A new q-analogue of Van Hamme's (A.2) supercongruence

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Abstract. We give a new q-analogue of the (A.2) supercongruence of Van Hamme. Our proof employs Andrews' multiseries generalization of Watson's ${}_8\phi_7$ transformation, Andrews' terminating q-analogue of Watson's ${}_3F_2$ summation, a q-Watson-type summation due to Wei–Gong–Li, and the creative microscoping method, developed by the author and Zudilin. As a conclusion, we confirm a weaker form of Conjecture 4.5 in [Integral Transforms Spec. Funct. 28 (2017), 888–899].

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1. Introduction

India's great mathematician Ramanujan mentioned the following formula

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^5}{k!^5}=\frac{2}{\Gamma(\frac{3}{4})^4} \tag{1.1}$$

in his second letter to Hardy on February 27, 1913. Here $\Gamma(x)$ stands for the Gamma function and $(a)_k=a(a+1)\cdots(a+k-1)$ is the rising factorial. In 1997, Van Hamme [15] observed that thirteen Ramanujan-type formulas possess neat p-adic analogues. For instance, the formula (1.1) corresponds to the following supercongruence modulo p^3:

$$\sum_{k=0}^{(p-1)/2}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^5}{k!^5}\equiv\begin{cases}-\dfrac{p}{\Gamma_p(\frac{3}{4})^4} \pmod{p^3}, & \text{if } p\equiv 1 \pmod{4},\\ 0 \pmod{p^3}, & \text{if } p\equiv 3 \pmod{4},\end{cases} \tag{1.2}$$

(tagged as (A.2) in [15]). Here and in what follows, p always denotes an odd prime and $\Gamma_p(x)$ is Morita's p-adic Gamma function (see, for example, [12, Chapter 7]). The supercongruence (1.2) was first confirmed by McCarthy and Osburn [11]. Swisher [13] further proved that (1.2) is true modulo p^5 for $p\equiv 1 \pmod 4$ and $p>5$. On the other hand, Liu [10] extended (1.2) for $p\equiv 3 \pmod 4$ to the modulus p^4 case. Recently, among other things, Wei [18] gave the following generalization of the second case of (1.2):

$$\sum_{k=0}^{(p-1)/2}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^5}{k!^5}\equiv p^2\frac{(\frac{3}{4})_{(p-1)/2}}{(\frac{5}{4})_{(p-1)/2}} \pmod{p^5} \quad \text{for } p\equiv 3 \pmod{4}.$$

During the past few years, more and more authors become interested in q-analogues of supercongruences. In particular, using the creative microscoping method introduced by the author and Zudilin [7], Wang and Yue [16], together with the author [5], gave a q-analogue of (1.2): modulo $[n]\Phi_n(q)^2$,

$$\sum_{k=0}^{M}(-1)^k[4k+1]\frac{(q;q^2)_k^4(q^2;q^4)_k}{(q^2;q^2)_k^4(q^4;q^4)_k}q^k \equiv \begin{cases} \dfrac{(q^2;q^4)_{(n-1)/4}^2}{(q^4;q^4)_{(n-1)/4}^2}[n], & \text{if } n\equiv 1 \pmod 4,\\ 0, & \text{if } n\equiv 3 \pmod 4, \end{cases} \tag{1.3}$$

where $M=(n-1)/2$ or $n-1$. Moreover, Wei [17, 18] further generalized (1.3) to the moduli $[n]\Phi_n(q)^3$ and $[n]\Phi_n(q)^4$ cases.

We now need to familiarize ourselves with the standard q-notation. The *q-shifted factorial* is defined by $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n\geqslant 1$ and $(a;q)_0=1$. For simplicity, we also use the abbreviated notation $(a_1,a_2,\dots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n$ for $n\geqslant 0$. The *q-integer* is defined as $[n]=[n]_q=(1-q^n)/(1-q^n)$. Moreover, the n-th *cyclotomic polynomial* $\Phi_n(q)$ is given by

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}}(q-\zeta^k),$$

where ζ is an n-th primitive root of unity.

