\sqrt{a}, aq/b_1, aq/c_1, \dots, aq/b_m, aq/c_m, aq^{N+1}; q)_k} \left(\frac{a^m q^{m+N}}{b_1 c_1 \cdots b_m c_m} \right)^k$$

$$= \frac{(aq, aq/b_m c_m; q)_N}{(aq/b_m, aq/c_m; q)_N} \sum_{l_1,\dots,l_{m-1}\geqslant 0} \frac{(aq/b_1c_1;q)_{l_1} \cdots (aq/b_{m-1}c_{m-1};q)_{l_{m-1}}}{(q;q)_{l_1} \cdots (q;q)_{l_{m-1}}}$$

$$\times \frac{(b_2, c_2; q)_{l_1} \dots (b_m, c_m; q)_{l_1+\cdots+l_{m-1}}}{(aq/b_1, aq/c_1; q)_{l_1} \dots (aq/b_{m-1}, aq/c_{m-1}; q)_{l_1+\cdots+l_{m-1}}}$$

$$\times \frac{(q^{-N};q)_{l_1+\cdots+l_{m-1}}}{(b_m c_m q^{-N}/a; q)_{l_1+\cdots+l_{m-1}}} \frac{(aq)^{l_{m-2}+\cdots+(m-2)l_1} q^{l_1+\cdots+l_{m-1}}}{(b_2c_2)^{l_1} \cdots (b_{m-1}c_{m-1})^{l_1+\cdots+l_{m-2}}}, \tag{2.2}$$

which is a multiseries generalization of Watson's ${}_8\phi_7$ transformation formula (see [5, Appendix (III.18)]):

$${}_8\phi_7\left[\begin{matrix} a, & qa^{\frac12}, & -qa^{\frac12}, & b, & c, & d, & e, & q^{-n} \\ & a^{\frac12}, & -a^{\frac12}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1} \end{matrix} ; q, \frac{a^2q^{n+2}}{bcde}\right]$$

$$= \frac{(aq, aq/de; q)_n}{(aq/d, aq/e; q)_n} {}_4\phi_3\left[\begin{matrix} aq/bc, \ d, \ e, \ q^{-n} \\ aq/b, \ aq/c, \ deq^{-n}/a \end{matrix} ; q, q\right].$$

For $a=q^{-2n}$ or $a=q^{2n}$, the left-hand side of (2.1) is equal to

$$\sum_{k=0}^{(n-1)/2}(-1)^k[4k+1]_{q^2}[4k+1]^{s-1}\frac{(q^2,q^{2-2n},q^{2+2n};q^4)_k}{(q^4,q^{4-2n},q^{4+2n};q^4)_k}q^{2k(k-s+1)}$$

$$=\lim_{N\to\infty}\sum_{k=0}^{(n-1)/2}\frac{(q^2,q^5,-q^5,\overbrace{q^5,\ldots,q^5}^{(s-1)\text{'s } q^5},q^{2-2n},q^{2+2n},q^{-4N};q^4)_k}{(q^4,q,-q,q,\ldots,q,q^{4+2n},q^{4-2n},q^{4N+6};q^4)_k}q^{(4N-2s+4)k},$$

which, by (2.2) with the parameter substitutions $m=(s+1)/2$, $q\mapsto q^4$, $a=q^2$, $b_1=c_1=\cdots=b_{m-1}=c_{m-1}=q^5$, $b_m=q^{2-2n}$ and $c_m=q^{2+2n}$, can be written as

$$\frac{(q^6,q^2;q^4)_\infty}{(q^{4+2n},q^{4-2n};q^4)_\infty}\sum_{l_1,\ldots,l_{m-1}\geqslant 0}\frac{(q^{-4};q^4)_{l_1}\cdots(q^{-4};q^4)_{l_{m-1}}}{(q^4;q^4)_{l_1}\cdots(q^4;q^4)_{l_{m-1}}}$$

$$\times\frac{(q^5;q^4)_{l_1}^2\cdots(q^5;q^4)_{l_1+\cdots+l_{m-2}}^2(q^{2-2n},q^{2+2n};q^4)_{l_1+\cdots+l_{m-1}}}{(q;q^4)_{l_1}^2\cdots(q;q^4)_{l_1+\cdots+l_{m-1}}^2}q^{2(l_1+\cdots+l_{m-1})-4(l_{m-2}+\cdots+(m-2)l_1)}.$$

