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Some q-supercongruences from a q-analogue of Watson's ${}_3F_2$ summation

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Abstract. We give some q-supercongruences from a q-analogue of Watson's ${}_3F_2$ summation and the method of "creative microscoping", introduced by the author and Zudilin. These q-supercongruences may be considered as further generalizations of the (A.2) supercongruence of Van Hamme modulo p^3 or p^2 for any odd prime p. Meanwhile, we confirm a supercongruence conjecture of Wang and Yue through establishing its q-analogue.

Keywords: cyclotomic polynomial; q-analogue of Watson's ${}_3F_2$ summation; q-congruences; supercongruences; creative microscoping

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1. Introduction

In 1997, Van Hamme [11, (A.2)] proposed the following conjecture:

$$\sum_{k=0}^{(p-1)/2}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^5}{k!^5} \equiv \begin{cases} -\dfrac{p}{\Gamma_p(\frac{3}{4})^4} \pmod{p^3}, & \text{if } p\equiv 1 \pmod 4,\\ 0 \pmod{p^3}, & \text{if } p\equiv 3 \pmod 4. \end{cases} \tag{1.1}$$

where p is an odd prime, $(a)_k = a(a+1)\cdots(a+k-1)$ is the rising factorial, and $\Gamma_p(x)$ stands for the p-adic Gamma function. Note that the following infinite series

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^5}{k!^5} = \frac{2}{\Gamma(\frac{3}{4})^4},$$

where $\Gamma(x)$ is the Gamma function, first appeared in Ramanujan's second letter to Hardy on February 27, 1913. The supercongruence (1.1) was confirmed by McCarthy and Osburn [7]. Swisher [9] further proved that (1.1) is true modulo p^5 for $p\equiv 1 \pmod 4$ and $p>5$. Later Liu [6] extended the second case of (1.1) to the modulus p^4 case.

Using the method of 'creative microscoping' introduced in [4] and a q-analogue of Watson's ${}_3F_2$ summation (see (1.9)), Wang and Yue [13] and the author [2] gave a q-analogue of (1.1) as follows: for odd n, modulo $[n]\Phi_n(q)^2$,

$$\sum_{k=0}^{(n-1)/2}(-1)^k[4k+1]\frac{(q;q^2)_k^4(q^2;q^4)_k}{(q^2;q^2)_k^4(q^4;q^4)_k}q^k \equiv \begin{cases} \dfrac{(q^2;q^4)_{(n-1)/4}^2}{(q^4;q^4)_{(n-1)/4}^2}[n], & \text{if } n\equiv 1 \pmod 4,\\ 0, & \text{if } n\equiv 3 \pmod 4. \end{cases} \tag{1.2}$$

At the moment we need to familiarize ourselves with the standard q-hypergeometric notation. The *q-shifted factorial* is defined as $(a;q)_0=1$ and $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n\geqslant 1$ or $n=\infty$. For simplicity, we also adopt the condensed notation $(a_1,a_2,\dots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n$ for $n\geqslant 0$ or $n=\infty$. The *q-integer* is defined by $[n]=[n]_q=(1-q^n)/(1-q)$. Moreover, the n-th *cyclotomic polynomial* is given by

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}}(q-\zeta^k),$$

where ζ is an n-th primitive root of unity. We refer the reader to [3, 5, 8, 12, 14–17] for some other q-supercongruences.

In this paper, we shall give some generalizations of (1.2), where the modulo $[n]\Phi_n(q)^2$ condition will be substituted with the weaker condition modulo $\Phi_n(q)^3$ or $\Phi_n(q)^2$. Our first result can be stated as follows.

Theorem 1.1. *Let $d\geqslant 2$ and $r\geqslant 1$ be integers with $\gcd(d,r)=1$. Let n be a positive integer with $n\equiv d+r \pmod{2d}$ and $n\geqslant d+r$. Then*

$$\sum_{k=0}^{(n-r)/d}(-1)^k[2dk+r]\frac{(q^r;q^d)_k^4(q^d;q^{2d})_kq^{(d-r)k}}{(q^d;q^d)_k^4(q^{d+2r};q^{2d})_k}\equiv 0 \pmod{\Phi_n(q)^3}. \tag{1.3}$$

