A symmetric generalization of an identity of Andrews and Yee

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Abstract. We give a symmetric generalization of an identity of Andrews and Yee, which may also be deemed a \( q \)-analogue of the Chaundy–Bullard identity for \( x = 1/2 \). We give two proofs of our identity. The first one uses the \( q \)-Wilf-Zeilberger method, while the second one is a combinatorial proof.

Keywords: \( q \)-Wilf–Zeilberger method; Chaundy–Bullard identity; partition; \( q \)-binomial theorem.

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1. Introduction

Andrews et al. [2] introduced two partition functions \( p_\omega(n) \) and \( p_\nu(n) \), where \( p_\omega(n) \) counts the number of partitions of \( n \) in which all odd parts are less than twice the smallest part, and \( p_\nu(n) \) counts the number of partitions of \( n \) with the same constraints as \( p_\omega(n) \) plus all parts being distinct. Recently, Andrews and Yee [3] studied two-variable generalizations of two identities involving the functions \( p_\omega(n) \) and \( p_\nu(n) \) and gave the following results:

\[
\sum_{n=1}^{\infty} \frac{q^n}{(zq^n; q)_{n+1}(zq^{2n+2}; q^2)_\infty} = \sum_{n=0}^{\infty} \frac{z^n q^{2n^2+n+1}}{(q; q^2)_{n+1}(zq; q^2)_{n+1}},
\]

(1.1)

\[
\sum_{n=0}^{\infty} q^n (-zq^{n+1}; q)_n (-zq^{2n+2}; q^2)_\infty = \sum_{n=0}^{\infty} \frac{z^n q^{n^2+n}}{(q; q^2)_{n+1}},
\]

(1.2)

where \((a; q)_n = (1-a)(1-aq) \cdots (1-aq^{n-1})\) and \((a; q)_\infty = \lim_{n \to \infty} (a; q)_n\).

The following intriguing identity [3, Lemma 6]

\[
\sum_{k=0}^{n} \frac{q^k(q; q)_{n+k}}{(q^2; q^2)_k} = (q^2; q^2)_n
\]

(1.3)
plays an important part in the proof of (1.1) and (1.2). Andrews and Yee’s proof of (1.3) depends on certain recurrence relations of 
\[ S_n(i) := \sum_{k=0}^{n} q^k(q; q)_{n+k}/(q^2; q^2)_k. \]
An interesting combinatorial proof of the equivalent form of (1.3):
\[ \sum_{k=0}^{n} q^k(-q^{k+1}; q)_{n-k} \left[ \frac{n+k}{k} \right] = (-q; q)_n^2 \quad (1.4) \]
was given by Chern [6]. Here
\[ \left[ \frac{n+k}{k} \right] = \frac{(q; q)_{n+k}}{(q; q)_n(q; q)_k} \]
denotes the \( q \)-binomial coefficient.

The aim of this paper is to give the following result.

**Theorem 1.1.** Let \( m \) and \( n \) be non-negative integers. Then
\[
\sum_{k=0}^{m} \frac{q^k}{(-q; q)_n(-q; q)_k} \left[ \frac{n+k}{k} \right] + \sum_{k=0}^{n} \frac{q^k}{(-q; q)_m(-q; q)_k} \left[ \frac{m+k}{k} \right] = 2. \quad (1.5)
\]
It is clear that, when \( m = n \), the identity (1.5) reduces to
\[
\sum_{k=0}^{n} \frac{q^k}{(-q; q)_n(-q; q)_k} \left[ \frac{n+k}{k} \right] = 1,
\]
which is equivalent to (1.3) and (1.4). On the other hand, the \( q = 1 \) case of (1.5) gives
\[
\sum_{k=0}^{m} 2^{-n-k} \left( \frac{n+k}{k} \right) + \sum_{k=0}^{n} 2^{-m-k} \left( \frac{m+k}{k} \right) = 2, \quad (1.6)
\]
which is the \( x = 1/2 \) case of the famous Chaundy–Bullard identity [4]:
\[
(1-x)^{n+1} \sum_{k=0}^{m} \left( \frac{n+k}{k} \right) x^k + x^{m+1} \sum_{k=0}^{n} \left( \frac{m+k}{k} \right) (1-x)^k = 1. \quad (1.7)
\]