It is easy to see that

$$\frac{(q^6,q^2;q^4)_\infty}{(q^{4+2n},q^{4-2n};q^4)_\infty}=\frac{(q^2;q^4)_{(n+1)/2}}{(q^{4-2n};q^4)_{(n+1)/2}}=[n]_{q^2}(-1)^{(n-1)/2}q^{(n-1)^2/2},$$

and

$$\frac{(q^{-4};q^4)_k}{(q^4;q^4)_k}=\begin{cases}(-1)^kq^{-4k}, & \text{if } k=0,1,\\ 0, & \text{otherwise},\end{cases}\tag{2.3}$$

This proves that the congruence (2.1) is true modulo $1-aq^{2n}$ or $a-q^{2n}$.

Moreover, by Lemma 2.1, it is easy to verify that, for $0\leqslant k\leqslant(n-1)/2$, the k-th and $((n-1)/2-k)$-th terms on the left-hand side of (2.1) cancel each other modulo $\Phi_n(q^2)$, i.e.,

$$(-1)^k[2n-4k-1]_{q^2}[2n-4k-1]^{s-1}\frac{(q^2,aq^2,q^2/a;q^4)_{(n-1)/2-k}}{(q^4;q^4)_{(n-1)/2-k}^3}$$

$$\times q^{2((n-1)/2-k)^2-2(s-1)((n-1)/2-k)}$$

$$\equiv-(-1)^k[4k+1]_{q^2}[4k+1]^{s-1}\frac{(q^2,aq^2,q^2/a;q^4)_k}{(q^4;q^4)_k^3}q^{2k^2-2(s-1)k}\pmod{\Phi_n(q^2)}.$$

Note that the above congruence also holds if $k=(n-1)/2-k$ (in this case the $(n-1)/4$-th term itself is clearly congruent to 0 modulo $\Phi_n(q^2)$). Thus, we conclude that the left-hand side of (2.1) is congruent to 0 modulo $\Phi_n(q^2)$, and therefore the congruence (2.1) holds modulo $\Phi_n(q^2)$. Since $\Phi_n(q^2)$, $1-aq^{2n}$, and $a-q^{2n}$ are relatively prime polynomials, we complete the proof of (2.1). □

Proof of Theorem 1.2. It is easy to see that the limits of the denominators on the left-hand side of (2.1) as $a \to 1$ are relatively prime to $\Phi_n(q^2)$, since $0 \leqslant k \leqslant (n-1)/2$. Moreover, the limit of $(1-aq^{2n})(a-q^{2n})$ as $a \to 1$ contains the factor $\Phi_n(q^2)^2$. It follows that the limiting case $a \to 1$ of (2.1) leads to the following congruence

$$\begin{aligned}\sum_{k=0}^{(n-1)/2} &(-1)^k [4k+1]_{q^2}[4k+1]^{s-1}\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3} q^{2k(k-s+1)} \\ &\equiv [n]_{q^2}(-1)^{(n-1)/2} q^{(n-1)^2/2} a_s(q) \pmod{\Phi_n(q^2)^3},\end{aligned} \tag{2.4}$$

where $a_s(q)$ is the Laurent polynomial in q defined in (1.7).

The congruence (2.4) also implies that

$$\begin{aligned}\sum_{k=0}^{n-1} &(-1)^k [4k+1]_{q^2}[4k+1]^{s-1}\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3} q^{2k(k-s+1)} \\ &\equiv [n]_{q^2}(-1)^{(n-1)/2} q^{(n-1)^2/2} a_s(q) \pmod{\Phi_n(q^2)^3},\end{aligned} \tag{2.5}$$

since $(q^2;q^4)_k^3/(q^4;q^4)_k^3 \equiv 0 \pmod{\Phi_n(q^2)^3}$ for k in the range $(n-1)/2 < k \leqslant n-1$.