Note that the $(d,r)=(2,1)$ case of (1.3) reduces to the second part of (1.2) modulo $\Phi_n(q)^3$, and the $r=1$ case of (1.3) was first obtained by Wang and Yue [13, Theorem 1.2]. It is easy to see that $\Phi_n(q^m)$ is divisible by $\Phi_{mn}(q)$ for all positive integers m and n. Hence, the q-supercongruence (1.3) is also true in the $\gcd(d,r)>1$ case. Letting $n=p^s$ be a prime power and $q\to 1$ in (1.3), we get the following result: for $d\geqslant 2$, $r,s\geqslant 1$ and any prime p with $p^s\equiv d+r \pmod{2d}$ and $p^s\geqslant d+r$,

$$\sum_{k=0}^{(p^s-r)/d}(-1)^k(2dk+r)\frac{(\frac{r}{d})_k^4(\frac12)_k}{k!^4(\frac{d+2r}{2d})_k}\equiv 0 \pmod{p^3}. \tag{1.4}$$

We shall also establish another two generalizations of the $n\equiv 3 \pmod 4$ case of (1.2) modulo $\Phi_n(q)^2$.

Theorem 1.2. *Let d and r be positive integers with $\gcd(d,r)=1$ and $d>r$. Let n be a positive integer with $n\equiv -1 \pmod{2d}$. Then*

$$\sum_{k=0}^{(dn-rn-r)/d}(-1)^k[2dk+r]\frac{(q^r;q^d)_k^4(q^d;q^{2d})_kq^{(d-r)k}}{(q^d;q^d)_k^4(q^{d+2r};q^{2d})_k}\equiv 0 \pmod{\Phi_n(q)^2}. \tag{1.5}$$

Similarly as before, when n is a prime power, taking $q\to 1$ in (1.5), we arrive at the following supercongruence: for $1\leqslant r<d$, $s\geqslant 1$ and any prime p with and $p^s\equiv -1 \pmod{2d}$,

$$\sum_{k=0}^{(dp^s-rp^s-r)/d}(-1)^k(2dk+r)\frac{(\frac{r}{d})_k^4(\frac12)_k}{k!^4(\frac{d+2r}{2d})_k}\equiv 0 \pmod{p^2}.$$

Theorem 1.3. *Let $d \geqslant 2$ and $r \geqslant 1$ be integers with r odd and* $\gcd(d,r)=1$. *Let $n>1$ be an integer with $n \equiv -r \pmod{2d}$ and $dn > n+r$. Then*

$$\sum_{k=0}^{(dn-n-r)/d} (-1)^k [2dk+r] \frac{(q^r;q^d)_k^4 (q^d;q^{2d})_k q^{(d-r)k}}{(q^d;q^d)_k^4 (q^{d+2r};q^{2d})_k} \equiv 0 \pmod{\Phi_n(q)^2}. \tag{1.6}$$

Likewise, the q-supercongruence (1.6) implies the following result: for $d \geqslant 2$, $r,s \geqslant 1$ and any odd prime p with $p^s \equiv -r \pmod{2d}$ and $(d-1)p^s > r$,

$$\sum_{k=0}^{(dp^s-p^s-r)/d} (-1)^k (2dk+r) \frac{(\frac{r}{d})_k^4 (\frac12)_k}{k!^4 (\frac{d+2r}{2d})_k} \equiv 0 \pmod{p^2}.$$

The fourth result of this paper is to build a generalization of (1.2) modulo $\Phi_n(q)^3$ for $n \equiv 1 \pmod 4$.

Theorem 1.4. *Let $d \geqslant 2$ and $r \geqslant 1$ be integers with* $\gcd(d,r)=1$. *Let n be a positive integer with $n \equiv r \pmod{2d}$ and $n>r$. Then*

$$\sum_{k=0}^{(n-r)/d} (-1)^k [2dk+r] \frac{(q^r;q^d)_k^4 (q^d;q^{2d})_k q^{(d-r)k}}{(q^d;q^d)_k^4 (q^{d+2r};q^{2d})_k} \equiv \frac{(q^{2r},q^d;q^{2d})_{(n-r)/(2d)} [n]}{(q^{d+2r},q^{2d};q^{2d})_{(n-r)/(2d)}} \pmod{\Phi_n(q)^3}. \tag{1.7}$$

