A \( q \)-analogue of the (A.2) supercongruence of Van Hamme for primes \( p \equiv 1 \pmod{4} \)

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Abstract. Recently, Wang and Yue gave a \( q \)-analogue of the (A.2) supercongruence of Van Hamme for any prime \( p \equiv 3 \pmod{4} \) by using a \( q \)-analogue of Watson’s \( 3F_2 \) summation and the creative microscoping method, devised by the author and Zudilin. In this note, we give the corresponding \( q \)-analogue of the (A.2) supercongruence of Van Hamme for all primes \( p \equiv 1 \pmod{4} \) in the same manner. A supercongruence conjecture similar to the (A.3) conjecture of Swisher is also presented.

Keywords: basic hypergeometric series; \( q \)-analogue of Watson’s \( 3F_2 \) summation; \( q \)-congruences; supercongruences; creative microscoping

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1. Introduction

In his second letter to Hardy on February 27, 1913, Ramanujan referred to the following identity

\[
\sum_{k=0}^{\infty} (-1)^k (4k + 1) \left( \frac{1}{2} \right)_k^5 = \frac{2}{\Gamma(\frac{3}{4})^4},
\]

where \( \Gamma(x) \) denotes the Gamma function and \( (a)_k = a(a+1)\cdots(a+k-1) \) is the rising factorial. In 1997, Van Hamme [27, Eq. (A.2)] observed that (1.1) posses a nice a \( p \)-adic analogue as follows:

\[
\sum_{k=0}^{(p-1)/2} (-1)^k (4k + 1) \left( \frac{1}{2} \right)_k^5 \equiv \begin{cases} \frac{p}{\Gamma_p(\frac{3}{4})^4} \pmod{p^3}, & \text{if } p \equiv 1 \pmod{4}, \\ 0 \pmod{p^3}, & \text{if } p \equiv 3 \pmod{4}. \end{cases}
\]

Here and in what follows, \( p \) always denotes an odd prime and \( \Gamma_p(x) \) stands for the \( p \)-adic Gamma function. The congruence (1.2) was first proved by McCarthy and Osburn [20]. Later Swisher [25] showed that the supercongruence (1.2) is also true modulo \( p^5 \) for \( p \equiv 1 \pmod{4} \) and \( p > 5 \). Recently, Liu [17] generalized the supercongruence (1.2) for \( p \equiv 3 \pmod{4} \) and \( p > 3 \) to the modulus \( p^4 \) case.

It should be mentioned that Van Hamme [27] proposed 13 supercongruence including (1.2). Nowadays all of these supercongruences have been confirmed by different authors
is defined by \((1.3)\) analogue of the supercongruence and a complete
\(n\)-th supercongruence:

\[
\sum_{k=0}^{(p-1)/2} \frac{(\frac{1}{2})_k^3}{k!^{15}} \equiv \begin{cases} 
-\Gamma_p \left( \frac{1}{4} \right)^4 \pmod{p^2}, & \text{if } p \equiv 1 \pmod{4}, \\
0 \pmod{p^2}, & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
\] (1.3)

Since \(\Gamma_p \left( \frac{1}{4} \right)^4 \Gamma_p \left( \frac{3}{4} \right)^4 = 1\), from (1.2) and (1.3) we immediately deduce that

\[
\sum_{k=0}^{(p-1)/2} (-1)^k (4k + 1) \frac{(\frac{1}{2})_k^4}{k!^{15}} \equiv p \sum_{k=0}^{(p-1)/2} \frac{(\frac{1}{2})_k^3}{k!^{15}} \pmod{p^3},
\] (1.4)

which was already noticed by Mortenson \[21\].