It remains to show that (2.4) and (2.5) are also true modulo $[n]_{q^2}$, i.e.,

$$\sum_{k=0}^{(n-1)/2} (-1)^k [4k+1]_{q^2}[4k+1]^{s-1}\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3} q^{2k(k-s+1)} \equiv 0 \pmod{[n]_{q^2}}, \tag{2.6}$$

$$\sum_{k=0}^{n-1} (-1)^k [4k+1]_{q^2}[4k+1]^{s-1}\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3} q^{2k(k-s+1)} \equiv 0 \pmod{[n]_{q^2}}. \tag{2.7}$$

Let $\zeta \neq 1$ be an n-th root of unity. Namely, ζ is a primitive root of unity of odd degree d with $d \mid n$. Denote by $c_q(k)$ the k-th term on the left-hand side of (2.6). In other words,

$$c_q(k) = (-1)^k [4k+1]_{q^2}[4k+1]^{s-1}\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3} q^{2k(k-s+1)}.$$

The congruences (2.4) and (2.5) with $n=d$ imply that

$$\sum_{k=0}^{(d-1)/2} c_\zeta(k) = \sum_{k=0}^{d-1} c_\zeta(k) = 0, \quad \text{and} \quad \sum_{k=0}^{(d-1)/2} c_{-\zeta}(k) = \sum_{k=0}^{d-1} c_{-\zeta}(k) = 0.$$

Noticing that

$$\frac{c_\zeta(\ell d+k)}{c_\zeta(\ell d)} = \lim_{q\to\zeta}\frac{c_q(\ell d+k)}{c_q(\ell d)} = c_\zeta(k),$$

we get

$$\sum_{k=0}^{(n-1)/2} c_\zeta(k) = \sum_{\ell=0}^{(n/d-3)/2} c_\zeta(\ell d) \sum_{k=0}^{d-1} c_\zeta(k) + \sum_{k=0}^{(d-1)/2} c_\zeta((n-d)/2+k) = 0,$$

$$\sum_{k=0}^{n-1} c_\zeta(k) = \sum_{\ell=0}^{n/d-1} \sum_{k=0}^{d-1} c_\zeta(\ell d+k) = \sum_{\ell=0}^{n/d-1} c_\zeta(\ell d) \sum_{k=0}^{d-1} c_\zeta(k) = 0.$$

This proves that the sums $\sum_{k=0}^{(n-1)/2} c_q(k)$ and $\sum_{k=0}^{n-1} c_q(k)$ are both congruent to 0 modulo $\Phi_d(q)$. In the same way we can show that they are also congruent to 0 modulo $\Phi_d(-q)$. Since d can be any divisor of n greater than 1, we conclude that these two sums are congruent to 0 modulo

$$\prod_{d|n,\, d>1} \Phi_d(q)\Phi_d(-q) = [n]_{q^2},$$

thus establishing (2.6) and (2.7). □

3. Proof of Theorem 1.4

We first establish the following parametric generalization of Theorem 1.4.

Theorem 3.1. *Let n and s be positive odd integers with $n > 1$ and let a be an indeterminate. Then, modulo $\Phi_n(q^2)(1-aq^{2n})(a-q^{2n})$,*

$$\sum_{k=0}^{(n-1)/2} [4k+1]_{q^2}[4k+1]^{s-1} \frac{(q^2,q^2,aq^2,q^2/a;q^4)_k}{(q^4,q^4,aq^4,q^4/a;q^4)_k} q^{(2-2s)k}$$

$$\equiv [n]_{q^2} q^{1-n} \sum_{l_1,\ldots,l_{m-1}=0}^{1} (-1)^{l_1+\cdots+l_{m-1}} q^{-4(l_{m-2}+\cdots+(m-2)l_1)}$$

$$\times \frac{(q^5;q^4)_{l_1}^2 \cdots (q^5;q^4)_{l_1+\cdots+l_{m-2}}^2 (q^2,aq^2,q^2/a;q^4)_{l_1+\cdots+l_{m-1}}}{(q;q^4)_{l_1}^2 \cdots (q;q^4)_{l_1+\cdots+l_{m-1}}^2 (q^4;q^4)_{l_1+\cdots+l_{m-1}}}. \tag{3.1}$$