Note that the $(d,r)=(2,1)$ case of (1.7) is just the first part of (1.2) modulo $\Phi_n(q)^3$, and the $r=1$ case of (1.7) was already obtained by the author in an earlier paper [2, Theorem 3.1]. Moreover, the q-supercongruence leads to the following result: for $d \geqslant 2$, $r,s \geqslant 1$ and any prime p with $p^s \equiv r \pmod{2d}$ and $p^s > r$,

$$\sum_{k=0}^{(p^s-r)/d} (-1)^k (2dk+r) \frac{(\frac{r}{d})_k^4 (\frac12)_k}{k!^4 (\frac{d+2r}{2d})_k} \equiv \frac{(\frac{r}{d})_{(p^s-r)/(2d)} (\frac12)_{(p^s-r)/(2d)}}{(\frac{d+2r}{2d})_{(p^s-r)/(2d)} (1)_{(p^s-r)/(2d)}} p^r \pmod{p^3}.$$

We shall also prove the following generalization of (1.4) for $r=1$, which was originally conjectured by Wang and Yue [13, Conjecture 5.1].

Theorem 1.5. *Let $d \geqslant 2$ and $s \geqslant 1$ be integers, and let p be a prime with $p \equiv d+1 \pmod{2d}$. Then*

$$\sum_{k=0}^{M} (-1)^k (2dk+1) \frac{(\frac{1}{d})_k^4 (\frac12)_k}{k!^4 (\frac{d+2}{2d})_k} \equiv 0 \pmod{p^{s+2}}, \tag{1.8}$$

where $M=(p^s-1)/d$ or p^s-1.

Recall that the *basic hypergeometric series* ${}_{r+1}\phi_r$ with $r+1$ upper parameters $a_1,\dots,a_{r+1}$, r lower parameters $b_1,\dots,b_r$, base q, and argument z is defined by (see [1]):

$${}_{r+1}\phi_r\left[\begin{matrix} a_1,a_2,\dots,a_{r+1} \\ b_1,b_2,\dots,b_r \end{matrix}; q,\ z\right] = \sum_{k=0}^{\infty} \frac{(a_1,a_2,\dots,a_{r+1};q)_k z^k}{(q,b_1,\dots,b_r;q)_k}.$$

Then a q-analogue of Watson's ${}_3F_2$ summation [1, Appendix (II.16)] can be stated as follows:

$$
{}_8\phi_7\left[\begin{matrix} \lambda, & q\lambda^{\frac12}, & -q\lambda^{\frac12}, & a, & b, & c, & -c, & \lambda q/c^2 \\ & \lambda^{\frac12}, & -\lambda^{\frac12}, & \lambda q/a, & \lambda q/b, & \lambda q/c, & -\lambda q/c, & c^2 \end{matrix} ; q, -\frac{\lambda q}{ab}\right]
$$
$$
= \frac{(\lambda q, c^2/\lambda; q)_\infty (aq, bq, c^2q/a, c^2q/b; q^2)_\infty}{(\lambda q/a, \lambda q/b; q)_\infty (q, abq, c^2q, c^2q/ab; q^2)_\infty}, \tag{1.9}
$$

where $\lambda = c(ab/q)^{\frac12}$.

We shall prove Theorems 1.1–1.4 by employing the creative microscoping method and the q-analogue of Watson's ${}_3F_2$ summation (1.9) once more. In order to prove Theorem 1.5, we shall first establish its q-analogue, which is on the basis of the respective $r = 1$ case of Theorems 1.1 and 1.4.

2. Proof of Theorem 1.1

We first build the following parametric version of Theorem 1.1.

Theorem 2.1. *Let $d \geqslant 2$ and $r \geqslant 1$ be integers with $\gcd(d, r) = 1$. Let n be a positive integer with $n \equiv d + r \pmod{2d}$ and $n \geqslant d + r$. Then, modulo $\Phi_n(q)(1 - aq^n)(a - q^n)$,*

$$
\sum_{k=0}^{(n-r)/d} (-1)^k [2dk + r] \frac{(aq^r, q^r/a; q)_k (q^r; q^d)_k^2 (q^d; q^{2d})_k q^{(d-r)k}}{(aq^d, q^d/a; q^d)_k (q^d; q^d)_k^2 (q^{d+2r}; q^{2d})_k} \equiv 0. \tag{2.1}
$$

