Proof of some q-supercongruences modulo the fourth power of a cyclotomic polynomial

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Abstract. We prove some q-supercongruences modulo the fourth power of a cyclotomic polynomial by making use of the Chinese remainder theorem for coprime polynomials, Watson's ${}_8\phi_7$ transformation, and the 'creative microscoping' method introduced by the author and Wadim Zudilin. In particular, we confirm Conjecture 1.5 in [Results Math. 74 (2019), Art. 131].

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1. Introduction

In 1997, Van Hamme [27, (C.2)] proved that, for any odd prime p,

$$\sum_{k=0}^{(p-1)/2} \frac{4k+1}{256^k} \binom{2k}{k}^4 \equiv p \pmod{p^3}. \tag{1.1}$$

In 2011, applying hypergeometric identities, Long [22] further proved that the above supercongruence also holds modulo p^4 for primes $p > 3$. In 2015, Swisher [25, (C.3)] made an interesting conjecture on a generalization of Long's result. In 2018, motivated by Zudilin's work [31] on proving supercongruences through the WZ (Wilf–Zeilberger) method [29], using the Zeilberger algorithm Wang [28] proved some generalizations of (1.1), such as

$$\sum_{k=0}^{(p-1)/2} \frac{(4k+1)^3}{256^k} \binom{2k}{k}^4 \equiv -p \pmod{p^4} \tag{1.2}$$

for any odd prime p with $p \equiv 2 \pmod 3$. In 2019, by applying a combinatorial identity Liu [20] confirmed (1.2) for all primes $p > 3$. Recently, Hou, Mu, and Zeilberger [19] investigated some supercongruences related to (1.2).

The aim of this paper is to confirm the following generalizations of (1.2), which were originally conjectured by the author (see [5, Conjecture 4.3]): for any prime $p>3$ and positive integer r,

$$\sum_{k=0}^{(p^r-1)/2}\frac{(4k+1)^3}{256^k}\binom{2k}{k}^4\equiv -p^r \pmod{p^{r+3}}, \tag{1.3}$$

$$\sum_{k=0}^{p^r-1}\frac{(4k+1)^3}{256^k}\binom{2k}{k}^4\equiv -p^r \pmod{p^{r+3}}. \tag{1.4}$$

Note that when $r=1$ the supercongruences (1.3) and (1.4) are equivalent to each other, since $\binom{2k}{k}\equiv 0 \pmod p$ for $(p+1)/2\leqslant k\leqslant p-1$. Moreover, in an early paper [6] the author has proved (1.3) and (1.4) modulo p^{r+2} by showing the q-supercongruences: for odd $n>1$,

$$\sum_{k=0}^{(n-1)/2}[4k+1]_{q^2}[4k+1]^2\frac{(q^2;q^4)_k^4}{(q^4;q^4)_k^4}q^{-4k}\equiv -[n]_{q^2}\frac{2q^{2-n}}{1+q^2} \pmod{[n]_{q^2}\Phi_n(q^2)^2}, \tag{1.5}$$

$$\sum_{k=0}^{n-1}[4k+1]_{q^2}[4k+1]^2\frac{(q^2;q^4)_k^4}{(q^4;q^4)_k^4}q^{-4k}\equiv -[n]_{q^2}\frac{2q^{2-n}}{1+q^2} \pmod{[n]_{q^2}\Phi_n(q^2)^2}, \tag{1.6}$$

where $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ denotes the *q-shifted factorial*, $[n]=[n]_q=1+q+\cdots+q^{n-1}$ is the *q-integer*, and $\Phi_n(q)$ stands for the n-th *cyclotomic polynomial* in q, i.e.,

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(n,k)=1}}(q-\zeta^k)$$

with ζ being an n-th primitive root of unity.

Throughout the paper, for polynomials $A_1(q),A_2(q),P(q)\in\mathbb{Z}[q]$, we say that $A_1(q)/A_2(q)\equiv 0 \pmod{P(q)}$ if $P(q)$ divides $A_1(q)$ but is relatively prime to $A_2(q)$. More generally, for rational functions $A(q),B(q)\in\mathbb{Z}(q)$, the congruence $A(q)\equiv B(q) \pmod{P(q)}$ is meant that $A(q)-B(q)\equiv 0 \pmod{P(q)}$. The reader is referred to [2–4, 6–11, 13–18, 21, 23, 24, 26, 30, 32] for some recent q-supercongruences.

We shall prove (1.3) and (1.4) by establishing the following complete q-analogues of them, which were originally conjectured by the author [6, Conjecture 1.5].

