A q-analogue of the (A.2) supercongruence of Van Hamme for primes $p \equiv 1 \pmod 4$

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Abstract. Recently, Wang and Yue gave a q-analogue of the (A.2) supercongruence of Van Hamme for any prime $p \equiv 3 \pmod 4$ by using a q-analogue of Watson's ${}_3F_2$ summation and the creative microscoping method, devised by the author and Zudilin. In this note, we give the corresponding q-analogue of the (A.2) supercongruence of Van Hamme for all primes $p \equiv 1 \pmod 4$ in the same manner. A supercongruence conjecture similar to the (A.3) conjecture of Swisher is also presented.

Keywords: basic hypergeometric series; q-analogue of Watson's ${}_3F_2$ summation; q-congruences; supercongruences; creative microscoping

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1. Introduction

In his second letter to Hardy on February 27, 1913, Ramanujan referred to the following identity

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^5}{k!^5} = \frac{2}{\Gamma(\frac{3}{4})^4}, \tag{1.1}$$

where $\Gamma(x)$ denotes the Gamma function and $(a)_k = a(a+1)\cdots(a+k-1)$ is the rising factorial. In 1997, Van Hamme [27, Eq. (A.2)] observed that (1.1) posses a nice a p-adic analogue as follows:

$$\sum_{k=0}^{(p-1)/2}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^5}{k!^5} \equiv \begin{cases} -\dfrac{p}{\Gamma_p(\frac{3}{4})^4} \pmod{p^3}, & \text{if } p \equiv 1 \pmod 4, \\ 0 \pmod{p^3}, & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{1.2}$$

Here and in what follows, p always denotes an odd prime and $\Gamma_p(x)$ stands for the p-adic Gamma function. The congruence (1.2) was first proved by McCarthy and Osburn [20]. Later Swisher [25] showed that the supercongruence (1.2) is also true modulo p^5 for $p \equiv 1 \pmod 4$ and $p > 5$. Recently, Liu [17] generalized the supercongruence (1.2) for $p \equiv 3 \pmod 4$ and $p > 3$ to the modulus p^4 case.

It should be mentioned that Van Hamme [27] proposed 13 supercongruence including (1.2). Nowadays all of these supercongruences have been confirmed by different authors

(see [23, 25]). For example, Van Hamme [27, Eq. (H.2)] himself proved the following supercongruence:

$$\sum_{k=0}^{(p-1)/2} \frac{(\frac{1}{2})_k^3}{k!^3} \equiv \begin{cases} -\Gamma_p\left(\frac{1}{4}\right)^4 \pmod{p^2}, & \text{if } p \equiv 1 \pmod 4, \\ 0 \pmod{p^2}, & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{1.3}$$

Since $\Gamma_p(\frac{1}{4})^4\Gamma_p(\frac{3}{4})^4 = 1$, from (1.2) and (1.3) we immediately deduce that

$$\sum_{k=0}^{(p-1)/2} (-1)^k(4k+1)\frac{(\frac{1}{2})_k^5}{k!^5} \equiv p \sum_{k=0}^{(p-1)/2} \frac{(\frac{1}{2})_k^3}{k!^3} \pmod{p^3}, \tag{1.4}$$

which was already noticed by Mortenson [21].

In recent years, q-analogues of supercongruences have attracted the interests of many authors (see, for example, [2–15,18,22,24,28–30]). In particular, the author and Zeng [12, Cor. 1.2] gave a q-analogue of (1.3) as follows:

$$\sum_{k=0}^{(p-1)/2} \frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k} q^{2k} \equiv \begin{cases} \dfrac{(q^2;q^4)_{(p-1)/4}^2}{(q^4;q^4)_{(p-1)/4}^2} q^{(p-1)/2} \pmod{[p]^2}, & \text{if } p \equiv 1 \pmod 4, \\ 0 \pmod{[p]^2}, & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{1.5}$$

Here we need to familiarize ourselves with the standard q-notation. The *q-shifted factorial* is defined by $(a;q)_0 = 1$ and $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n \geqslant 1$ or $n = \infty$. For convenience, we also compactly write $(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n(a_2;q)_n \cdots (a_m;q)_n$ for $n \geqslant 0$ or $n = \infty$. The *q-integer* is given by $[n] = [n]_q = 1 + q + \cdots + q^{n-1}$. Moreover, the n-th *cyclotomic polynomial*, denoted by $\Phi_n(q)$, is defined by

$$\Phi_n(q) = \prod_{\substack{1 \leqslant k \leqslant n \\ \gcd(n,k)=1}} (q - \zeta^k),$$

where ζ is an n-th primitive root of unity.