Proof. The $s=1$ case is just a special case of [24, Theorem 4.2]. We now suppose that $s \geqslant 3$. For $a=q^{-2n}$ or $a=q^{2n}$, the left-hand side of (3.1) is equal to

$$\sum_{k=0}^{(n-1)/2} [4k+1]_{q^2}[4k+1]^{s-1} \frac{(q^2,q^2,q^{2-2n},q^{2+2n};q^4)_k}{(q^4,q^4,q^{4-2n},q^{4+2n};q^4)_k} q^{(2-2s)k}$$

$$= \sum_{k=0}^{(n-1)/2} \frac{(q^2,q^5,-q^5,\overbrace{q^5,\ldots,q^5}^{(s-1)\text{'s } q^5},q^2,q^{2+2n},q^{2-2n};q^4)_k}{(q^4,q,-q,q,\ldots,q,q^4,q^{4-2n},q^{4+2n};q^4)_k} q^{(2-2s)k}, \tag{3.2}$$

which, by Andrews' transformation formula (2.2) with the parameter substitutions $m = (s+1)/2$, $q \mapsto q^4$, $a = q^2$, $b_1 = c_1 = \cdots = b_{m-1} = c_{m-1} = q^5$, $b_m = q^2$, $c_m = q^{2+2n}$ and $N = (n-1)/2$, can be written as

$$\frac{(q^6, q^{2-2n}; q^4)_{(n-1)/2}}{(q^4, q^{4-2n}; q^4)_{(n-1)/2}} \sum_{l_1,\dots,l_{m-1}\geqslant 0} \frac{(q^{-4};q^4)_{l_1}\cdots(q^{-4};q^4)_{l_{m-1}}}{(q^4;q^4)_{l_1}\cdots(q^4;q^4)_{l_{m-1}}} q^{4(l_1+\cdots+l_{m-1})-4(l_{m-2}+\cdots+(m-2)l_1)}$$
$$\times \frac{(q^5;q^4)_{l_1}^2\cdots(q^5;q^4)_{l_1+\cdots+l_{m-2}}^2 (q^2, q^{2-2n}, q^{2+2n}; q^4)_{l_1+\cdots+l_{m-1}}}{(q;q^4)_{l_1}^2\cdots(q;q^4)_{l_1+\cdots+l_{m-1}}^2 (q^4;q^4)_{l_1+\cdots+l_{m-1}}}.$$

By (2.3) and the following identity

$$\frac{(q^6, q^{2-2n}; q^4)_{(n-1)/2}}{(q^4, q^{4-2n}; q^4)_{(n-1)/2}} = [n]_{q^2} q^{1-n},$$

we can simplify the above expression as follows:

$$[n]_{q^2} q^{1-n} \sum_{l_1,\dots,l_{m-1}=0}^{1} (-1)^{l_1+\cdots+l_{m-1}} q^{-4(l_{m-2}+\cdots+(m-2)l_1)}$$
$$\times \frac{(q^5;q^4)_{l_1}^2\cdots(q^5;q^4)_{l_1+\cdots+l_{m-2}}^2 (q^2, q^{2-2n}, q^{2+2n}; q^4)_{l_1+\cdots+l_{m-1}}}{(q;q^4)_{l_1}^2\cdots(q;q^4)_{l_1+\cdots+l_{m-1}}^2 (q^4;q^4)_{l_1+\cdots+l_{m-1}}}.$$

Thus we have proved that the congruence (3.1) holds modulo $1 - aq^{2n}$ or $a - q^{2n}$.

Furthermore, by Lemma 2.1, it is easily seen that the k-th and $((n-1)/2-k)$-th terms on the left-hand side of (3.1) cancel each other modulo $\Phi_n(q^2)$ for $0 \leqslant k \leqslant (n-1)/2$. This means that the left-hand side of (3.1) is congruent to 0 modulo $\Phi_n(q^2)$, and therefore the congruence (3.1) also holds modulo $\Phi_n(q^2)$. This completes the proof of (3.1). □

Proof of Theorem 1.4. The limits of the denominators on the left-hand side of (3.1) as $a \to 1$ are relatively prime to $\Phi_n(q^2)$. It is easy to see that the denominators of the reduced forms of fractions in (3.1) are relatively prime to $\Phi_n(q^2)$ (for odd n) as well. Letting $a \to 1$ in (3.1), we see that the congruences (1.11) and (1.12) hold modulo $\Phi_n(q^2)^3$. It remains to show that they are also true modulo $[n]_{q^2}$. This is exactly the same as the proof of (2.6) and (2.7), and is left to the interested reader. □

4. More similar results

In this section, we give some q-congruences similar to Theorems 1.2 and 1.4. The author proved in [12, Theorem 1.3] and [13, Theorem 5.1] that, for odd $n > 1$, modulo