Proof. For $a = q^{-n}$ or $a = q^n$, the left-hand side of (2.1) may be written as

$$
\sum_{k=0}^{(n-r)/d} (-1)^k [2dk + r] \frac{(q^{r-n}, q^{r+n}; q^d)_k (q^r; q^d)_k^2 (q^d; q^{2d})_k}{(q^{d-n}, q^{d+n}; q^d)_k (q^d; q^d)_k^2 (q^{d+2r}; q^{2d})_k} q^{(d-r)k}
$$
$$
= [r]_8\phi_7\left[\begin{matrix} q^r, & q^{d+\frac{r}{2}}, & -q^{d+\frac{r}{2}}, & q^{r-n}, & q^{r+n}, & q^r, & -q^{\frac{d}{2}}, & q^{\frac{d}{2}} \\ & q^{\frac{r}{2}}, & -q^{\frac{r}{2}}, & q^{d+n}, & q^{d-n}, & q^d & -q^{\frac{d}{2}+r}, & q^{\frac{d}{2}+r} \end{matrix} ; q^d, -q^{d-r}\right],
$$

where we have used $(q^{r-n}; q^d)_k = 0$ for $k > (n-r)/d$. Letting $q \mapsto q^d$, $a \mapsto q^{r-n}$, $b \mapsto q^{r+n}$, $c \mapsto q^{\frac{d}{2}}$ (and consequently $\lambda = q^r$) in (1.9), we see that the ${}_8\phi_7$ summation on the right-hand side equals

$$
\frac{(q^{d+r}, q^{d-r}; q^d)_\infty (q^{d+r-n}, q^{d+r+n}, q^{2d-r+n}, q^{2d-r-n}; q^{2d})_\infty}{(q^{d+n}, q^{d-n}; q^d)_\infty (q^d, q^{d+2r}, q^{2d}, q^{2d-2r}; q^{2d})_\infty} = 0,
$$

since $(q^{d+r-n}; q^{2d})_\infty = 0$. This proves that the q-congruence (2.1) holds modulo $1 - aq^n$ and $a - q^n$.

Moreover, performing the substitutions $q \mapsto q^d$, $a \mapsto aq^r$, $b \mapsto q^r/a$, $c = q^{\frac{d}{2}-n}$ (and so $\lambda = q^{r-n}$) in (1.9) and observing that $(q^{d+r-n}; q^d)_\infty = 0$, we obtain

$$\sum_{k=0}^{(n-r)/d} (-1)^k \frac{(1-q^{2dk+r-n})(aq^r, q^r/a; q)_k (q^{r-n}, q^{r+n}; q^d)_k (q^{d-2n}; q^{2d})_k q^{(d-r-n)k}}{(1-q^{r-n})(aq^{d-n}, q^{d-n}/a; q^d)_k (q^d, q^{d-2n}; q^d)_k (q^{d+2r}; q^{2d})_k} = 0. \quad (2.2)$$

By the condition $\gcd(d, r) = 1$ and $n \equiv d + r \pmod{2d}$, we have $\gcd(d, n) = 1$. Note that $1 - q^N \equiv 0 \pmod{\Phi_n(q)}$ if and only if N is divisible by n. The minimum positive integer k such that $(q^{d-2n}; q^d)_k \equiv 0 \pmod{\Phi_n(q)}$ is n. The minimum k for $(q^{d+2r}; q^{2d})_k \equiv 0 \pmod{\Phi_n(q)}$ is $(dn + 2n - d - 2r)/(2d) + 1$ if n is odd, and does not exist otherwise. This means that the polynomial $(q^{d-n}; q^d)_k (q^{d+2r}; q^{2d})_k$ is coprime with $\Phi_n(q)$ for $0 \leqslant k \leqslant (n-r)/d$ because $0 < (n-r)/d \leqslant (dn + 2n - d - 2r)/(2d)$. Since $q^n \equiv 1 \pmod{\Phi_n(q)}$, from (2.2) we deduce that (2.1) is true modulo $\Phi_n(q)$.

Noticing that $1 - aq^n$, $a - q^n$ and $\Phi_n(q)$ are pairwise coprime polynomials in q, we complete the proof of the theorem. □

Proof of Theorem 1.1. For $a = 1$, the denominators on both sides of (2.1) are coprime with $\Phi_n(q)$, since $0 \leqslant k \leqslant (n-r)/d$. On the other hand, the polynomial $(1-q^n)^2$ contains the factor $\Phi_n(q)^2$. Thus, taking $a = 1$ in (2.1), we immediately get the desired q-congruence (1.3). □

3. Proof of Theorem 1.2

Likewise, we first establish the following parametric version of Theorem 1.2.