The author and Zudilin [13] devised a method, called 'creative microscoping', to prove many q-supercongruences modulo powers of a cyclotomic polynomial, including a partial q-analogue of the (J.2) supercongruence and a complete q-analogue of the (L.2) supercongruence of Van Hamme [27] (q-analogues of the (B.2), (C.2), (E.2), (F.2), (I.2) supercongruences of Van Hamme were already given in [3,5,6,11]). The author and Zudilin [14, Thm. 2] also gave a slighter generalization of (1.5). The author and Schlosser [10, Thm. 2.2] used the creative microscoping method to give more q-supercongruences, such as

$$\sum_{k=0}^{(n-1)/2} (-1)^k[4k+1]\frac{(q;q^2)_k^5}{(q^2;q^2)_k^5} q^{k^2+k} \equiv [n]q^{(1-n)/2} \sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k^3}{(q^2;q^2)_k^3} q^{2k} \pmod{[n]\Phi_n(q)^2},$$

which is a q-analogue of (1.4). However, they did not find a direct q-analogue of Van Hamme's (A.2) supercongruence (1.2).

Very recently, using the creative microscoping method and a q-analogue of Watson's ${}_3F_2$ summation [1, Appendix (II.16)], Wang and Yue [29] gave a q-analogue of (1.2) for $p \equiv 3 \pmod 4$ as follows: for $n \equiv 3 \pmod 4$,

$$\sum_{k=0}^{(n-1)/2} (-1)^k [4k+1] \frac{(q;q^2)_k^4 (q^2;q^4)_k}{(q^2;q^2)_k^4 (q^4;q^4)_k} q^k \equiv 0 \pmod{[n]\Phi_n(q)^2}, \tag{1.6}$$

$$\sum_{k=0}^{n-1} (-1)^k [4k+1] \frac{(q;q^2)_k^4 (q^2;q^4)_k}{(q^2;q^2)_k^4 (q^4;q^4)_k} q^k \equiv 0 \pmod{[n]\Phi_n(q)^2}. \tag{1.7}$$

In this note, we shall give a q-analogue of (1.2) for $p \equiv 1 \pmod 4$, as a complement to Wang and Yue's results (1.6) and (1.7).

Theorem 1.1. *Let $n > 1$ be an integer with $n \equiv 1 \pmod 4$. Then*

$$\sum_{k=0}^{(n-1)/2} (-1)^k [4k+1] \frac{(q;q^2)_k^4 (q^2;q^4)_k}{(q^2;q^2)_k^4 (q^4;q^4)_k} q^k \equiv \frac{(q^2;q^4)_{(n-1)/4}^2}{(q^4;q^4)_{(n-1)/4}^2} [n] \pmod{[n]\Phi_n(q)^2}, \tag{1.8}$$

$$\sum_{k=0}^{n-1} (-1)^k [4k+1] \frac{(q;q^2)_k^4 (q^2;q^4)_k}{(q^2;q^2)_k^4 (q^4;q^4)_k} q^k \equiv \frac{(q^2;q^4)_{(n-1)/4}^2}{(q^4;q^4)_{(n-1)/4}^2} [n] \pmod{[n]\Phi_n(q)^2}. \tag{1.9}$$

Let $n = p \equiv 1 \pmod 4$ be a prime and let $q \to 1$. Then (1.8) reduces to

$$\sum_{k=0}^{(p-1)/2} (-1)^k (4k+1) \frac{(\frac{1}{2})_k^5}{k!^5} \equiv \frac{(\frac{1}{2})_{(p-1)/4}^2}{(1)_{(p-1)/4}^2} p = \binom{-1/2}{(p-1)/4}^2 p \pmod{p^3}. \tag{1.10}$$

Note that Van Hamme [26, Theorem 3] proved that

$$\binom{-1/2}{(p-1)/4} \equiv -\frac{\Gamma_p(\frac{1}{4})^2}{\Gamma_p(\frac{1}{2})} \pmod{p^2}.$$

Since $\Gamma_p(\frac{1}{2})^2 = -1$ for $p \equiv 1 \pmod 4$, by the aforementioned identity $\Gamma_p(\frac{1}{4})^4 \Gamma_p(\frac{3}{4})^4 = 1$, we see that the supercongruence (1.10) is just (1.2) for $p \equiv 1 \pmod 4$. This means that (1.8) (or (1.9)) is indeed a q-analogue of the (A.2) supercongruence of Van Hamme for $p \equiv 1 \pmod 4$.