Theorem 1.1. *Let $n>1$ be an odd integer. Then, modulo $[n]_{q^2}\Phi_n(q^2)^3$,*

$$\sum_{k=0}^{(n-1)/2}[4k+1]_{q^2}[4k+1]^2\frac{(q^2;q^4)_k^4}{(q^4;q^4)_k^4}q^{-4k}\equiv -[n]_{q^2}\frac{2q^{2-n}}{1+q^2}-[n]_{q^2}^3\frac{(n^2-1)(1-q^2)^2q^{2-n}}{12(1+q^2)}, \tag{1.7}$$

$$\sum_{k=0}^{n-1}[4k+1]_{q^2}[4k+1]^2\frac{(q^2;q^4)_k^4}{(q^4;q^4)_k^4}q^{-4k}\equiv -[n]_{q^2}\frac{2q^{2-n}}{1+q^2}-[n]_{q^2}^3\frac{(n^2-1)(1-q^2)^2q^{2-n}}{12(1+q^2)}. \tag{1.8}$$

Note that, for $k\geqslant 0$ and any prime power p^r, there hold

$$\lim_{q\to 1}\frac{(q^2;q^4)_k}{(q^4;q^4)_k}=\frac{1}{4^k}\binom{2k}{k}\quad\text{and}\quad \Phi_{p^r}(1)=p.$$

Hence, letting $n=p^r$ and $q\to 1$ in (1.7) and (1.8), we immediately obtain (1.3) and (1.4). Furthermore, if we let $n=p^r$ and $q\to -1$ in (1.7) and (1.8), then we get

$$\sum_{k=0}^{(p^r-1)/2}\frac{4k+1}{256^k}\binom{2k}{k}^4\equiv p^r \pmod{p^{r+3}}, \tag{1.9}$$

$$\sum_{k=0}^{p^r-1}\frac{4k+1}{256^k}\binom{2k}{k}^4\equiv p^r \pmod{p^{r+3}}, \tag{1.10}$$

which are generalizations of (1.1). The supercongruence (1.9) was first observed by Long [22] (she only proved the $r=1$ case) and proved by the author and Wang [16] along with the following q-analogue:

$$\sum_{k=0}^{(n-1)/2}[4k+1]\frac{(q;q^2)_k^4}{(q^2;q^2)_k^4}\equiv [n]q^{(1-n)/2}+\frac{(n^2-1)(1-q)^2}{24}[n]^3q^{(1-n)/2}\pmod{[n]\Phi_n(q)^3}$$

for odd n. The same q-analogue of (1.10) was formulated by the author and Wang [16, Conjecture 5.1] and was later confirmed by the author and Schlosser in the proof of [15, Theorem 12.9].

We shall also prove the following q-congruences.

Theorem 1.2. *Let $n>1$ be an odd integer. Then*

$$\sum_{k=0}^{(n+1)/2}[4k-1]_{q^2}[4k-1]^2\frac{(q^{-2};q^4)_k^4}{(q^4;q^4)_k^4}q^{4k}\equiv 0\pmod{[n]_{q^2}\Phi_n(q^2)^3}, \tag{1.11}$$

$$\sum_{k=0}^{n-1}[4k-1]_{q^2}[4k-1]^2\frac{(q^{-2};q^4)_k^4}{(q^4;q^4)_k^4}q^{4k}\equiv 0\pmod{[n]_{q^2}\Phi_n(q^2)^3}. \tag{1.12}$$

Note that the q-supercongruences (1.11) and (1.12) modulo $[n]_{q^2}\Phi_n(q^2)^2$ were proved by the author [6, Theorem 5.4]. Moreover, letting $n=p^r$ and $q\to 1$ in Theorem 1.2, we are led to the following conclusion, which confirms the $m=3$ case of [12, Conjecture 5.2].

Corollary 1.3. *Let p be an odd prime and let r be a positive integer. Then*

$$\sum_{k=0}^{(p^r+1)/2}\frac{(4k-1)^3}{256^k(2k-1)^4}\binom{2k}{k}^4 \equiv 0 \pmod{p^{r+3}},$$

$$\sum_{k=0}^{p^r-1}\frac{(4k-1)^3}{256^k(2k-1)^4}\binom{2k}{k}^4 \equiv 0 \pmod{p^{r+3}}.$$

The rest of the paper is organized as follows. We recall some known auxiliary results in Section 2. We shall prove Theorems 1.1 and 1.2 in Sections 3 and 4, respectively. More precisely, we shall first use Watson's ${}_8\phi_7$ transformation, the method of creative microscoping recently introduced by the author and Zudilin [17] and the Chinese remainder theorem for coprime polynomials to establish the corresponding q-congruences modulo $[n]_{q^2}(1-aq^{2n})(a-q^{2n})(b-q^{2n})$ (see [10] for some similar q-congruences). Then we deduce Theorems 1.1 and 1.2 from these q-congruences with parameters a and b by taking the limits as $a,b\to 1$. We give a remark in Section 5 for further study.