Theorem 3.1. *Let d and r be positive integers with $\gcd(d, r) = 1$ and $d > r$. Let n be a positive integer with $n \equiv -1 \pmod{2d}$. Then, modulo $(1 - aq^{(d-r)n})(a - q^{(d-r)n})$,*

$$\sum_{k=0}^{(dn-rn-r)/d} (-1)^k [2dk + r] \frac{(aq^r, q^r/a; q)_k (q^r; q^d)_k^2 (q^d; q^{2d})_k q^{(d-r)k}}{(aq^d, q^d/a; q^d)_k (q^d; q^d)_k^2 (q^{d+2r}; q^{2d})_k} \equiv 0. \quad (3.1)$$

Proof. The proof is very similar to that of Theorem 2.1. For $a = q^{-(d-r)n}$ or $a = q^{(d-r)n}$, the left-hand side of (3.1) may be written as

$$\sum_{k=0}^{(dn-rn-r)/d} (-1)^k [2dk + r] \frac{(q^{r-(d-r)n}, q^{r+(d-r)n}; q^d)_k (q^r; q^d)_k^2 (q^d; q^{2d})_k}{(q^{d-(d-r)n}, q^{d+(d-r)n}; q^d)_k (q^d; q^d)_k^2 (q^{d+2r}; q^{2d})_k} q^{(d-r)k}$$

$$= [r]\, {}_8\phi_7 \left[\begin{matrix} q^r, & q^{d+\frac{r}{2}}, & -q^{d+\frac{r}{2}}, & q^{r-(d-r)n}, & q^{r+(d-r)n}, & q^r, & -q^{\frac{d}{2}}, & q^{\frac{d}{2}} \\ & q^{\frac{r}{2}}, & -q^{\frac{r}{2}}, & q^{d+(d-r)n}, & q^{d-(d-r)n}, & q^d & -q^{\frac{d}{2}+r}, & q^{\frac{d}{2}+r} \end{matrix} ; q^d, -q^{d-r} \right],$$

Letting $q \mapsto q^d$, $a \mapsto q^{r-(d-r)n}$, $b \mapsto q^{r+(d-r)n}$, $c \mapsto q^{\frac{d}{2}}$ (and consequently $\lambda = q^r$) in (1.9), we see that the ${}_8\phi_7$ summation on the right-hand side equals

$$\frac{(q^{d+r}, q^{d-r}; q^d)_\infty (q^{d+r-(d-r)n}, q^{d+r+(d-r)n}, q^{2d-r+(d-r)n}, q^{2d-r-(d-r)n}; q^{2d})_\infty}{(q^{d+(d-r)n}, q^{d-(d-r)n}; q^d)_\infty (q^d, q^{d+2r}, q^{2d}, q^{2d-2r}; q^{2d})_\infty} = 0,$$

since $(q^{d+r-(d-r)n};q^{2d})_\infty=0$. This proves that the q-congruence (3.1) holds modulo $1-aq^{(d-r)n}$ and $a-q^{(d-r)n}$. $\square$

Proof of Theorem 1.2. For $a=1$, the denominators on both sides of (3.1) are coprime with $\Phi_n(q)$, since $0\leqslant k\leqslant (dn-rn-r)/d$. Moreover, the polynomial $(1-q^{(d-r)n})^2$ has the factor $\Phi_n(q)^2$. Thus, taking $a=1$ in (3.1), we obtain the desired q-congruence (1.5). $\square$

4. Proof of Theorem 1.3

We need to establish the following parametric version of Theorem 1.3.