The paper is organized as follows. We shall prove Theorem 1.1 in the next section. A generalization of Theorem 1.1 will be given in Section 3. Finally, in Section 4, we shall propose a related conjecture on supercongruences.

2. Proof of Theorem 1.1

The following easily proved result (see [10, Lemma 3.1]) will be used in our proof.

Lemma 2.1. *Let n be a positive odd integer. Then, for $0 \leqslant k \leqslant (n-1)/2$, we have*

$$\frac{(aq;q^2)_{(n-1)/2-k}}{(q^2/a;q^2)_{(n-1)/2-k}} \equiv (-a)^{(n-1)/2-2k}\frac{(aq;q^2)_k}{(q^2/a;q^2)_k}q^{(n-1)^2/4+k} \pmod{\Phi_n(q)}.$$

Following Gasper and Rahman [1], the *basic hypergeometric series* ${}_{r+1}\phi_r$ is defined as

$${}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, b_2, \ldots, b_r \end{matrix}; q,\ z\right] = \sum_{k=0}^{\infty}\frac{(a_1, a_2, \ldots, a_{r+1}; q)_k z^k}{(q, b_1, \ldots, b_r; q)_k}.$$

We will make use of a q-analogue of Watson's ${}_3F_2$ summation [1, Appendix (II.16)]:

$$\begin{aligned} &{}_8\phi_7\left[\begin{matrix} \lambda, & q\lambda^{\frac12}, & -q\lambda^{\frac12}, & a, & b, & c, & -c, & \lambda q/c^2 \\ & \lambda^{\frac12}, & -\lambda^{\frac12}, & \lambda q/a, & \lambda q/b, & \lambda q/c, & -\lambda q/c, & c^2 \end{matrix}; q,\ -\frac{\lambda q}{ab}\right] \\ &\quad = \frac{(\lambda q, c^2/\lambda; q)_\infty (aq, bq, c^2q/a, c^2q/b; q^2)_\infty}{(\lambda q/a, \lambda q/b; q)_\infty (q, abq, c^2q, c^2q/ab; q^2)_\infty}, \end{aligned} \tag{2.1}$$

where $\lambda = c(ab/q)^{\frac12}$.

We first give the following parametric version of Theorem 1.1.

Theorem 2.2. *Let $n > 1$ be an integer with $n \equiv 1 \pmod 4$. Then, modulo $\Phi_n(q)(1-aq^n)(a-q^n)$,*

$$\sum_{k=0}^{(n-1)/2}(-1)^k[4k+1]\frac{(aq, q/a; q^2)_k (q;q^2)_k^2 (q^2;q^4)_k}{(aq^2, q^2/a; q^2)_k (q^2;q^2)_k^2 (q^4;q^4)_k}q^k \equiv \frac{(q^2;q^4)_{(n-1)/4}^2}{(q^4;q^4)_{(n-1)/4}^2}[n]. \tag{2.2}$$

Proof. For $a = q^{-n}$ or $a = q^n$, the left-hand side of (2.2) may be written as

$$\begin{aligned} &\sum_{k=0}^{(n-1)/2}(-1)^k[4k+1]\frac{(q^{1-n}, q^{1+n}; q^2)_k (q;q^2)_k^2 (q^2;q^4)_k}{(q^{2-n}, q^{2+n}; q^2)_k (q^2;q^2)_k^2 (q^4;q^4)_k}q^k \\ &\quad = {}_8\phi_7\left[\begin{matrix} q, & q^{\frac52}, & -q^{\frac52}, & q^{1-n}, & q^{1+n}, & q, & -q, & q \\ & q^{\frac12}, & -q^{\frac12}, & q^{2+n}, & q^{2-n}, & q^2 & -q^2, & q^2 \end{matrix}; q^2,\ -q\right]. \end{aligned}$$

Letting $q \mapsto q^2$, $a \mapsto q^{1-n}$, $b \mapsto q^{1+n}$, $c \mapsto q$ (and so $\lambda = q$) in (2.1), we see that the above ${}_8\phi_7$ summation is equal to

$$\frac{(q^3, q; q^2)_\infty (q^{3-n}, q^{3+n}; q^4)_\infty^2}{(q^{2+n}, q^{2-n}; q^2)_\infty (q^2, q^4; q^4)_\infty^2} = \frac{(q;q^2)_{(n+1)/2}(q^{3-n};q^4)_{(n-1)/4}^2}{(q^{2-n};q^2)_{(n+1)/2}(q^4;q^4)_{(n-1)/4}^2} = \frac{(q^2;q^4)_{(n-1)/4}^2}{(q^4;q^4)_{(n-1)/4}^2}[n].$$