2. Some auxiliary results

We will make use of Watson's ${}_8\phi_7$ transformation formula (see [1, Appendix (III.18)]):

$$\begin{aligned}
&{}_8\phi_7\left[\begin{matrix} a, & qa^{\frac12}, & -qa^{\frac12}, & b, & c, & d, & e, & q^{-n} \\ & a^{\frac12}, & -a^{\frac12}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1}\end{matrix}; q,\ \frac{a^2q^{n+2}}{bcde}\right] \\
&\quad = \frac{(aq;q)_n(aq/de;q)_n}{(aq/d;q)_n(aq/e;q)_n}\,{}_4\phi_3\left[\begin{matrix} aq/bc,\ d,\ e,\ q^{-n} \\ aq/b,\ aq/c,\ deq^{-n}/a\end{matrix}; q,\ q\right], \qquad (2.1)
\end{aligned}$$

where the *basic hypergeometric series* ${}_{r+1}\phi_r$ is defined as

$${}_{r+1}\phi_r\left[\begin{matrix} a_1,a_2,\dots,a_{r+1} \\ b_1,b_2,\dots,b_r\end{matrix}; q,\ z\right] = \sum_{k=0}^{\infty}\frac{(a_1;q)_k(a_2;q)_k\cdots(a_{r+1};q)_k}{(q;q)_k(b_1;q)_k\cdots(b_r;q)_k}z^k.$$

We shall also use the following easily proved results (see [15, Lemma 3.1] and [10, Lemma 2.1]).

Lemma 2.1. *Let n be a positive odd integer. Then, for $0\leqslant k\leqslant (n-1)/2$,*

$$\frac{(aq;q^2)_{(n-1)/2-k}}{(q^2/a;q^2)_{(n-1)/2-k}} \equiv (-a)^{(n-1)/2-2k}\frac{(aq;q^2)_k}{(q^2/a;q^2)_k}q^{(n-1)^2/4+k} \pmod{\Phi_n(q)}.$$

Lemma 2.2. *Let n be a positive odd integer. Then*

$$(aq^2, q^2)_{(n-1)/2}(q^2/a, q^2)_{(n-1)/2} \equiv (-1)^{(n-1)/2}\frac{(1-a^n)q^{-(n-1)^2/4}}{(1-a)a^{(n-1)/2}} \pmod{\Phi_n(q)},$$

$$(aq, q^2)_{(n-1)/2}(q/a, q^2)_{(n-1)/2} \equiv (-1)^{(n-1)/2}\frac{(1-a^n)q^{(1-n^2)/4}}{(1-a)a^{(n-1)/2}} \pmod{\Phi_n(q)}.$$

3. Proof of Theorem 1.1

We first present the following two-parametric generalization of (1.5). Note that the $b=1$ case has already been given in [6, Theorem 4.1].

Theorem 3.1. *Let $n>1$ be an odd integer and let a, b be indeterminates. Then, modulo $\Phi_n(q^2)(1-aq^{2n})(a-q^{2n})$,*

$$\sum_{k=0}^{(n-1)/d}[4k+1]_{q^2}[4k+1]^2\frac{(aq^2;q^4)_k(q^2/a;q^4)_k(q^2/b;q^4)_k(q^2;q^4)_k}{(aq^4;q^4)_k(q^4/a;q^4)_k(bq^4;q^4)_k(q^4;q^4)_k}\left(\frac{b}{q^4}\right)^k$$
$$\equiv b^{(n-1)/2}q^{1-n}[n]_{q^2}\frac{(q^4/b;q^4)_{(n-1)/2}}{(bq^4;q^4)_{(n-1)/2}}\left(1-\frac{(1-aq^2)(1-q^2/a)(1-q^2/b)}{(1-q)^2(1-q^4/b)}\right), \tag{3.1}$$

where $d=1,2$.

Proof. For $a=q^{-2n}$ or $a=q^{2n}$, the left-hand side of (3.1) is equal to

$$\sum_{k=0}^{(n-1)/2}[4k+1]_{q^2}[4k+1]^2\frac{(q^{2-2n};q^4)_k(q^{2+2n};q^4)_k(q^2/b;q^4)_k(q^2;q^4)_k}{(q^{4-2n};q^4)_k(q^{4+2n};q^4)_k(bq^4;q^4)_k(q^4;q^4)_k}b^kq^{-4k}$$
$$= {}_8\phi_7\left[\begin{matrix} q^2, & q^5, & -q^5, & q^5, & q^5, & q^2/b, & q^{2+2n}, & q^{2-2n} \\ & q, & -q, & q, & q, & bq^4, & q^{4-2n}, & q^{4+2n} \end{matrix}; q^4, bq^{-4}\right]. \tag{3.2}$$

By Watson's ${}_8\phi_7$ transformation (2.1), we can write the right-hand side of (3.2) as

$$\frac{(q^6;q^4)_{(n-1)/2}(bq^{2-2n};q^4)_{(n-1)/2}}{(bq^4;q^4)_{(n-1)/2}(q^{4-2n};q^4)_{(n-1)/2}}{}_4\phi_3\left[\begin{matrix} q^{-4}, & q^2/b, & q^{2+2n}, & q^{2-2n} \\ & q, q, & q^4/b & \end{matrix}; q^4, q^4\right]$$
$$= b^{(n-1)/2}q^{1-n}[n]_{q^2}\frac{(q^4/b;q^4)_{(n-1)/2}}{(bq^4;q^4)_{(n-1)/2}}\left(1-\frac{(1-q^{2-2n})(1-q^{2+2n})(1-q^2/b)}{(1-q)^2(1-q^4/b)}\right). \tag{3.3}$$

This proves that the congruence (3.1) holds modulo $1-aq^{2n}$ and $a-q^{2n}$.