$[n]_{q^2}\Phi_n(q^2)^2$,

$$\sum_{k=0}^{M}(-1)^k[4k-1]_{q^2}\frac{(q^{-2};q^4)_k^3}{(q^4;q^4)_k^3}q^{2k^2+4k}\equiv[n]_{q^2}(-q^2)^{(n-3)(n+1)/4}, \tag{4.1}$$

$$\sum_{k=0}^{M}(-1)^k[4k-1]_{q^2}[4k-1]^2\frac{(q^{-2};q^4)_k^3}{(q^4;q^4)_k^3}q^{2k^2}\equiv[n]_{q^2}(-1)^{(n-1)/2}q^{(n-1)^2/2}\frac{q+2}{q^3}, \tag{4.2}$$

where $M=(n+1)/2$ or $n-1$. We shall give the following generalization of (4.1) and (4.2), which is very similar to Theorem 1.2.

Theorem 4.1. *Let n and s be positive odd integers with $n>1$. Then, modulo $[n]_{q^2}\Phi_n(q^2)^2$,*

$$\sum_{k=0}^{M}(-1)^k[4k-1]_{q^2}[4k-1]^{s-1}\frac{(q^{-2};q^4)_k^3}{(q^4;q^4)_k^3}q^{2k(k-s+3)}\equiv[n]_{q^2}(-1)^{(n-1)/2}q^{(n-1)^2/2}C_s(q), \tag{4.3}$$

where $M=(n+1)/2$ or $n-1$, and $C_s(q)$ is a Laurent polynomial in q given by

$$C_s(q)=-q^{-s-1}\sum_{l_1,\dots,l_{m-1}=0}^{1}(-1)^{l_1+\cdots+l_{m-1}}q^{2(l_1+\cdots+l_{m-1})-4(l_{m-2}+\cdots+(m-2)l_1)}$$
$$\times\frac{(q^3;q^4)_{l_1}^2\cdots(q^3;q^4)_{l_1+\cdots+l_{m-2}}^2(q^{-2};q^4)_{l_1+\cdots+l_{m-1}}^2}{(q^{-1};q^4)_{l_1}^2\cdots(q^{-1};q^4)_{l_1+\cdots+l_{m-1}}^2}\quad \text{with } m=\frac{s+1}{2}. \tag{4.4}$$

The reason why (4.4) gives a Laurent polynomial in q is similar to $A_s(q)$. Using the formula (4.4), we can easily obtain the first values of $C_s(q)$ as follows: $C_1(q)=-q^{-2}$, $C_3(q)=(q+2)q^{-3}$, and

$$C_5(q)=\frac{7q^2+3q^4+8q^3+4q+1}{q^8},$$
$$C_7(q)=\frac{(2q^2+2q+1)(3q^6+6q^5+2q^4-2q^3-5q^2-4q-1)}{q^{14}}.$$

Sketch of Proof of Theorem 4.1. Like before, we need to establish the following parametric generalization. Modulo $\Phi_n(q^2)(1-aq^{2n})(a-q^{2n})$,

$$\sum_{k=0}^{(n+1)/2}(-1)^k[4k-1]_{q^2}[4k-1]^{s-1}\frac{(q^{-2},aq^{-2},q^{-2}/a;q^4)_k}{(q^4,aq^4,q^4/a;q^4)_k}q^{2k(k-s+3)}$$
$$\equiv-[n]_{q^2}(-1)^{(n-1)/2}q^{(n-1)^2/2-s-1}\sum_{l_1,\dots,l_{m-1}=0}^{1}(-1)^{l_1+\cdots+l_{m-1}}q^{2(l_1+\cdots+l_{m-1})-4(l_{m-2}+\cdots+(m-2)l_1)}$$
$$\times\frac{(q^3;q^4)_{l_1}^2\cdots(q^3;q^4)_{l_1+\cdots+l_{m-2}}^2(aq^{-2},q^{-2}/a;q^4)_{l_1+\cdots+l_{m-1}}}{(q^{-1};q^4)_{l_1}^2\cdots(q^{-1};q^4)_{l_1+\cdots+l_{m-1}}^2}, \tag{4.5}$$

where $m=(s+1)/2$.