Theorem 4.1. *Let $d\geqslant 2$ and $r\geqslant 1$ be integers with r odd and $\gcd(d,r)=1$. Let $n>1$ be an integer with $n\equiv -r \pmod{2d}$ and $dn>n+r$. Then, modulo $(1-aq^{(d-1)n})(a-q^{(d-1)n})$,*

$$\sum_{k=0}^{(dn-n-r)/d}(-1)^k[2dk+r]\frac{(aq^r,q^r/a;q)_k(q^r;q^d)_k^2(q^d;q^{2d})_kq^{(d-r)k}}{(aq^d,q^d/a;q^d)_k(q^d;q^d)_k^2(q^{d+2r};q^{2d})_k}\equiv 0. \tag{4.1}$$

Proof. Similarly as before, for $a=q^{-(d-1)n}$ or $a=q^{(d-1)n}$, the left-hand side of (4.1) may be written as

$$\sum_{k=0}^{(dn-n-r)/d}(-1)^k[2dk+r]\frac{(q^{r-(d-1)n},q^{r+(d+1)n};q^d)_k(q^r;q^d)_k^2(q^d;q^{2d})_k}{(q^{d-(d-1)n},q^{d+(d+1)n};q^d)_k(q^d;q^d)_k^2(q^{d+2r};q^{2d})_k}q^{(d-r)k}$$
$$=[r]_8\phi_7\left[\begin{matrix} q^r, & q^{d+\frac{r}{2}}, & -q^{d+\frac{r}{2}}, & q^{r-(d-1)n}, & q^{r+(d-1)n}, & q^r, & -q^{\frac{d}{2}}, & q^{\frac{d}{2}} \\ & q^{\frac{r}{2}}, & -q^{\frac{r}{2}}, & q^{d+(d-1)n}, & q^{d-(d-1)n}, & q^d & -q^{\frac{d}{2}+r}, & q^{\frac{d}{2}+r}\end{matrix};q^d,\,-q^{d-r}\right],$$

Letting $q\mapsto q^d$, $a\mapsto q^{r-(d-1)n}$, $b\mapsto q^{r+(d-1)n}$, $c\mapsto q^{\frac{d}{2}}$ (and so $\lambda=q^r$) in (1.9), we see that the ${}_8\phi_7$ summation on the right-hand side becomes

$$\frac{(q^{d+r},q^{d-r};q^d)_\infty(q^{d+r-(d-1)n},q^{d+r+(d-1)n},q^{2d-r+(d-1)n},q^{2d-r-(d-1)n};q^{2d})_\infty}{(q^{d+(d-1)n},q^{d-(d-1)n};q^d)_\infty(q^d,q^{d+2r},q^{2d},q^{2d-2r};q^{2d})_\infty}=0,$$

since $(q^{d+r-(d-1)n};q^{2d})_\infty=0$. This proves that the q-congruence (4.1) holds modulo $1-aq^{(d-1)n}$ and $a-q^{(d-1)n}$. $\square$

Proof of Theorem 1.2. Putting $a=1$ in (4.1), we are led to the desired q-congruence (1.6). $\square$

5. Proof of Theorem 1.4

We first build the following parametric version of Theorem 1.4.

Theorem 5.1. *Let $d \geqslant 2$ and $r \geqslant 1$ be integers with* $\gcd(d,r)=1$. *Let n be a positive integer with $n \equiv r \pmod{2d}$ and $n > r$. Then, modulo $\Phi_n(q)(1-aq^n)(a-q^n)$,*

$$\sum_{k=0}^{(n-r)/d}(-1)^k[2dk+r]\frac{(aq^r,q^r/a;q)_k(q^r;q^d)_k^2(q^d;q^{2d})_k q^{(d-r)k}}{(aq^d,q^d/a;q^d)_k(q^d;q^d)_k^2(q^{d+2r};q^{2d})_k}\equiv\frac{(q^{2r},q^d;q^{2d})_{(n-r)/(2d)}[n]}{(q^{d+2r},q^{2d};q^{2d})_{(n-r)/(2d)}}. \tag{5.1}$$

Proof. The proof is similar to that of Theorem (2.1). For $a=q^{-n}$ or $a=q^n$, the left-hand side of (5.1) may be written as

$$\begin{aligned}&\sum_{k=0}^{(n-r)/d}(-1)^k[2dk+r]\frac{(q^{r-n},q^{r+n};q^d)_k(q^r;q^d)_k^2(q^d;q^{2d})_k}{(q^{d-n},q^{d+n};q^d)_k(q^d;q^d)_k^2(q^{d+2r};q^{2d})_k}q^{(d-r)k}\\&=[r]_8\phi_7\left[\begin{matrix}q^r, & q^{d+\frac{r}{2}}, & -q^{d+\frac{r}{2}}, & q^{r-n}, & q^{r+n}, & q^r, & -q^{\frac{d}{2}}, & q^{\frac{d}{2}}\\ & q^{\frac{r}{2}}, & -q^{\frac{r}{2}}, & q^{d+n}, & q^{d-n}, & q^d & -q^{\frac{d}{2}+r}, & q^{\frac{d}{2}+r}\end{matrix};q^d,\ -q^{d-r}\right].\end{aligned} \tag{5.2}$$