This proves that the q-congruence (2.2) holds modulo $1-aq^n$ or $a-q^n$. On the other hand, by Lemma 2.1 it is easy to check that the k-th summand and $((n-1)/2-k)$-th summand cancel each other modulo $\Phi_n(q)$ for any odd n. It follows that

$$\sum_{k=0}^{(n-1)/2}(-1)^k[4k+1]\frac{(aq, q/a; q^2)_k (q;q^2)_k^2 (q^2;q^4)_k}{(aq^2, q^2/a; q^2)_k (q^2;q^2)_k^2 (q^4;q^4)_k}q^k \equiv 0 \pmod{\Phi_n(q)} \tag{2.3}$$

(for a more general form, see Wang and Yue [29, Lemma 2.2]). Since $[n] \equiv 0 \pmod{\Phi_n(q)}$ for $n > 1$, we have proved that the q-congruence (2.2) also holds modulo $\Phi_n(q)$. Noticing that $1 - aq^n$, $a - q^n$ and $\Phi_n(q)$ are pairwise relatively prime polynomials in q, we finish the proof of the theorem. □

Proof of Theorem 1.1. The limits of the denominators on both sides of (2.2) as $a \to 1$ are relatively prime to $\Phi_n(q)$, since $0 \leqslant k \leqslant (n-1)/2$. On the other hand, the limit of $(1-aq^n)(a-q^n)$ as $a \to 1$ contains the factor $\Phi_n(q)^2$. Therefore, the limiting case $a \to 1$ of (2.2) leads to the following congruence

$$\sum_{k=0}^{(n-1)/2} (-1)^k [4k+1] \frac{(q;q^2)_k^4 (q^2;q^4)_k}{(q^2;q^2)_k^4 (q^4;q^4)_k} q^k \equiv \frac{(q^2;q^4)_{(n-1)/4}^2}{(q^4;q^4)_{(n-1)/4}^2} [n] \pmod{\Phi_n(q)^3}, \tag{2.4}$$

which also means that

$$\sum_{k=0}^{n-1} (-1)^k [4k+1] \frac{(q;q^2)_k^4 (q^2;q^4)_k}{(q^2;q^2)_k^4 (q^4;q^4)_k} q^k \equiv \frac{(q^2;q^4)_{(n-1)/4}^2}{(q^4;q^4)_{(n-1)/4}^2} [n] \pmod{\Phi_n(q)^3}, \tag{2.5}$$

since $(q;q^2)_k^4 (q^2;q^4)_k / ((q^2;q^2)_k^4 (q^4;q^4)_k) \equiv 0 \pmod{\Phi_n(q)^5}$ for k in the range $(n-1)/2 < k \leqslant n-1$. It remains to show that the above two congruences also hold modulo $[n]$, i.e.,

$$\sum_{k=0}^{(n-1)/2} (-1)^k [4k+1] \frac{(q;q^2)_k^4 (q^2;q^4)_k}{(q^2;q^2)_k^4 (q^4;q^4)_k} q^k \equiv 0 \pmod{[n]}, \tag{2.6}$$

$$\sum_{k=0}^{n-1} (-1)^k [4k+1] \frac{(q;q^2)_k^4 (q^2;q^4)_k}{(q^2;q^2)_k^4 (q^4;q^4)_k} q^k \equiv 0 \pmod{[n]}. \tag{2.7}$$

For $n > 1$, let $\zeta \neq 1$ be an n-th root of unity, perhaps not primitive. Namely, ζ is a primitive root of unity of odd degree d with $d \mid n$. Let $c_q(k)$ stand for the k-th term on the left-hand side of the congruences (2.6) and (2.7), i.e.,

$$c_q(k) = (-1)^k [4k+1] \frac{(q;q^2)_k^4 (q^2;q^4)_k}{(q^2;q^2)_k^4 (q^4;q^4)_k} q^k.$$