The congruence (4.5) modulo $1-aq^{2n}$ or $a-q^{2n}$ follows from Andrews' transformation formula (2.2) by taking $m=(s+1)/2$, $q\mapsto q^4$, $a=q^{-2}$, $b_1=c_1=\cdots=c_{m-1}=d_{m-1}=q^3$, $b_m=q^{-2-2n}$ and $c_m=q^{-2+2n}$, and then letting $N\to\infty$.

Furthermore, by Lemma 2.1 we have

$$\begin{aligned}\frac{(aq^{-1};q^2)_{(n+1)/2-k}}{(q^2/a;q^2)_{(n+1)/2-k}}&=\frac{(1-aq^{-1})(aq;q^2)_{(n-1)/2-k}}{(1-q^{n+1-2k}/a)(q^2/a;q^2)_{(n-1)/2-k}}\\&\equiv(-a)^{(n-1)/2-2k}\frac{(1-aq^{-1})(aq;q^2)_k}{(1-q^{1-2k}/a)(q^2/a;q^2)_k}q^{(n-1)^2/4+k}\\&=(-a)^{(n+1)/2-2k}\frac{(aq^{-1};q^2)_k}{(q^2/a;q^2)_k}q^{(n-1)^2/4+3k-1}\pmod{\Phi_n(q)}\end{aligned}\tag{4.6}$$

for $0\leqslant k\leqslant(n+1)/2$. Using the above congruence with q replaced by q^2, we see that the k-th and $((n+1)/2-k)$-th terms on the left-hand side of (4.5) cancel each other modulo $\Phi_n(q^2)$. Hence the congruence (4.5) hold modulo $\Phi_n(q^2)$, and we finish the proof of (4.5).

The process of the proof of (4.3) from (4.5) is exactly the same as the proof of (1.5) and (1.6). □

Letting $n=p^r$ and $q\to1$ in Theorem 4.1 and noticing that

$$\lim_{q\to1}\frac{(q^{-2};q^4)_k}{(q^4;q^4)_k}=\frac{-1}{4^k(2k-1)}\binom{2k}{k}=\frac{(-\frac12)_k}{k!},\tag{4.7}$$

where $(a)_k=a(a+1)\cdots(a+k-1)$, we obtain the following conclusion, which was originally conjectured by the author and Liu [16, Conjecture 5.1].

Corollary 4.2. *For any odd positive integer s, there exists an integer $c_s=C_s(1)$ such that, for any odd prime p and positive integer r, there hold*

$$\sum_{k=0}^{(p^r+1)/2}(-1)^k(4k-1)^s\frac{(-\frac12)_k^3}{k!^3}\equiv c_sp^r(-1)^{\frac{(p-1)r}{2}}\pmod{p^{r+2}},$$

$$\sum_{k=0}^{p^r-1}(-1)^k(4k-1)^s\frac{(-\frac12)_k^3}{k!^3}\equiv c_sp^r(-1)^{\frac{(p-1)r}{2}}\pmod{p^{r+2}}.$$

In particular, we have $c_1=-1$, $c_3=3$, $c_5=23$, $c_7=-5$, $c_9=1647$, and $c_{11}=-96973$.

We shall also give the following result similar to Theorem 1.4.

Theorem 4.3. *Let n and s be positive odd integers with $n>1$. Then, modulo $[n]_{q^2}\Phi_n(q^2)^2$,*

$$\sum_{k=0}^{M}[4k-1]_{q^2}[4k-1]^{s-1}\frac{(q^{-2};q^4)_k^4}{(q^4;q^4)_k^4}q^{(10-2s)k}\equiv[n]_{q^2}q^{1-n}D_s(q),\tag{4.8}$$

where $M=(n+1)/2$ *or* $n-1$, *and* $D_s(q)$ *is a rational function of* q *given by*

$$D_s(q)=-\frac{q^{3-s}}{(1-q^2)^2}\sum_{l_1,\dots,l_{m-1}=0}^{1}(-1)^{l_1+\cdots+l_{m-1}}q^{-4(l_{m-2}+\cdots+(m-2)l_1)}$$
$$\times\frac{(q^3;q^4)_{l_1}^2\cdots(q^3;q^4)_{l_1+\cdots+l_{m-2}}^2(q^{-2};q^4)_{l_1+\cdots+l_{m-1}}^3}{(q^{-1};q^4)_{l_1}^2\cdots(q^{-1};q^4)_{l_1+\cdots+l_{m-1}}^2(q^4;q^4)_{l_1+\ldots+l_{m-1}-2}} \tag{4.9}$$

with $m=(s+1)/2$ *and* $1/(q^4;q^4)_k=0$ *for any negative integer* k.