Letting $q\mapsto q^d$, $a\mapsto q^{r-n}$, $b\mapsto q^{r+n}$, $c\mapsto q^{\frac{d}{2}}$ (and consequently $\lambda=q^r$) in (1.9), we see that the right-hand side of (5.2) equals

$$\begin{aligned}&[r]\frac{(q^{d+r},q^{d-r};q^d)_\infty(q^{d+r-n},q^{d+r+n},q^{2d-r+n},q^{2d-r-n};q^{2d})_\infty}{(q^{d+n},q^{d-n};q^d)_\infty(q^d,q^{d+2r},q^{2d},q^{2d-2r};q^{2d})_\infty}\\&=[r]\frac{(q^{d+r};q^d)_{(n-r)/d}(q^{d+r-n},q^{2d-r-n};q^{2d})_{(n-r)/(2d)}}{(q^{d-n};q^d)_{(n-r)/d}(q^{d+2r},q^{2d};q^{2d})_{(n-r)/(2d)}}\\&=\frac{(q^{2r},q^d;q^{2d})_{(n-r)/(2d)}}{(q^{d+2r},q^{2d};q^{2d})_{(n-r)/(2d)}}[n].\end{aligned}$$

This proves that the q-congruence (5.1) holds modulo $1-aq^n$ and $a-q^n$.

Moreover, the identity (2.2) still holds for $n\equiv r \pmod{2d}$ and $n>r$. It follows that the left-hand side of (5.1) is congruent to 0 modulo $\Phi_n(q)$. On the other hand, it is easy to see that the right-hand side of (5.1) is also congruent to 0 modulo $\Phi_n(q)$. This means that the q-congruence (5.1) holds modulo $\Phi_n(q)$. □

Proof of Theorem 1.4. Letting $a=1$ in (5.1), we arrive at the q-congruence (1.7). □

6. Proof of Theorem 1.5

We first give the following q-supercongruence.

Theorem 6.1. *Let $d, n \geqslant 2$ and $s \geqslant 1$ be integers with d even and $n \equiv d+1 \pmod{2d}$. Then, modulo $\Phi_{n^s}(q)^2 \prod_{j=1}^{s} \Phi_{n^j}(q)$,*

$$\sum_{k=0}^{M}(-1)^k[2dk+1]\frac{(q;q^d)_k^4(q^d;q^{2d})_k q^{(d-1)k}}{(q^d;q^d)_k^4(q^{d+2};q^{2d})_k} \equiv \begin{cases} 0, & \text{if } s \text{ is odd,} \\ \dfrac{(q^2,q^d;q^{2d})_{(n^s-1)/(2d)}}{(q^{d+2},q^{2d};q^{2d})_{(n^s-1)/(2d)}}[n^s], & \text{if } s \text{ is even.} \end{cases} \tag{6.1}$$

where $M = (n^s-1)/d$ or n^s-1.

Proof. Since d is even and $n \equiv d+1 \pmod{2d}$, we know that $n^s \equiv d+1 \pmod{2d}$ if s is odd, and $n^s \equiv 1 \pmod{2d}$ if s is even. By the respective $r=1$ case of Theorems 1.1 and 1.4, the q-supercongruence (6.1) is true modulo $\Phi_{n^s}(q)^3$ for $M=(n^s-1)/d$. Moreover, for $(n^s-1)/d < k \leqslant n^s-1$, the q-shifted factorial $(q;q^d)_k$ is divisible by $\Phi_{n^s}(q)$, and $(q^d;q^d)_k^4(q^{d+2};q^{2d})_k$ cannot be divisible by $\Phi_{n^s}(q)^2$, and therefore the k-th summand on the left-hand side of (6.1) is congruent to 0 modulo $\Phi_{n^s}(q)^3$. Thus, the q-supercongruence (6.1) is also true modulo $\Phi_{n^s}(q)^3$ for $M=n^s-1$.