Note that, the above identity was also found by Kesava Menon [11]. For a survey on different proofs of (1.7) and historical remarks, see Koornwinder and Schlosser [9, 10]. Koornwinder and Schlosser later (in a talk by Schlosser) gave the following \( q \)-analogue of (1.7):
\[
(x; q)_{n+1} \sum_{k=0}^{m} \left[ \frac{n+k}{k} \right] x^k + x^{m+1} \sum_{k=0}^{n} \left[ \frac{m+k}{k} \right] q^k(x; q)_k = 1. \quad (1.8)
\]
It is obvious that (1.5) is not a special case of (1.8). Recently, a generalization of (1.7) with more variables was given by Ma [12, Corollary 4.2]:

\[
\sum_{k=0}^{m} \binom{n+k}{k} \frac{(c; q)_{m-k}(aq^{-n}/b; q)_{m-k}}{(-aq; q)_{m-k}(-cq/b; q)_{m-k}} q^{(n+1)(m-k)} \frac{b+1}{b+c} \sum_{k=0}^{m} \binom{n+k}{k} \frac{(c; q)_{m-k}(aq^{-m}/a; q)_{m-k}}{(-bq; q)_{m-k}(-cq/a; q)_{m-k}} q^{(m+1)(n-k)} \frac{a+1}{a+c} \sum_{k=0}^{n} \binom{m+k}{k} \frac{(c; q)_{m-k}(bq^{-m}/a; q)_{m-k}}{(-bq; q)_{m-k}(-cq/a; q)_{m-k}} q^{(m+1)(n-k)}
\]

\[
= \left( \frac{1}{b} - \frac{1}{a} \right) \frac{(aq/b; q)_{m}(aq/b; q)_{n}(c; q)_{m+n+1}}{(-aq; q)_{m}(-bq; q)_{n}(-c/b; q)_{m+1}(-c/a; q)_{n+1}}.
\]

Moreover, a proof of (1.7) using the Wilf–Zeilberger method [13] was given by Chen [5]. We shall give two different proofs of Theorem 1.1. The first one is based on the \(q\)-Wilf–Zeilberger method [14], which is motivated by Chen’s proof of (1.7). We point out that the \(q\)-Wilf–Zeilberger method was also used by the first author [7, 8] to prove two other curious \(q\)-series identities recently. The second one is purely combinatorial and is very similar to Chern’s combinatorial proof of (1.4). Not like that there are seven different proofs of (1.7) in [9], we cannot even find a simple inductive proof of Theorem 1.1. The reader is encouraged to find any other proof of this theorem.

2. A proof using the \(q\)-Wilf–Zeilberger method

Let \([n] := (1 - q^n)/(1 - q)\) be the \(q\)-integer. We define two functions in \(q\):

\[
F(n, k) = \frac{q^k}{(-q; q)_n(-q; q)_k} \binom{n+k}{k},
\]

\[
G(n, k) = -\frac{q^{n+1}[2k]}{[2n+2](-q; q)_n(-q; q)_k} \binom{n+k}{k}.
\]

It is easy to check that they satisfy the following relation

\[
F(n + 1, k) - F(n, k) = G(n, k + 1) - G(n, k).
\]  

(2.1)

Namely, the functions \(F(n, k)\) and \(G(n, k)\) form a \(q\)-Wilf–Zeilberger pair. Summing (2.1) over \(k\) from 0 to \(m\), we obtain

\[
\sum_{k=0}^{m} (F(n + 1, k) - F(n, k)) = G(n, m + 1) - G(n, 0).
\]

Letting \(n \to j\) in the above identity and then summing it over \(j = 0, 1, \ldots, n - 1\), we get

\[
\sum_{k=0}^{m} (F(n, k) - F(0, k)) = \sum_{j=0}^{n-1} (G(j, m + 1) - G(j, 0)) = \sum_{j=0}^{n-1} G(j, m + 1),
\]
where we have used the fact that $G(j, 0) = 0$ for any non-negative integer $j$.

Namely, we have

$$
\sum_{k=0}^{m} \frac{q^k}{(-q; q)_n(-q; q)_k} \left[ \frac{n+k}{k} \right] - \sum_{k=0}^{m} \frac{q^k}{(-q; q)_k} = \sum_{j=0}^{n-1} \frac{-q^{j+1}[2m+2]}{[2j+2](-q; q)_j(-q; q)_{m+1}} \left[ \frac{j+m+1}{j} \right]
$$

$$
= \sum_{j=1}^{n} \frac{q^j[2m+2]}{[2j](-q; q)_{j-1}(-q; q)_{m+1}} \left[ \frac{j+m}{j-1} \right]
$$

$$
= \sum_{j=1}^{n} \frac{q^j}{(-q; q)_j(-q; q)_{m}} \left[ \frac{j+m}{j} \right]. \quad (2.2)
$$

Finally, it is easy to prove by induction that

$$
\sum_{k=0}^{m} \frac{q^k}{(-q; q)_k} = 2 - \frac{1}{(-q; q)_m}. \quad (2.3)
$$

Combining (2.2) and (2.3), we obtain the desired identity (1.5).

3. A combinatorial proof

We shall give a combinatorial proof of the following equivalent form of (1.5):

$$
\sum_{k=0}^{m} q^k(-q^{k+1}; q)_{m-k} \left[ \frac{n+k}{k} \right] + \sum_{k=0}^{n} q^k(-q^{k+1}; q)_{n-k} \left[ \frac{m+k}{k} \right] = 2(-q; q)_m(-q; q)_n. \quad (3.1)
$$

In order to prove (3.1), we need the following result, which is a generalization of [6, Lemma 1].