Using the formula (4.9), we obtain the first values of $D_s(q)$: $D_1(q)=D_3(q)=0$, $D_5(q)=(q+1)^4/q^8$, and

$$D_7(q)=\frac{2(2q^2+q+2)(q+1)^4}{(q^2+1)q^{10}},$$
$$D_9(q)=\frac{(10q^8+8q^7+19q^6+4q^5+14q^4+4q^3+19q^2+8q+10)(q+1)^4}{(q^4+1)(q^2+1)^2q^{12}}.$$

Noticing (4.7) and the fact that $1/(2k-1)\binom{2k}{k}$ is always an integer, we can show that $D_s(1)$ is an integer as before.

Sketch of Proof of Theorem 4.3. The $s=1$ case was already proved by the author and Schlosser [21, Theorem 5.3]. For $s\geqslant 3$, we first give a parametric generalization of (4.8) as follows. Modulo $\Phi_n(q^2)(1-aq^{2n})(a-q^{2n})$,

$$\sum_{k=0}^{(n+1)/2}[4k-1]_{q^2}[4k-1]^{s-1}\frac{(q^{-2},q^{-2},aq^{-2},q^{-2}/a;q^4)_k}{(q^4,q^4,aq^4,q^4/a;q^4)_k}q^{(10-2s)k}$$
$$\equiv-\frac{[n]_{q^2}q^{4-n-s}}{(1-aq^2)(1-q^2/a)}\sum_{l_1,\dots,l_{m-1}=0}^{1}(-1)^{l_1+\cdots+l_{m-1}}q^{-4(l_{m-2}+\cdots+(m-2)l_1)}$$
$$\times\frac{(q^3;q^4)_{l_1}^2\cdots(q^3;q^4)_{l_1+\cdots+l_{m-2}}^2(q^{-2},aq^{-2},q^{-2}/a;q^4)_{l_1+\cdots+l_{m-1}}}{(q^{-1};q^4)_{l_1}^2\cdots(q^{-1};q^4)_{l_1+\cdots+l_{m-1}}^2(q^4;q^4)_{l_1+\ldots+l_{m-1}-2}}. \tag{4.10}$$

The congruence (4.5) modulo $1-aq^{2n}$ or $a-q^{2n}$ follows from Andrews' transformation formula (2.2) by first performing the parameter substitutions $m=(s+1)/2$, $q\mapsto q^4$, $a=q^{-2}$, $b_1=c_1=\cdots=c_{m-1}=d_{m-1}=q^3$, $b_m=q^{-2}$, $c_m=q^{-2+2n}z$ and $N=(n+1)/2$, and then letting $z\to 1$.

Applying (4.6) with $q\mapsto q^2$ again, we can show that the k-th and $((n+1)/2-k)$-th terms on the left-hand side of (4.10) cancel each other modulo $\Phi_n(q^2)$. Namely, the congruence (4.10) hold modulo $\Phi_n(q^2)$.

The proof of (4.8) is then exactly the same as that of Theorem 1.2. □

Letting $n=p^r$ and $q\to 1$ in Theorem 4.3, we arrive at the following conclusion, which confirms a weaker from of [16, Conjecture 5.2].

Corollary 4.4. *For any odd positive integer s, there exists an integer $d_s=D_s(1)$ such that, for any odd prime p and positive integer r, there hold*

$$\sum_{k=0}^{(p^r+1)/2}(4k-1)^s\frac{(-\frac{1}{2})_k^4}{k!^4}\equiv d_s p^r \pmod{p^{r+2}},$$

$$\sum_{k=0}^{p^r-1}(4k-1)^s\frac{(-\frac{1}{2})_k^4}{k!^4}\equiv d_s p^r \pmod{p^{r+2}}.$$

In particular, we have $d_1=d_3=0$, $d_5=16$, $d_7=80$, $d_9=192$, $d_{11}=640$, $d_{13}=-3472$, and $d_{15}=138480$.