Further, by Lemma 2.1 it is not hard to check that the k-th and $((n-1)/2-k)$-th terms on the left-hand side of (3.1) modulo $\Phi_n(q^2)$ cancel each other for $0 \leqslant k \leqslant (n-1)/2$. This indicates that the left-hand side of (3.1) is congruent to 0 modulo $\Phi_n(q^2)$, and therefore the congruence (3.1) is true modulo $\Phi_n(q^2)$. □

We now give a simpler congruence as follows.

Theorem 3.2. *Let $n>1$ be an odd integer and let a, b be indeterminates. Then, modulo $b-q^{2n}$,*

$$\sum_{k=0}^{(n-1)/d}[4k+1]_{q^2}[4k+1]^2\frac{(aq^2;q^4)_k(q^2/a;q^4)_k(q^2/b;q^4)_k(q^2;q^4)_k}{(aq^4;q^4)_k(q^4/a;q^4)_k(bq^4;q^4)_k(q^4;q^4)_k}\left(\frac{b}{q^4}\right)^k$$
$$\equiv\frac{[n]_{q^2}(q^2;q^4)_{(n-1)/2}^2}{(aq^4;q^4)_{(n-1)/2}(q^4/a;q^4)_{(n-1)/2}}\left(1-\frac{(1-aq^2)(1-q^2/a)(1-q^2/b)}{(1-q)^2(1-q^4/b)}\right), \tag{3.4}$$

where $d=1,2$.

Proof. For $b=q^{2n}$, the left-hand side of (3.1) is equal to

$$\sum_{k=0}^{(n-1)/2}[4k+1]_{q^2}[4k+1]^2\frac{(aq^2;q^4)_k(q^2/a;q^4)_k(q^{2-2n};q^4)_k(q^2;q^4)_k}{(aq^4;q^4)_k(q^4/a;q^4)_k(q^{4+2n};q^4)_k(q^4;q^4)_k}q^{(2n-4)k}$$
$$={}_8\phi_7\left[\begin{matrix}q^2, & q^5, & -q^5, & q^5, & q^5, & aq^2, & q^2/a, & q^{2-2n}\\ & q, & -q, & q, & q, & q^4/a, & aq^4, & q^{4+2n}\end{matrix};q^4,\ q^{2n-4}\right]. \tag{3.5}$$

By Watson's transformation (2.1), the right-hand side of (3.5) may be written as

$$\frac{(q^6;q^4)_{(n-1)/2}(q^2;q^4)_{(n-1)/2}}{(aq^4;q^4)_{(n-1)/2}(q^4/a;q^4)_{(n-1)/2}}{}_4\phi_3\left[\begin{matrix}q^{-4}, & aq^2, & q^2/a, & q^{2-2n}\\ & q, & q, & q^{4-2n}\end{matrix};q^4,\ q^4\right]$$
$$=\frac{[n]_{q^2}(q^2;q^4)_{(n-1)/2}^2}{(aq^4;q^4)_{(n-1)/2}(q^4/a;q^4)_{(n-1)/2}}\left(1-\frac{(1-aq^2)(1-q^2/a)(1-q^{2-2n})}{(1-q)^2(1-q^{4-2n})}\right). \tag{3.6}$$

This proves that the congruence (3.4) is true modulo $b-q^{2n}$. □

With the help of Theorems 3.1, 3.2 and the Chinese remainder theorem for coprime polynomials, we are able to prove Theorem 1.1. More precisely, we shall prove Theorem 1.1 by establishing the following parametric generalization.

Theorem 3.3. *Let $n>1$ be an odd integer and a an indeterminate. Then, modulo $\Phi_n(q^2)^2(1-aq^{2n})(a-q^{2n})$,*

$$\sum_{k=0}^{(n-1)/d}[4k+1]_{q^2}[4k+1]^2\frac{(aq^2;q^4)_k(q^2/a;q^4)_k(q^2;q^4)_k^2}{(aq^4;q^4)_k(q^4/a;q^4)_k(q^4;q^4)_k^2}q^{-4k}$$
$$\equiv q^{1-n}[n]_{q^2}\left(1-\frac{(1-aq^2)(1-q^2/a)}{(1+q^2)(1-q)^2}\right)$$
$$\times\left\{1+\frac{(1-aq^{2n})(a-q^{2n})}{(1-a)^2}\left(1-\frac{n(1-a)a^{(n-1)/2}}{1-a^n}\right)\right\}, \tag{3.7}$$

where $d=1,2$.