In recent years, \(q\)-analogues of supercongruences have attracted the interests of many authors (see, for example, \[2–15,18,22,24,28–30\]). In particular, the author and Zeng \[12, Cor. 1.2\] gave a \(q\)-analogue of (1.3) as follows:

\[
\sum_{k=0}^{(p-1)/2} \frac{(q; q^2)_k^5(q^2; q^4)_k}{(q^2; q^2)_k^5(q^4; q^4)_k^2} k^{2k} \equiv \begin{cases} 
\frac{(q^2; q^4)^2}{(q^4; q^4)^2} q^{(p-1)/4} g^{(p-1)/4} \pmod{[p]^2}, & \text{if } p \equiv 1 \pmod{4}, \\
0 \pmod{[p]^2}, & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
\] (1.5)

Here we need to familiarize ourselves with the standard \(q\)-notation. The \(q\)-shifted factorial is defined by \((a; q)_0 = 1\) and \((a; q)_n = (1 + a)(1 + aq) \cdots (1 + aq^{n-1})\) for \(n \geq 1\) or \(n = \infty\). For convenience, we also compactly write \((a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n(a_2; q)_n \cdots (a_m; q)_n\) for \(n \geq 0\) or \(n = \infty\). The \(q\)-integer is given by \([n] = [n]_q = 1 + q + \cdots + q^{n-1}\). Moreover, the \(n\)-th cyclotomic polynomial, denoted by \(\Phi_n(q)\), is defined by

\[
\Phi_n(q) = \prod_{\substack{1 \leq k \leq n \gcd(n,k) = 1}} (q - \zeta^k),
\]

where \(\zeta\) is an \(n\)-th primitive root of unity.

The author and Zudilin \[13\] devised a method, called ‘creative microscoping’, to prove many \(q\)-supercongruences modulo powers of a cyclotomic polynomial, including a partial \(q\)-analogue of the (J.2) supercongruence and a complete \(q\)-analogue of the (L.2) supercongruence of Van Hamme \[27\] \((q\)-analogues of the (B.2), (C.2), (E.2), (F.2), (I.2) supercongruences of Van Hamme were already given in \[3,5,6,11\]). The author and Zudilin \[14, Thm. 2\] also gave a slighter generalization of (1.5). The author and Schlosser \[10, Thm. 2.2\] used the creative microscoping method to give more \(q\)-supercongruences, such as

\[
\sum_{k=0}^{(n-1)/2} (-1)^k [4k + 1] \frac{(q; q^2)_k^5}{(q^2; q^2)_k^5} q^k \equiv [n] q^{1-n}/2 \sum_{k=0}^{(n-1)/2} \frac{(q; q^2)_k^3}{(q^2; q^2)_k^3} q^k \pmod{[n][\Phi_n(q)]^2},
\]
which is a $q$-analogue of (1.4). However, they did not find a direct $q$-analogue of Van Hamme’s (A.2) supercongruence (1.2).

Very recently, using the creative microscoping method and a $q$-analogue of Watson’s $3F_2$ summation [1, Appendix (II.16)], Wang and Yue [29] gave a $q$-analogue of (1.2) for $p \equiv 3 \pmod{4}$ as follows: for $n \equiv 3 \pmod{4},$

$$
\sum_{k=0}^{(n-1)/2} (-1)^k [4k + 1] \frac{(q; q^2)_k^4 (q^2; q^4)_k}{(q^2; q^4)^4_k} q^k \equiv 0 \pmod{[n] \Phi_n(q)^2},
$$

(1.6)

$$
\sum_{k=0}^{n-1} (-1)^k [4k + 1] \frac{(q; q^2)_k^4 (q^2; q^4)_k}{(q^2; q^4)^4_k} q^k \equiv 0 \pmod{[n] \Phi_n(q)^2}.
$$

(1.7)

In this note, we shall give a $q$-analogue of (1.2) for $p \equiv 1 \pmod{4}$, as a complement to Wang and Yue’s results (1.6) and (1.7).