We now assume that $s \geqslant 2$ and $1 \leqslant j \leqslant s-1$. Let ζ denote an n^j-th primitive root of unity, and let $c_q(k)$ be the k-th summand on the left-hand side of (6.1), i.e.,

$$c_q(k) = (-1)^k[2dk+1]\frac{(q;q^d)_k^4(q^d;q^{2d})_k q^{(d-1)k}}{(q^d;q^d)_k^4(q^{d+2};q^{2d})_k}.$$

Then the q-supercongruence (6.1) modulo $\Phi_{n^s}(q)$ for $s=j$ indicates that

$$\sum_{k=0}^{(n^j-1)/d} c_\zeta(k) = \sum_{k=0}^{n^j-1} c_\zeta(k) = 0.$$

Observing that, for any non-negative integer ℓ,

$$\frac{c_\zeta(\ell n^j+k)}{c_\zeta(\ell n^j)} = \lim_{q\to\zeta}\frac{c_q(\ell n^j+k)}{c_q(\ell n^j)} = c_\zeta(k),$$

we get

$$\begin{aligned}
\sum_{k=0}^{(n^s-1)/d} c_\zeta(k) &= \sum_{\ell=0}^{(n^{s-j}-1)/d-1}\sum_{k=0}^{n^j-1} c_\zeta(\ell n^j+k) + \sum_{k=0}^{(n^j-1)/d} c_\zeta((n^s-n^j)/d+k) \\
&= \sum_{\ell=0}^{(n^{s-j}-1)/d-1} c_\zeta(\ell n^j)\sum_{k=0}^{n^j-1} c_\zeta(k) + c_\zeta((n^s-n^j)/d)\sum_{k=0}^{(n^j-1)/d} c_\zeta(k) \\
&= 0,
\end{aligned}$$

and

$$\sum_{k=0}^{n^s-1} c_\zeta(k) = \sum_{\ell=0}^{n^{s-j}-1}\sum_{k=0}^{n^j-1} c_\zeta(\ell n^j+k) = \sum_{\ell=0}^{n^{s-j}-1} c_\zeta(\ell n^j)\sum_{k=0}^{n^j-1} c_\zeta(k) = 0.$$

This means that, for $M=(n^s-1)/d$ and n^s-1, the sum $\sum_{k=0}^{M} c_q(k)$ is congruent to 0 modulo $\Phi_{n^j}(q)$. It is well known that each $\Phi_{n^j}(q)$ is an irreducible polynomial in $\mathbb{Z}[q]$, we conclude that the q-congruence (6.1) holds. □

We are now able to prove Theorem 1.5.

Proof of Theorem 1.5. It is easy to see that $\Phi_m(1)=1$ if m has at least two distinct prime factors, and $\Phi_{p^r}(1)=p$. Moreover, the denominators on the left-hand side of (6.1) are products of cyclotomic polynomials coprime with $\Phi_{n^j}(q)$ for integers $j>s$. So is the denominator on the right-hand side of (6.1). Thus, letting $n=p$ be a prime and $q\to 1$ in (6.1), we obtain

$$\sum_{k=0}^{M}(-1)^k(2dk+1)\frac{(\frac{1}{d})_k^4(\frac{1}{2})_k}{k!^4(\frac{d+2}{2d})_k} \equiv \begin{cases} 0, & \text{if } s \text{ is odd},\\ \dfrac{(\frac{1}{d})_{(p^s-1)/(2d)}(\frac{1}{2})_{(p^s-1)/(2d)}}{(\frac{d+2}{2d})_{(p^s-1)/(2d)}(1)_{(p^s-1)/(2d)}}p^s, & \text{if } s \text{ is even}, \end{cases} \pmod{p^{s+2}}, \tag{6.2}$$

where $M=(p^s-1)/d$ or p^s-1. It is easy to see that, for $s\geqslant 2$,

$$\frac{(\frac{1}{d})_{(p^s-1)/(2d)}}{(\frac{d+2}{2d})_{(p^s-1)/(2d)}} \equiv \frac{(\frac{1}{2})_{(p^s-1)/(2d)}}{(1)_{(p^s-1)/(2d)}} \equiv 0 \pmod{p}.$$

Substituting the above congruences into (6.2), we are led to (1.8). □

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