Proof. It is easy to see that $\Phi_n(q^2)(1-aq^{2n})(a-q^{2n})$ and $b-q^{2n}$ are relatively prime polynomials. By the Chinese reminder theorem for coprime polynomials, we can uniquely determine the remainder of the left-hand side of (3.1) modulo $\Phi_n(q^2)(1-aq^{2n})(a-q^{2n})(b-q^{2n})$ from the congruences (3.1) and (3.4). To this end, we need the following q-congruences:

$$\frac{(b-q^{2n})(ab-1-a^2+aq^{2n})}{(a-b)(1-ab)} \equiv 1 \pmod{(1-aq^{2n})(a-q^{2n})}, \tag{3.8}$$

$$\frac{(1-aq^{2n})(a-q^{2n})}{(a-b)(1-ab)} \equiv 1 \pmod{b-q^{2n}}. \tag{3.9}$$

Thus, from (3.1) and (3.4) we deduce that

$$\begin{aligned}
&\sum_{k=0}^{(n-1)/d} [4k+1]_{q^2}[4k+1]^2 \frac{(aq^2;q^4)_k(q^2/a;q^4)_k(q^2/b;q^4)_k(q^2;q^4)_k}{(aq^4;q^4)_k(q^4/a;q^4)_k(bq^4;q^4)_k(q^4;q^4)_k} b^k q^{-4k} \\
&\equiv b^{(n-1)/2}q^{1-n}[n]_{q^2}\frac{(q^4/b;q^4)_{(n-1)/2}}{(bq^4;q^4)_{(n-1)/2}}\left(1-\frac{(1-aq^2)(1-q^2/a)(1-q^2/b)}{(1-q)^2(1-q^4/b)}\right) \\
&\quad\times \frac{(b-q^{2n})(ab-1-a^2+aq^{2n})}{(a-b)(1-ab)} \\
&\quad+\frac{[n]_{q^2}(q^2;q^4)_{(n-1)/2}^2}{(aq^4;q^4)_{(n-1)/2}(q^4/a;q^4)_{(n-1)/2}}\left(1-\frac{(1-aq^2)(1-q^2/a)(1-q^2/b)}{(1-q)^2(1-q^4/b)}\right) \\
&\quad\times \frac{(1-aq^{2n})(a-q^{2n})}{(a-b)(1-ab)} \pmod{\Phi_n(q^2)(1-aq^{2n})(a-q^{2n})(b-q^{2n})}.
\end{aligned} \tag{3.10}$$

By Lemma 2.2, we have

$$\frac{(q^2;q^4)_{(n-1)/2}^2}{(aq^4;q^4)_{(n-1)/2}(q^4/a;q^4)_{(n-1)/2}} \equiv \frac{n(1-a)a^{(n-1)/2}}{(1-a^n)q^{n-1}} \pmod{\Phi_n(q^2)}. \tag{3.11}$$

It is clear that $1-q^{2n}$ has the factor $\Phi_n(q^2)$. Moreover, the factor $(q^4;q^4)_{(n-1)/d}$ in the denominator of the left-hand side of (3.10) is relatively prime to $\Phi_n(q^2)$. Thus, letting $b=1$ in (3.10), applying the congruence (3.11) and using

$$(1-q^{2n})(1+a^2-a-aq^{2n}) = (1-a)^2+(1-aq^{2n})(a-q^{2n}),$$

we conclude that (3.7) holds modulo $\Phi_n(q^2)^2(1-aq^{2n})(a-q^{2n})$. □

Proof of Theorem 1.1. By l'Hôpital's rule, we have

$$\lim_{a\to 1}\frac{(1-aq^{2n})(a-q^{2n})}{(1-a)^2}\frac{(1-a^n-n(1-a)a^{(n-1)/2})}{(1-a^n)} = \frac{(n^2-1)(1-q^2)^2}{24}[n]_{q^2}^2.$$

Thus, letting $a\to 1$ in (3.7), we see that the congruences (1.7) and (1.8) hold modulo $\Phi_n(q^2)^4$. In view of (1.5) and (1.6), they are true modulo $[n]_{q^2}$. The proof then follows from the fact that the least common multiple of $[n]_{q^2}$ and $\Phi_n(q^2)^4$ is just $[n]_{q^2}\Phi_n(q^2)^3$. □

4. Proof of Theorem 1.2

Similarly as before, we first give a two-parametric q-congruence. Note that the $b=1$ case is just [6, eq. (5.5)].

Theorem 4.1. *Let $n>1$ be an odd integer and let a, b be indeterminates. Then, modulo $\Phi_n(q^2)(1-aq^{2n})(a-q^{2n})$,*

$$\sum_{k=0}^{M}[4k-1]_{q^2}[4k-1]^2\frac{(aq^{-2};q^4)_k(q^{-2}/a;q^4)_k(q^{-2}/b;q^4)_k(q^{-2};q^4)_k}{(aq^4;q^4)_k(q^4/a;q^4)_k(bq^4;q^4)_k(q^4;q^4)_k}\left(bq^4\right)^k$$
$$\equiv b^{(n-1)/2}q^{n-5}[n]_{q^2}\frac{(1/b;q^4)_{(n-1)/2}}{(bq^8;q^4)_{(n-1)/2}}\left(\frac{(1-aq^{-2})(1-q^{-2}/a)(1-q^{-2}/b)}{(1-q^{-1})^2(1-q^{-4}/b)}-1\right), \qquad (4.1)$$

where $M=(n+1)/2$ or $n-1$.