**Theorem 1.1.** Let $n > 1$ be an integer with $n \equiv 1 \pmod{4}$. Then

$$
\sum_{k=0}^{(n-1)/2} (-1)^k [4k + 1] \frac{(q; q^2)_k^4 (q^2; q^4)_k}{(q^2; q^4)^4_k} q^k \equiv \frac{(q^2; q^4)^2_{(n-1)/4}}{(q^4; q^4)^2_{(n-1)/4}} [n] \pmod{[n] \Phi_n(q)^2},
$$

(1.8)

$$
\sum_{k=0}^{n-1} (-1)^k [4k + 1] \frac{(q; q^2)_k^4 (q^2; q^4)_k}{(q^2; q^4)^4_k} q^k \equiv \frac{(q^2; q^4)^2_{(n-1)/4}}{(q^4; q^4)^2_{(n-1)/4}} [n] \pmod{[n] \Phi_n(q)^2}.
$$

(1.9)

Let $n = p \equiv 1 \pmod{4}$ be a prime and let $q \to 1$. Then (1.8) reduces to

$$
\sum_{k=0}^{(p-1)/2} (-1)^k (4k + 1) \frac{\left(\frac{1}{2}\right)^5_k}{k!^4} \equiv \frac{\left(\frac{1}{2}\right)^2_{(p-1)/4}}{(p^2-1)^2_{(p-1)/4}} p = \left(-\frac{1}{2}/(p-1)/4\right)^2 p \pmod{p^3}.
$$

(1.10)

Note that Van Hamme [26, Theorem 3] proved that

$$
\left(-\frac{1}{2}/(p-1)/4\right) = -\frac{\Gamma_p(\frac{1}{4})^2}{\Gamma_p(\frac{3}{4})} \pmod{p^2}.
$$

(1.11)

Since $\Gamma_p(\frac{3}{2}) = -1$ for $p \equiv 1 \pmod{4}$, by the aforementioned identity $\Gamma_p(\frac{1}{4})^4 \Gamma_p(\frac{3}{4})^4 = 1$, we see that the supercongruence (1.10) is just (1.2) for $p \equiv 1 \pmod{4}$. This means that (1.8) (or (1.9)) is indeed a $q$-analogue of the (A.2) supercongruence of Van Hamme for $p \equiv 1 \pmod{4}$.

The paper is organized as follows. We shall prove Theorem 1.1 in the next section. A generalization of Theorem 1.1 will be given in Section 3. Finally, in Section 4, we shall propose a related conjecture on supercongruences.

## 2. Proof of Theorem 1.1

The following easily proved result (see [10, Lemma 3.1]) will be used in our proof.
Lemma 2.1. Let \( n \) be a positive odd integer. Then, for \( 0 \leq k \leq (n-1)/2 \), we have
\[
\frac{(aq; q^2)_{(n-1)/2-k}}{(q^2/a; q^2)_{(n-1)/2-k}} \equiv (-a)^{(n-1)/2-2k} \frac{(aq; q^2)^{k}}{(q^2/a; q^2)^{k}} q^{(n-1)/4 + k} \pmod{\Phi_n(q)}.
\]

Following Gasper and Rahman [1], the basic hypergeometric series \( r+1 \phi_r \) is defined as
\[
\sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_{k} z^k}{(q, b_1, b_2, \ldots, b_r; q)_{k}}.
\]

We will make use of a \( q \)-analogue of Watson’s \( 3F_2 \) summation [1, Appendix (II.16)]:
\[
s_7\phi_7 \left[ \begin{array}{c} \lambda, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, a, b, c, -c, \lambda q/c^2 \\ \lambda^2, -\lambda^2, \lambda q/a, \lambda q/b, \lambda q/c, -\lambda q/c, c^2 \\ q, -\frac{\lambda q}{ab} \end{array} \right] = \left( \frac{\lambda q, c^2/\lambda; q)_\infty (aq, bq,c_2q/a, c^2q/b, q^2)_\infty}{(\lambda q/a, \lambda q/b, q)_\infty (aq, c^2q, c^2q/ab, q^2)_\infty} \right), \tag{2.1}
\]

where \( \lambda = c(ab/q)^{\frac{1}{2}} \).

We first give the following parametric version of Theorem 1.1.