5. Some open problems

The author [10, Conjecture 4.3] further conjectured that the number b_s in (1.9) and (1.10) is the coefficient of $x^{(s-1)/2}$ in the expansion

$$\exp\left(\sum_{n=1}^{\infty}(-1)^n E_{2n}\frac{x^n}{n}\right), \tag{5.1}$$

where E_{2n} is the $2n$-th Euler number, i.e.,

$$\frac{2}{\exp(x)+\exp(-x)}=\sum_{n=0}^{\infty}\frac{E_n}{n!}x^n.$$

Letting $q\to 1$ in Theorem 1.4, one sees that this assertion is equivalent to the following conjecture.

Conjecture 5.1. *For any integer $m\geqslant 1$, the coefficient of x^m in (5.1) is equal to*

$$\sum_{l_1,\dots,l_m=0}^{1}(-1)^{l_1+\cdots+l_m}\frac{(\frac{5}{4})_{l_1}^2\cdots(\frac{5}{4})_{l_1+\cdots+l_{m-1}}^2(\frac{1}{2})_{l_1+\cdots+l_m}^3}{(\frac{1}{4})_{l_1}^2\cdots(\frac{1}{4})_{l_1+\cdots+l_m}^2(1)_{l_1+\cdots+l_m}}.$$

The author and Wang [23] have proved (1.9) for $s=1$ by establishing the following q-analogue: for odd n,

$$\sum_{k=0}^{(n-1)/2}[4k+1]\frac{(q;q^2)_k^4}{(q^2;q^2)_k^4}\equiv [n]q^{(1-n)/2}+\frac{(n^2-1)(1-q)^2}{24}[n]^3q^{(1-n)/2} \pmod{[n]\Phi_n(q)^3}. \tag{5.2}$$

The corresponding q-analogue of (1.9) for $s=1$ was also conjectured by the author and Wang [23, Conjecture 5.1] and has been confirmed by the author and Schlosser in the proof of [21, Theorem 12.9].

Motivated by the congruence (5.2), we would like to propose the following generalization of Theorem 1.4, which is also a complete q-analogue of Conjecture 1.3.

Conjecture 5.2. *Let n and s be positive odd integers with $n\geqslant (s+1)/2$. Let $B_s(q)$ be given by* (1.13)*. Then, modulo $[n]_{q^2}\Phi_n(q^2)^3$,*

$$\sum_{k=0}^{m}[4k+1]_{q^2}[4k+1]^{s-1}\frac{(q^2;q^4)_k^4}{(q^4;q^4)_k^4}q^{(2-2s)k}$$
$$\equiv [n]_{q^2}q^{1-n}B_s(q)+\frac{(n^2-1)(1-q^2)^2}{24}[n]_{q^2}^3q^{1-n}B_s(q),$$

where $m=(n-1)/2$ or $n-1$.

Similarly, we have the following generalization of Theorem 4.3, which is also a complete q-analogue of [16, Conjecture 5.2].

Conjecture 5.3. *Let n and s be positive odd integers with $n\geqslant (s-1)/2$. Let $D_s(q)$ be given by* (4.9)*. Then, modulo $[n]_{q^2}\Phi_n(q^2)^3$,*

$$\sum_{k=0}^{M}[4k-1]_{q^2}[4k-1]^{s-1}\frac{(q^{-2};q^4)_k^4}{(q^4;q^4)_k^4}q^{(10-2s)k}$$
$$\equiv [n]_{q^2}q^{1-n}D_s(q)+\frac{(n^2-1)(1-q^2)^2}{24}[n]_{q^2}^3q^{1-n}D_s(q),$$

where $M=(n+1)/2$ or $n-1$.

There is a stronger version of Conjecture 5.3 for $s=1,3$. Using the q-WZ method [38], the author and Schlosser [19] have proved that

$$\sum_{k=0}^{M}[4k-1]\frac{(q^{-1};q^2)_k^4}{(q^2;q^2)_k^4}q^{4k}\equiv -(1+3q+q^2)[n]^4 \pmod{[n]^4\Phi_n(q)}.$$

Moreover, the author [13] conjectured that

$$\sum_{k=0}^{M}[4k-1]_{q^2}[4k-1]^2\frac{(q^{-2};q^4)_k^4}{(q^4;q^4)_k^4}q^{4k}\equiv 0 \pmod{[n]_{q^2}^4}.$$

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