Proof. For $a=q^{-2n}$ or $a=q^{2n}$, by Watson's ${}_8\phi_7$ transformation (2.1), the left-hand side of (4.1) is equal to

$$-q^{-4}{}_8\phi_7\left[\begin{matrix} q^{-2}, & q^3, & -q^3, & q^3, & q^3, & q^{-2}/b, & q^{-2+2n}, & q^{-2-2n} \\ & q^{-1}, & -q^{-1}, & q^{-1}, & q^{-1}, & bq^4, & q^{4-2n}, & q^{4+2n} \end{matrix};q^4,\, bq^4\right]$$
$$=-q^{-4}\frac{(q^2;q^4)_{(n+1)/2}(bq^{6-2n};q^4)_{(n+1)/2}}{(bq^4;q^4)_{(n+1)/2}(q^{4-2n};q^4)_{(n+1)/2}}{}_4\phi_3\left[\begin{matrix} q^{-4}, & q^{-2}/b, & q^{-2+2n}, & q^{-2-2n} \\ & q^{-1}, & q^{-1}, & q^{-4}/b \end{matrix};q^4,\, q^4\right]$$
$$=b^{(n-1)/2}q^{n-5}[n]_{q^2}\frac{(1/b;q^4)_{(n-1)/2}}{(bq^8;q^4)_{(n-1)/2}}\left(\frac{(1-q^{-2-2n})(1-q^{-2+2n})(1-q^{-2}/b)}{(1-q^{-1})^2(1-q^{-4}/b)}-1\right).$$

Namely, the congruence (4.1) holds modulo $1-aq^{2n}$ or $a-q^{2n}$.

On the other hand, by Lemma 2.1, for odd $n>1$ and $0\leqslant k\leqslant (n+1)/2$, we get

$$\begin{aligned}\frac{(aq^{-1};q^2)_{(n+1)/2-k}}{(q^2/a;q^2)_{(n+1)/2-k}} &= \frac{(1-aq^{-1})(aq;q^2)_{(n-1)/2-k}}{(1-q^{n+1-2k}/a)(q^2/a;q^2)_{(n-1)/2-k}} \\ &\equiv (-a)^{(n-1)/2-2k}\frac{(1-aq^{-1})(aq;q^2)_k}{(1-q^{1-2k}/a)(q^2/a;q^2)_k}q^{(n-1)^2/4+k} \\ &= (-a)^{(n+1)/2-2k}\frac{(aq^{-1};q^2)_k}{(q^2/a;q^2)_k}q^{(n-1)^2/4+3k-1} \pmod{\Phi_n(q)}. \end{aligned} \qquad (4.2)$$

Utilizing (4.2) we can verify that the k-th and $((n+1)/2-k)$-th terms on the left-hand side of (4.1) cancel each other modulo $\Phi_n(q^2)$. Namely, the congruence (4.1) is also true modulo $\Phi_n(q^2)$ for $M=(n+1)/2$. Since $[4k-1]_{q^2}$ is congruent to 0 modulo $\Phi_n(q^2)$ for any integer k in the range $(n+1)/2<k\leqslant n-1$, we conclude that (4.1) is still true modulo $\Phi_n(q^2)$ for $M=n-1$. This proves the theorem. ☐

We also need the following simpler q-congruence.

Theorem 4.2. *Let $n>1$ be an odd integer and let a, b be indeterminates. Then, modulo $b-q^{2n}$,*

$$\sum_{k=0}^{M}[4k-1]_{q^2}[4k-1]^2\frac{(aq^{-2};q^4)_k(q^{-2}/a;q^4)_k(q^{-2}/b;q^4)_k(q^{-2};q^4)_k}{(aq^4;q^4)_k(q^4/a;q^4)_k(bq^4;q^4)_k(q^4;q^4)_k}\left(bq^4\right)^k$$

$$\equiv\frac{[n]_{q^2}(q^2;q^4)_{(n-1)/2}(q^2;q^4)_{(n+3)/2}}{q^4(aq^4;q^4)_{(n+1)/2}(q^4/a;q^4)_{(n+1)/2}}\left(\frac{(1-aq^{-2})(1-q^{-2}/a)(1-q^{-2}/b)}{(1-q^{-1})^2(1-q^{-4}/b)}-1\right), \quad (4.3)$$

where $M=(n+1)/2$ or $n-1$.