Theorem 2.2. Let \( n > 1 \) be an integer with \( n \equiv 1 \pmod{4} \). Then, modulo \( \Phi_n(q)(1-aq^n)(a-q^n) \),
\[
\sum_{k=0}^{(n-1)/2} (-1)^k [4k + 1] \frac{(aq, q/a; q^2)_k(q; q^2)_k(q^2; q^4)_k}{(aq^2, q^2/a; q^2)_k(q^2; q^4)_k(q^4; q^4)_k} q^k \equiv \frac{(q^2; q^4)^2}{(q^4; q^4)^2(n-1)/4}[n]. \tag{2.2}
\]

Proof. For \( a = q^{-n} \) or \( a = q^n \), the left-hand side of (2.2) may be written as
\[
\sum_{k=0}^{(n-1)/2} (-1)^k [4k + 1] \frac{(aq^{-n}, q^{-1+n}; q^2)_k(q^2; q^4)_k}{(q^{-2+n}, q; q^2)_k(q^2; q^4)_k(q^4; q^4)_k} q^{-n} q^k = s_7\phi_7 \left[ \begin{array}{c} q, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, q^{-\frac{1}{2}}, q^{1-n}, q^{-1-n}, q, -q, q, q^2, -q \end{array} \right].
\]

Letting \( q \mapsto q^2, a \mapsto q^{-n}, b \mapsto q^{1+n}, c \mapsto q \) (and so \( \lambda = q \)) in (2.1), we see that the above \( s_7\phi_7 \) summation is equal to
\[
\frac{(q^3; q^3)_\infty (q^{3-n}; q^2)_\infty (q^3; q^4)_\infty}{(q^{2+n}; q^2)_\infty (q^3; q^4)_\infty (q^4; q^4)_\infty} = \frac{(q^2; q^4)^{3-n}_\infty}{(q^{2-n}; q^2)_\infty (q; q^2)_\infty (q^2; q^4)_\infty (q^4; q^4)_\infty} = \frac{(q^2; q^4)^2(n-1)/4}[n].
\]

This proves that the \( q \)-congruence (2.2) holds modulo \( 1-aq^n \) or \( a-q^n \). On the other hand, by Lemma 2.1 it is easy to check that the \( k \)-th summand and \( ((n-1)/2-k) \)-th summand cancel each other modulo \( \Phi_n(q) \) for any odd \( n \). It follows that
\[
\sum_{k=0}^{(n-1)/2} (-1)^k [4k + 1] \frac{(aq, q/a; q^2)_k(q; q^2)_k(q^2; q^4)_k}{(aq^2, q^2/a; q^2)_k(q^2; q^4)_k(q^4; q^4)_k} q^k \equiv 0 \pmod{\Phi_n(q)}. \tag{2.3}
\]
(for a more general form, see Wang and Yue [29, Lemma 2.2]). Since \( [n] \equiv 0 \pmod{\Phi_n(q)} \) for \( n > 1 \), we have proved that the \( q \)-congruence (2.2) also holds modulo \( \Phi_n(q) \). Noticing that \( 1 - aq^n \), \( a - q^n \) and \( \Phi_n(q) \) are pairwise relatively prime polynomials in \( q \), we finish the proof of the theorem. \( \square \)

**Proof of Theorem 1.1.** The limits of the denominators on both sides of (2.2) as \( a \to 1 \) are relatively prime to \( \Phi_n(q) \), since \( 0 \leq k \leq (n - 1)/2 \). On the other hand, the limit of \( (1 - aq^n)(a - q^n) \) as \( a \to 1 \) contains the factor \( \Phi_n(q)^2 \). Therefore, the limiting case \( a \to 1 \) of (2.2) leads to the following congruence

\[
\sum_{k=0}^{(n-1)/2} (-1)^k[4k + 1] \left( \frac{q^2}{q^2} \right)_k^4 \left( \frac{q^4}{q^4} \right)_k \equiv \left( \frac{q^4}{q^4} \right)_{(n-1)/4}^2 \pmod{n} \quad \text{(mod } \Phi_n(q)^3), \tag{2.4}
\]