Note that (2.3) holds for any odd $n > 1$. Thus, letting $a \to 1$ and $n = d$ in (2.3) yields

$$\sum_{k=0}^{(d-1)/2} c_\zeta(k) = \sum_{k=0}^{d-1} c_\zeta(k) = 0.$$

Observing that

$$\frac{c_\zeta(\ell d + k)}{c_\zeta(\ell d)} = \lim_{q \to \zeta} \frac{c_q(\ell d + k)}{c_q(\ell d)} = c_\zeta(k),$$

we have

$$\sum_{k=0}^{n-1} c_\zeta(k) = \sum_{\ell=0}^{n/d-1}\sum_{k=0}^{d-1} c_\zeta(\ell d+k) = \sum_{\ell=0}^{n/d-1} c_\zeta(\ell d)\sum_{k=0}^{d-1} c_\zeta(k) = 0,$$

and

$$\sum_{k=0}^{(n-1)/2} c_\zeta(k) = \sum_{\ell=0}^{(n/d-3)/2} c_\zeta(\ell d)\sum_{k=0}^{d-1} c_\zeta(k) + \sum_{k=0}^{(d-1)/2} c_\zeta((n-d)/2+k) = 0.$$

This implies that the sums $\sum_{k=0}^{n-1} c_q(k)$ and $\sum_{k=0}^{(n-1)/2} c_q(k)$ are both divisible $\Phi_d(q)$. Since each cyclotomic polynomial $\Phi_d(q)$ is irreducible in $\mathbb{Z}[q]$, we conclude that they are divisible by

$$\prod_{d|n,\, d>1} \Phi_d(q) = [n],$$

thus establishing (2.6) and (2.7). □

3. A generalization of Theorem 1.1

Wang and Yue [29] observed that, for $d, n > 1$ with $n \equiv 1 \pmod{d}$,

$$\sum_{k=0}^{(n-1)/d} (-1)^k[2dk+1]\frac{(q;q^d)_k^2(aq,q/a;q^d)_k(q^d;q^{2d})_k}{(q^d;q^d)_k^2(q^d/a,aq^d;q^d)_k(q^{d+2};q^{2d})_k}q^{(d-1)k} \equiv 0 \pmod{\Phi_n(q)}. \quad (3.1)$$

They further proved that, for $d, n > 1$ with $n \equiv d+1 \pmod{2d}$,

$$\sum_{k=0}^{(n-1)/d} (-1)^k[2dk+1]\frac{(q;q^d)_k^4(q^d;q^{2d})_k}{(q^d;q^d)_k^4(q^{d+2};q^{2d})_k}q^{(d-1)k} \equiv 0 \pmod{\Phi_n^3(q)}. \quad (3.2)$$

In this section, we shall give a q-congruence on the left-hand side of (3.2) for $n \equiv 1 \pmod{2d}$, which is also a generalization of the q-congruence (1.8) modulo $\Phi_n(q)^3$.

Theorem 3.1. *Let $d, n > 1$ be integers with $n \equiv 1 \pmod{2d}$. Then*

$$\sum_{k=0}^{(n-1)/d} (-1)^k[2dk+1]\frac{(q;q^d)_k^4(q^d;q^{2d})_k}{(q^d;q^d)_k^4(q^{d+2};q^{2d})_k}q^{(d-1)k} \equiv \frac{(q^2,q^d;q^{2d})_{(n-1)/2d}}{(q^{d+2},q^{2d};q^{2d})_{(n-1)/2d}}[n] \pmod{\Phi_n(q)^3}. \quad (3.3)$$

Proof. We first establish the following parametric generalization of (3.3):

$$\sum_{k=0}^{(n-1)/d} (-1)^k[2dk+1]\frac{(aq,q/a;q^d)_k(q;q^d)_k^2(q^d;q^{2d})_k}{(aq^d,q^d/a;q^d)_k(q^d;q^d)_k^2(q^{d+2};q^{2d})_k}q^{(d-1)k}$$
$$\equiv \frac{(q^2,q^d;q^{2d})_{(n-1)/2d}}{(q^{d+2},q^{2d};q^{2d})_{(n-1)/2d}}[n] \pmod{\Phi_n(q)(1-aq^n)(a-q^n)}. \quad (3.4)$$