Proof. For $b=q^{2n}$, by (2.1) the left-hand side of (4.1) is equal to

$$-q^{-4}{}_8\phi_7\left[\begin{matrix}q^{-2}, & q^3, & -q^3, & q^3, & q^3, & aq^{-2}, & q^{-2}/a, & q^{-2-2n}\\ & q^{-1}, & -q^{-1}, & q^{-1}, & q^{-1}, & q^4/a, & aq^4, & q^{4+2n}\end{matrix};q^4,\, bq^4\right]$$

$$=-q^{-4}\frac{(q^2;q^4)_{(n+1)/2}(q^6;q^4)_{(n+1)/2}}{(aq^4;q^4)_{(n+1)/2}(q^4/a;q^4)_{(n+1)/2}}{}_4\phi_3\left[\begin{matrix}q^{-4},\ aq^{-2},\ q^{-2}/a,\ q^{-2-2n}\\ q^{-1},\ q^{-1},\ q^{-4-2n}\end{matrix};q^4,\, q^4\right]$$

$$=\frac{[n]_{q^2}(q^2;q^4)_{(n-1)/2}(q^2;q^4)_{(n+3)/2}}{q^4(aq^4;q^4)_{(n+1)/2}(q^4/a;q^4)_{(n+1)/2}}\left(\frac{(1-aq^{-2})(1-q^{-2}/a)(1-q^{-2-2n})}{(1-q^{-1})^2(1-q^{-4-2n})}-1\right).$$

This means that the congruence (4.3) is true modulo $b-q^{2n}$. □

With the help of Theorems 4.1 and 4.2, we can prove the following parametric generalization of Theorem 1.2.

Theorem 4.3. *Let $n>1$ be an odd integer and a an indeterminate. Then, modulo $\Phi_n(q^2)^2(1-aq^{2n})(a-q^{2n})$,*

$$\sum_{k=0}^{M}[4k-1]_{q^2}[4k-1]^2\frac{(aq^{-2};q^4)_k(q^{-2}/a;q^4)_k(q^{-2};q^4)_k^2}{(aq^4;q^4)_k(q^4/a;q^4)_k(q^4;q^4)_k^2}q^{4k}\equiv 0, \quad (4.4)$$

where $M=(n+1)/2$ or $n-1$.

Proof. Using (3.8) and (3.9), we deduce from (4.1) and (4.3) that

$$\sum_{k=0}^{M}[4k-1]_{q^2}[4k-1]^2\frac{(aq^{-2};q^4)_k(q^{-2}/a;q^4)_k(q^{-2}/b;q^4)_k(q^{-2};q^4)_k}{(aq^4;q^4)_k(q^4/a;q^4)_k(bq^4;q^4)_k(q^4;q^4)_k}\left(bq^4\right)^k$$

$$\equiv b^{(n-1)/2}q^{n-5}[n]_{q^2}\frac{(1/b;q^4)_{(n-1)/2}}{(bq^8;q^4)_{(n-1)/2}}\left(\frac{(1-aq^{-2})(1-q^{-2}/a)(1-q^{-2}/b)}{(1-q^{-1})^2(1-q^{-4}/b)}-1\right)$$

$$\times\frac{(b-q^{2n})(ab-1-a^2+aq^{2n})}{(a-b)(1-ab)}$$

$$+\frac{[n]_{q^2}(q^2;q^4)_{(n-1)/2}(q^2;q^4)_{(n+3)/2}}{q^4(aq^4;q^4)_{(n+1)/2}(q^4/a;q^4)_{(n+1)/2}}\left(\frac{(1-aq^{-2})(1-q^{-2}/a)(1-q^{-2}/b)}{(1-q^{-1})^2(1-q^{-4}/b)}-1\right)$$

$$\times\frac{(1-aq^{2n})(a-q^{2n})}{(a-b)(1-ab)}\pmod{\Phi_n(q^2)(1-aq^{2n})(a-q^{2n})(b-q^{2n})}. \tag{4.5}$$

It is clear that $1-q^{2n}\equiv 0\pmod{\Phi_n(q^2)}$, $(1;q^4)_{(n-1)/2}=0$ and

$$\frac{(q^2;q^4)_{(n-1)/2}(q^2;q^4)_{(n+3)/2}}{(aq^4;q^4)_{(n+1)/2}(q^4/a;q^4)_{(n+1)/2}}\equiv 0\pmod{\Phi_n(q^2)}.$$

Hence, taking $b=1$ in (4.5) we conclude that (3.7) holds modulo $\Phi_n(q^2)^2(1-aq^{2n})(a-q^{2n})$. □

5. A remark

The author and Schlosser [14, Conjecture 3] proposed the following generalization of Theorem 1.2, which remains open:

Conjecture 5.1. *Let $n>1$ be an odd integer. Then*

$$\sum_{k=0}^{M}[4k-1]_{q^2}[4k-1]^2\frac{(q^{-2};q^4)_k^4}{(q^4;q^4)_k^4}q^{4k}\equiv(2q+2q^{-1}-1)[n]_{q^2}^4\pmod{[n]_{q^2}^4\Phi_n(q^2)},$$

where $M=(n+1)/2$ or $n-1$.

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References

[1] G. Gasper and M. Rahman, Basic hypergeometric series, second edition, Encyclopedia of Mathematics and Its Applications 96, Cambridge University Press, Cambridge, 2004.

[2] O. Gorodetsky, q-Congruences, with applications to supercongruences and the cyclic sieving phenomenon, Int. J. Number Theory 15 (2019), 1919–1968.