which also means that

\[
\sum_{k=0}^{n-1} (-1)^k[4k + 1] \left( \frac{q^2}{q^2} \right)_k^4 \left( \frac{q^4}{q^4} \right)_k \equiv \left( \frac{q^4}{q^4} \right)_{(n-1)/4}^2 \pmod{n} \quad \text{(mod } \Phi_n(q)^3), \tag{2.5}
\]

since \( (q^2; q^2)_k^4 ((q^2; q^2)_k^4 (q^4; q^4)_k) \equiv 0 \pmod{\Phi_n(q)^5} \) for \( k \) in the range \((n - 1)/2 < k \leq n - 1\). It remains to show that the above two congruences also hold modulo \([n]\), i.e.,

\[
\sum_{k=0}^{(n-1)/2} (-1)^k[4k + 1] \left( \frac{q^2}{q^2} \right)_k^4 \left( \frac{q^4}{q^4} \right)_k \equiv 0 \pmod{n}, \tag{2.6}
\]

\[
\sum_{k=0}^{n-1} (-1)^k[4k + 1] \left( \frac{q^2}{q^2} \right)_k^4 \left( \frac{q^4}{q^4} \right)_k \equiv 0 \pmod{n}. \tag{2.7}
\]

For \( n > 1 \), let \( \zeta \neq 1 \) be an \( n \)-th root of unity, perhaps not primitive. Namely, \( \zeta \) is a primitive root of unity of odd degree \( d \) with \( d \mid n \). Let \( c_q(k) \) stand for the \( k \)-th term on the left-hand side of the congruences (2.6) and (2.7), i.e.,

\[
c_q(k) = (-1)^k[4k + 1] \left( \frac{q^2}{q^2} \right)_k^4 \left( \frac{q^4}{q^4} \right)_k q^k .
\]

Note that (2.3) holds for any odd \( n > 1 \). Thus, letting \( a \to 1 \) and \( n = d \) in (2.3) yields

\[
\sum_{k=0}^{(d-1)/2} c_\zeta(k) = \sum_{k=0}^{d-1} c_\zeta(k) = 0.
\]

Observing that

\[
\frac{c_\zeta(\ell d + k)}{c_\zeta(\ell d)} = \lim_{q \to \zeta} \frac{c_q(\ell d + k)}{c_q(\ell d)} = c_\zeta(k),
\]
we have
\[ \sum_{k=0}^{n-1} c_\zeta(k) = \sum_{\ell=0}^{n/d-1} \sum_{k=0}^{d-1} c_\zeta(\ell d + k) = \sum_{\ell=0}^{n/d-1} c_\zeta(\ell d) \sum_{k=0}^{d-1} c(k) = 0, \]
and
\[ \sum_{k=0}^{(n-1)/2} c_\zeta(k) = \sum_{\ell=0}^{(n/d-3)/2} c_\zeta(\ell d) \sum_{k=0}^{d-1} c(k) + \sum_{k=0}^{(d-1)/2} c((n - d)/2 + k) = 0. \]
This implies that the sums \( \sum_{k=0}^{n-1} c_\zeta(k) \) and \( \sum_{k=0}^{(n-1)/2} c_\zeta(k) \) are both divisible \( \Phi_d(q) \). Since each cyclotomic polynomial \( \Phi_d(q) \) is irreducible in \( \mathbb{Z}[q] \), we conclude that they are divisible by
\[ \prod_{d|n, d>1} \Phi_d(q) = [n], \]
thus establishing (2.6) and (2.7).