**Lemma 3.1.** Let $m$ and $n$ be non-negative integers. Then

$$
\sum_{k=0}^{m} q^k(-q^{k+1}; q)_{m-k} \left[ \frac{n+k}{k} \right] = \sum_{k=0}^{m} q^{(\frac{k+1}{2})} \left[ \frac{m+n+1}{m-k} \right]. \quad (3.2)
$$

**Proof.** We follow the idea of Chern’s proof of [6, Lemma 1]. Let $\mathcal{B}_1$ be the set of bipartitions $(\lambda, \pi)$ such that $\lambda$ is a partition into distinct parts (perhaps empty) with largest part $\leq m$ and $\pi$ is a partition with at most $n+1$ parts and largest part less than the smallest part of $\lambda$ whenever $\lambda$ is not empty. Moreover, if $\lambda$ is empty, then the largest part of $\pi$ is assumed to be at most $m$. Let $|\lambda|$ denote the sum of all parts of $\lambda$. Then

$$
\sum_{(\lambda, \pi) \in \mathcal{B}_1} q^{|\lambda|+|\pi|} = \sum_{k=0}^{m} q^k(-q^{k+1}; q)_{m-k} \left[ \frac{n+k}{k} \right].
$$


We denote by $\mathcal{B}_2$ the set of bipartitions $(\mu, \nu)$ such that $\mu$ is a partition with parts being the first $k$ consecutive positive integers for some $0 \leq k \leq m$ (i.e., $\mu = (k, k - 1, \ldots, 1)$) and $\nu$ is a partition with at most $n + 1 + k$ parts and largest part being at most $m - k$. Then

$$\sum_{(\mu, \nu) \in \mathcal{B}_2} q^{\mu + \nu} = \sum_{k=0}^{m} q^{\binom{k+1}{2}} \left[ \frac{m + n + 1}{m - k} \right].$$

Let $\ell = \ell(\lambda)$ denote the number of parts of $\lambda$, and let $\lambda_1$ and $\lambda_\ell$ stand for the largest part and smallest part of $\lambda$, respectively. For $(\lambda, \pi) \in \mathcal{B}_1$, let $\phi((\lambda, \pi)) = (\mu, \nu)$ given by

- $\mu = (\ell(\lambda), \ell(\lambda) - 1, \ldots, 1)$;
- $\nu = (\lambda_1 - \ell(\lambda), \lambda_2 - (\ell(\lambda) - 1), \ldots, \lambda_\ell - 1, \pi_1, \pi_2, \ldots, \pi_\ell)$.

For example, when $m = 7$ and $n = 5$, if $\lambda = (7, 5, 4)$ and $\pi = (3, 2, 2, 1, 1)$ then $\phi((\lambda, \pi)) = (\mu, \nu)$ with $\mu = (3, 2, 1)$ and $\nu = (4, 3, 3, 2, 2, 1, 1)$.

It is easy to see that $\nu$ is a partition (i.e., a decreasing sequence) with at most $n + 1 + \ell(\lambda)$ parts and largest part at most $m - \ell(\lambda)$. Namely, $\phi$ is a map from $\mathcal{B}_1$ to $\mathcal{B}_2$. It is clear that this map is weight-preserving ($|\lambda| + |\pi| = |\mu| + |\nu|$). It is also easy to verify that the map $\phi$ is invertible, and therefore it is a bijection. This proves the lemma. \qed

Proof of (1.5). By Lemma 3.1, it remains to prove

$$\sum_{k=0}^{m} q^{\binom{k+1}{2}} \left[ \frac{m + n + 1}{m - k} \right] + \sum_{k=0}^{n} q^{\binom{k+1}{2}} \left[ \frac{m + n + 1}{n - k} \right] = 2(-q; q)_m(-q; q)_n.$$  \hspace{1cm} (3.3)

But this is just a special case of the $q$-binomial theorem (cf. [1, p. 36, Theorem 3.3]):

$$(z; q)_N = \sum_{k=0}^{N} \binom{N}{k} (-1)^k q^{\binom{k}{2}} z^k$$  \hspace{1cm} (3.4)

with $z = -q^{-m}$ and $N = m + n + 1$. \qed

A combinatorial proof of (3.4) is well known and can be given by just comparing the coefficients of $z^k$ on both sides of (3.4) and using the partition interpretation of $q$-binomial coefficients (see [1, Theorem 3.1]). When $z = -q^{-m}$ and $N = m + n + 1$, we may consider (3.4) as the $z = 1$ case of the following identity

$$(-zq^{-m}; q)_{m+n+1} = \sum_{k=0}^{m+n+1} \binom{m+n+1}{k} z^k q^{-mk},$$

whose combinatorial proof can also be given by comparing the coefficients of $z^k$. For this reason, we omit the combinatorial proof of (3.3) here. (When $m = n$, such a proof has already been given explicitly by Chern [6].)
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References