Since Wang and Yue [29] have proved (3.1), it remains to show that the above q-congruence holds modulo $(1-aq^n)(a-q^n)$, or equivalently, both sides of (3.4) are equal when $a=q^{-n}$ or $a=q^n$. But this immediately follows from (2.1) by setting $q\mapsto q^d$, $a\mapsto q^{1-n}$, $b\mapsto q^{1+n}$, $c\mapsto q^{d/2}$ (and so $\lambda=q$) and noticing that

$$\begin{aligned}
&\frac{(q^{d+1},q^{d-1};q^d)_\infty(q^{d+1-n},q^{d+1+n},q^{2d-1-n},q^{2d-1+n};q^{2d})_\infty}{(q^{d-n},q^{d+n};q^d)_\infty(q^d,q^{d+2},q^{2d},q^{2d-2};q^{2d})_\infty}\\
&\quad=\frac{(q^{d+1};q^d)_{(n-1)/d}(q^{d+1-n},q^{2d-1-n};q^{2d})_{(n-1)/2d}}{(q^{d-n};q^d)_{(n-1)/d}(q^{d+2},q^{2d};q^{2d})_{(n-1)/2d}}\\
&\quad=\frac{(q^2,q^d;q^{2d})_{(n-1)/2d}}{(q^{d+2},q^{2d};q^{2d})_{(n-1)/2d}}[n].
\end{aligned}$$

The proof of (3.3) then follows by letting $a\to 1$ in (3.4). □

We point out that (3.3) also holds for $d=1$, since both sides are equal in this case.

Letting n be an odd prime power and $q\to 1$ in Theorem 3.1, we obtain the following generalization of the supercongruence (1.2) for $p\equiv 1 \pmod 4$.

Corollary 3.2. *Let $d>1$ be an integer. Let p be an odd prime and let $r\geqslant 1$ with $p^r\equiv 1 \pmod{2d}$. Then*

$$\sum_{k=0}^{(p^r-1)/d}(-1)^k(2dk+1)\frac{(\frac{1}{d})_k^4(\frac{1}{2})_k}{k!^4(\frac{d+2}{2d})_k}\equiv\frac{(\frac{1}{d})_{(p^r-1)/2d}(\frac{1}{2})_{(p^r-1)/2d}}{(\frac{d+2}{2d})_{(p^r-1)/2d}(1)_{(p^r-1)/2d}}p^r \pmod{p^3}.$$

4. An open problem

Swisher [25] has proposed many amazing general Van Hamme-type conjectures. For instance, she [25, (A.3)] conjectured that

$$\begin{aligned}
&\sum_{k=0}^{(p^r-1)/2}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^5}{k!^5}\\
&\quad\equiv\begin{cases}
-p\Gamma_p\left(\frac{1}{4}\right)^4\displaystyle\sum_{k=0}^{(p^{r-1}-1)/2}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^5}{k!^5} \pmod{p^{5r}}, & p\equiv 1 \pmod 4,\ r\geqslant 1,\\
p^4\displaystyle\sum_{k=0}^{(p^{r-2}-1)/2}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^5}{k!^5} \pmod{p^{5r-2}}, & p\equiv 3 \pmod 4\ r\geqslant 2,
\end{cases}
\end{aligned}$$

where $p>5$ if $p\equiv 1 \pmod 4$.

Motivated by the above conjecture of Swisher, we shall put forward the following supercongruence conjecture.

Conjecture 4.1. *We have*

$$\sum_{k=0}^{p^r-1}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^5}{k!^5}$$
$$\equiv\begin{cases}-p\Gamma_p\left(\frac{1}{4}\right)^4\sum_{k=0}^{p^{r-1}-1}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^5}{k!^5} \pmod{p^{5r}}, & p\equiv 1 \pmod 4,\ r\geqslant 1,\\ p^4\sum_{k=0}^{p^{r-2}-1}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^5}{k!^5} \pmod{p^{5r-2}}, & p\equiv 3 \pmod 4\ r\geqslant 2,\end{cases} \tag{4.1}$$

where $p>5$ *if* $p\equiv 1 \pmod 4$.

Recently, the author and Zudilin [16, Conjecture 4.1] have proposed a challenging q-analogue of (4.1) modulo p^{3r} for $p\equiv 1 \pmod 4$.

Note that, letting $n=p^r$ and $q\to 1$ in Theorem 1.1, we obtain the following results: If $p^r\equiv 1 \pmod 4$, then

$$\sum_{k=0}^{(p^r-1)/2}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^5}{k!^5}\equiv\binom{(p^r-1)/2}{(p^r-1)/4}^2\frac{p^r}{2^{p^r-1}} \pmod{p^{r+2}},$$
$$\sum_{k=0}^{p^r-1}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^5}{k!^5}\equiv\binom{(p^r-1)/2}{(p^r-1)/4}^2\frac{p^r}{2^{p^r-1}} \pmod{p^{r+2}}.$$

Similar supercongruences can be deduced from Wang and Yue's q-congruences (1.6) and (1.7) for $p^r\equiv 3 \pmod 4$.

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