3. A generalization of Theorem 1.1

Wang and Yue [29] observed that, for \( d, n > 1 \) with \( n \equiv 1 \pmod{d} \),
\[ \sum_{k=0}^{(n-1)/d} (-1)^k [2dk + 1] \frac{(q; q^d)_k^2(aq, q/a; q^d)_k(q^d; q^{2d})_k}{(q^d; q^d)_k^2(q^d/a, qa; q^d)_k(q^d+a; q^d)_k(q^{d+2}; q^{2d})_k} \equiv 0 \pmod{\Phi_n(q)}. \tag{3.1} \]
They further proved that, for \( d, n > 1 \) with \( n \equiv d + 1 \pmod{2d} \),
\[ \sum_{k=0}^{(n-1)/d} (-1)^k [2dk + 1] \frac{(q; q^d)_k^4(q^d; q^{2d})_k}{(q^d; q^d)_k^4(q^d+2; q^{2d})_k} \equiv 0 \pmod{\Phi_n(q)^3}. \tag{3.2} \]
In this section, we shall give a \( q \)-congruence on the left-hand side of (3.2) for \( n \equiv 1 \pmod{2d} \), which is also a generalization of the \( q \)-congruence (1.8) modulo \( \Phi_n(q)^3 \).

**Theorem 3.1.** Let \( d, n > 1 \) be integers with \( n \equiv 1 \pmod{2d} \). Then
\[ \sum_{k=0}^{(n-1)/d} (-1)^k [2dk + 1] \frac{(aq; q^d)_k(q^d+a; q^d)_k(q^d; q^{2d})_k}{(aq^2, q^d+a; q^d+a)_k(q^d+2; q^{2d})_k(q^d+a; q^d)_k(q^d+2+a; q^{2d})_k} \equiv \frac{(q^2, q^d, q^{2d})(n-1)/2d}{(q^{d+2}, q^{2d}, q^{2d})(n-1)/2d} [n] \pmod{\Phi_n(q)^3}. \tag{3.3} \]

**Proof.** We first establish the following parametric generalization of (3.3):
\[ \sum_{k=0}^{(n-1)/d} (-1)^k [2dk + 1] \frac{(aq, q/a; q^d)_k(q^2; q^{2d})_k}{(aq^2, q^2/a; q^2)_k(q^d+2; q^{2d})_k(q^2; q^{2d})_k} \equiv \frac{(q^2, q^2; q^{2d})(n-1)/2d}{(q^{d+2}, q^{2d}, q^{2d})(n-1)/2d} [n] \pmod{\Phi_n(q)(1 - aq^n)(a - q^n)}. \tag{3.4} \]
Since Wang and Yue [29] have proved (3.1), it remains to show that the above $q$-congruence holds modulo $(1-aq^n)(a-q^n)$, or equivalently, both sides of (3.4) are equal when $a = q^{-n}$ or $a = q^n$. But this immediately follows from (2.1) by setting $q \mapsto q^d$, $a \mapsto q^{1-n}$, $b \mapsto q^{1+n}$, $c \mapsto q^{d/2}$ (and so $\lambda = q$) and noticing that

\[
\frac{(q^{d+1}, q^{d_1-n}, q^{d+1+n}, q^{2d-1-n}, q^{2d-1+n}, q^{2d})}{(q^{d-n}, q^{d+n}, q^{2d-2d}; q^{2d})} = \frac{(q^{d+1}, q^{d_1-n}, q^{2d-1-n}, q^{2d})}{(q^{d-n}, q^{d_1-n}, q^{2d})}\frac{(q^{d+1}, q^{d_1-n}, q^{2d-1-n}, q^{2d})}{(q^{d-n}, q^{d_1-n}, q^{2d})}
\]

Letting $n \equiv 1$ in (3.4).

\[
\frac{(q^{d+1}, q^{d_1-n}, q^{d+1+n}, q^{2d-1-n}, q^{2d-1+n}, q^{2d})}{(q^{d-n}, q^{d+n}, q^{2d-2d}; q^{2d})} = \frac{(q^{d+1}, q^{d_1-n}, q^{2d-1-n}, q^{2d})}{(q^{d-n}, q^{d_1-n}, q^{2d})}[n].
\]

The proof of (3.3) then follows by letting $a \to 1$ in (3.4).