[3] C.-Y. Gu and V.J.W. Guo, q-Analogues of two supercongruences of Z.-W. Sun, Czechoslovak Math. J., in press; `https://doi.org/10.21136/CMJ.2020.0516-18`

[4] J. Guillera, WZ pairs and q-analogues of Ramanujan series for $1/\pi$, J. Difference Equ. Appl. 24 (2018), 1871–1879.

[5] V.J.W. Guo, Some generalizations of a supercongruence of van Hamme, Integral Transforms Spec. Funct. 28 (2017), 888–899.

[6] V.J.W. Guo, Common q-analogues of some different supercongruences, Results Math. 74 (2019), Art. 131.

[7] V.J.W. Guo, Proof of a generalization of the (B.2) supercongruence of Van Hamme through a q-microscope, Adv. Appl. Math. 116 (2020), Art. 102016.

[8] V.J.W. Guo, q-Analogues of Dwork-type supercongruences, J. Math. Anal. Appl. 487 (2020), Art. 124022.

[9] V.J.W. Guo, A q-analogue of the (A.2) supercongruence of Van Hamme for primes $p \equiv 1 \pmod 4$, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM 114 (2020), Art. 123.

[10] V.J.W. Guo, q-Supercongruences modulo the fourth power of a cyclotomic polynomial via creative microscoping, preprint, 2019, arXiv:1912.00765.

[11] V.J.W. Guo and J.-C. Liu, q-Analogues of two Ramanujan-type formulas for $1/\pi$, J. Difference Equ. Appl. 24 (2018), 1368–1373.

[12] V.J.W. Guo and J.-C. Liu, Some congruences related to a congruence of Van Hamme, Integral Transforms Spec. Funct. 31 (2020), 221–231.

[13] V.J.W. Guo and M.J. Schlosser, Some new q-congruences for truncated basic hypergeometric series: even powers, Results Math. 75 (2020), Art. 1.

[14] V.J.W. Guo and M.J. Schlosser, A family of q-hypergeometric congruences modulo the fourth power of a cyclotomic polynomial, Israel J. Math. (to appear).

[15] V.J.W. Guo and M.J. Schlosser, Some q-supercongruences from transformation formulas for basic hypergeometric series, preprint, Constr. Approx. (to appear).

[16] V.J.W. Guo and S.-D. Wang, Some congruences involving fourth powers of central q-binomial coefficients, Proc. Roy. Soc. Edinburgh Sect. A 150 (2020), 1127–1138.

[17] V.J.W. Guo and W. Zudilin, A q-microscope for supercongruences, Adv. Math. 346 (2019), 329–358.

[18] V.J.W. Guo and W. Zudilin, A common q-analogue of two supercongruences, Results Math. 75 (2020), Art. 46.

[19] Q.-H. Hou, Y.-P. Mu, and D. Zeilberger, Polynomial reduction and supercongruences, J. Symbolic Comput., in press; https://doi.org/10.1016/j.jsc.2019.11.004

[20] J.-C. Liu, Semi-automated proof of supercongruences on partial sums of hypergeometric series, J. Symbolic Comput. 93 (2019), 221–229.

[21] J.-C. Liu and F. Petrov, Congruences on sums of q-binomial coefficients, Adv. Appl. Math. 116 (2020), Art. 102003.

[22] L. Long, Hypergeometric evaluation identities and supercongruences, Pacific J. Math. 249 (2011), 405–418.

[23] H.-X. Ni and H. Pan, On a conjectured q-congruence of Guo and Zeng, Int. J. Number Theory 14 (2018), 1699–1707.

[24] A. Straub, Supercongruences for polynomial analogs of the Apéry numbers, Proc. Amer. Math. Soc. 147 (2019), 1023–1036.

[25] H. Swisher, On the supercongruence conjectures of van Hamme, Res. Math. Sci. 2 (2015), Art. 18.

[26] R. Tauraso, Some q-analogs of congruences for central binomial sums, Colloq. Math. 133 (2013), 133–143.

[27] L. Van Hamme, Some conjectures concerning partial sums of generalized hypergeometric series, in: p-Adic Functional Analysis (Nijmegen, 1996), Lecture Notes in Pure and Appl. Math. 192, Dekker, New York, 1997, pp. 223–236.

[28] S.-D. Wang, Some supercongruences involving $\binom{2k}{k}^4$, J. Difference Equ. Appl. 24 (2018), 1375–1383.

[29] H.S. Wilf and D. Zeilberger, An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities, Invent. Math. 108 (1992), 575–633.

[30] X. Wang and M. Yue, A q-analogue of the (A.2) supercongruence of Van Hamme for any prime $p \equiv 3 \pmod 4$, Int. J. Number Theory, in press; `https://doi.org/10.1142/S1793042120500694`

[31] W. Zudilin, Ramanujan-type supercongruences, J. Number Theory 129 (2009), 1848–1857.

[32] W. Zudilin, Congruences for q-binomial coefficients, Ann. Combin. 23 (2019), 1123–1135.