We point out that (3.3) also holds for $d = 1$, since both sides are equal in this case.

Letting $n$ be an odd prime power and $q \to 1$ in Theorem 3.1, we obtain the following generalization of the supercongruence (1.2) for $p \equiv 1 \pmod{4}$.

**Corollary 3.2.** Let $d > 1$ be an integer. Let $p$ be an odd prime and let $r \geq 1$ with $p^r \equiv 1 \pmod{2d}$. Then

\[
\sum_{k=0}^{(p^r-1)/d} (-1)^k(2dk+1)\left(\frac{1}{2}\right)^k\frac{(\frac{1}{2})}{K_{k+1}(\frac{1}{2})} \equiv \frac{(1)}{2^{(p^r-1)/2d}}(\frac{1}{2})^{(p^r-1)/2d}p^{r^2} \pmod{p^3}.
\]

### 4. An open problem

Swisher [25] has proposed many amazing general Van Hamme-type conjectures. For instance, she [25, (A.3)] conjectured that

\[
\sum_{k=0}^{(p^r-1)/2} (-1)^k(4k+1)\left(\frac{1}{2}\right)^k\frac{(\frac{1}{2})}{K_{k+1}(\frac{1}{2})} \equiv \begin{cases} -p!\Gamma_p \left(\frac{1}{4}\right)^{p^r-1/2} \sum_{k=0}^{(p^r-2)/2} (-1)^k(4k+1)(\frac{1}{2})^k \pmod{p^5r}, & p \equiv 1 \pmod{4}, \ r \geq 1, \\
p! \sum_{k=0}^{(p^r-2)/2} (-1)^k(4k+1)(\frac{1}{2})^k \pmod{p^{5r-2}}, & p \equiv 3 \pmod{4} \ r \geq 2, \end{cases}
\]

where $p > 5$ if $p \equiv 1 \pmod{4}$.

Motivated by the above conjecture of Swisher, we shall put forward the following supercongruence conjecture.
Conjecture 4.1. We have

\[
\sum_{k=0}^{p^r-1} (-1)^k (4k+1) \left(\frac{1}{2}\right)_k \frac{5}{k!^5} \equiv \begin{cases} \\
-p \Gamma_p \left(\frac{1}{4}\right) 4^{p^r-1-1} \sum_{k=0}^{p^r-2-1} (-1)^k (4k+1) \frac{5}{k!^5} & \text{mod } p^{5r}, \quad p \equiv 1 \pmod{4}, \quad r \geq 1, \\
p^4 \sum_{k=0}^{p^r-2-1} (-1)^k (4k+1) \frac{5}{k!^5} & \text{mod } p^{5r-2}, \quad p \equiv 3 \pmod{4}, \quad r \geq 2,
\end{cases}
\]

(4.1)

where \( p > 5 \) if \( p \equiv 1 \pmod{4} \).

Recently, the author and Zudilin [16, Conjecture 4.1] have proposed a challenging \( q \)-analogue of (4.1) modulo \( p^{3r} \) for \( p \equiv 1 \pmod{4} \).

Note that, letting \( n = p^r \) and \( q \to 1 \) in Theorem 1.1, we obtain the following results: If \( p^r \equiv 1 \pmod{4} \), then

\[
\sum_{k=0}^{(p^r-1)/2} (-1)^k (4k+1) \frac{5}{k!^5} \equiv \left(\frac{(p^r - 1)/2}{p^r - 1}/4\right)^2 \frac{p^r}{2^{p^r-1}} \pmod{p^{r+2}},
\]

\[
\sum_{k=0}^{p^r-1} (-1)^k (4k+1) \frac{5}{k!^5} \equiv \left(\frac{(p^r - 1)/2}{p^r - 1}/4\right)^2 \frac{p^r}{2^{p^r-1}} \pmod{p^{r+2}}.
\]

Similar supercongruences can be deduced from Wang and Yue’s \( q \)-congruences (1.6) and (1.7) for \( p^r \equiv 3 \pmod{4} \).

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References


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