Letting $n=p\equiv 1 \pmod 4$ and taking $q\to 1$ in (1.3), we obtain

$$\sum_{k=0}^{(p-1)/2}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^5}{k!^5}\equiv \frac{(\frac{1}{2})_{(p-1)/4}^2}{(1)_{(p-1)/4}^2}p=\binom{-1/2}{(p-1)/4}^2 p \pmod{p^3}. \tag{1.4}$$

From [14, Theorem 3] we know that

$$\binom{-1/2}{(p-1)/4}\equiv \frac{\Gamma_p(\frac{1}{4})^2}{\Gamma_p(\frac{1}{2})} \pmod{p^2}.$$

Since $\Gamma_p(\frac{1}{2})^2=-1$ for $p\equiv 1 \pmod 4$, by the identity $\Gamma_p(\frac{1}{4})^4\Gamma_p(\frac{3}{4})^4=1$, we see that the supercongruence (1.4) is just (1.2) for $p\equiv 1 \pmod 4$. This implies that (1.3) for $M=(n-1)/2$ is really a q-analogue of the (A.2) supercongruence of Van Hamme.

Note that supercongruences may have different q-analogues. See [8] for such examples. In this note, we shall give the following new q-analogue of (1.2).

Theorem 1.1. *Let $n>1$ be an odd integer. Then, modulo $[n]_{q^2}\Phi_n(q^2)^2$,*

$$\sum_{k=0}^{M}(-1)^k[4k+1]_{q^2}[4k+1]^2\frac{(q^2;q^4)_k^4(q^4;q^8)_k}{(q^4;q^4)_k^4(q^8;q^8)_k}q^{-2k}$$
$$\equiv \begin{cases} -\dfrac{2q(q^4;q^8)_{(n-1)/4}^2}{(1+q^2)(q^8;q^8)_{(n-1)/4}^2}[n]_{q^2}, & \textit{if } n\equiv 1 \pmod 4,\\ \dfrac{(1+q)^2(q^4,q^{12};q^8)_{(n-3)/4}}{(1+q^2)(1+q^4)(q^8,q^{16};q^8)_{(n-3)/4}}[n]_{q^2}, & \textit{if } n\equiv 3 \pmod 4. \end{cases} \tag{1.5}$$

where $M=(n-1)/2$ *or* $n-1$.

For n prime, letting $q\to -1$ in Theorem 1.1, we get (1.2). On the other hand, for n prime and $q\to 1$ in Theorem 1.1, we arrive at

$$\sum_{k=0}^{(p-1)/2}(-1)^k(4k+1)^3\frac{(\frac{1}{2})_k^5}{k!^5}\equiv\begin{cases}\dfrac{p}{\Gamma_p(\frac{3}{4})^4} \pmod{p^3}, & \text{if } p\equiv 1 \pmod 4,\\[2mm] \dfrac{(\frac{1}{2})_{(p-3)/4}(\frac{3}{2})_{(p-3)/4}p}{(\frac{p-3}{4})!(\frac{p+1}{4})!} \pmod{p^3}, & \text{if } p\equiv 3 \pmod 4.\end{cases} \tag{1.6}$$

Thus, Theorem 1.1 may be considered as a common q-analogue of (1.2) and (1.6).

Moreover, letting n be an odd prime power and $q\to 1$ in (1.3) and (1.5), respectively, we are led to the following results: If $p^r\equiv 1 \pmod 4$, then

$$\sum_{k=0}^{(p^r-1)/d}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^5}{k!^5}\equiv\binom{(p^r-1)/2}{(p^r-1)/4}^2\frac{p^r}{2^{p^r-1}} \pmod{p^{r+2}}, \tag{1.7}$$

$$\sum_{k=0}^{(p^r-1)/d}(-1)^k(4k+1)^3\frac{(\frac{1}{2})_k^5}{k!^5}\equiv-\binom{(p^r-1)/2}{(p^r-1)/4}^2\frac{p^r}{2^{p^r-1}} \pmod{p^{r+2}}, \tag{1.8}$$

where $d=1$ or 2. Since $4+1+(4k+1)^3=2(4k+1)(8k^2+4k+1)$, combining (1.7) and (1.8) we obtain the following conclusion.

Corollary 1.2. *If* $p^r\equiv 1 \pmod 4$*, then*

$$\sum_{k=0}^{(p^r-1)/d}(-1)^k(4k+1)(8k^2+4k+1)\frac{(\frac{1}{2})_k^5}{k!^5}\equiv 0 \pmod{p^{r+2}}, \tag{1.9}$$

where $d=1$ *or* 2.

Note that the author [4, Conjecture 4.5] conjectured that (1.9) is true modulo p^{3r} for $p\equiv 1 \pmod 4$.

We shall prove Theorem 1.1 in the next section. In Section 3, we raise two related conjectures on supercongruences.

2. Proof of Theorem 1.1

We first give the following q-congruence. See [6, Lemma 3.1] for a short proof.

Lemma 2.1. *Let n be a positive odd integer. Then, for* $0\leqslant k\leqslant (n-1)/2$*, we have*

$$\frac{(aq;q^2)_{(n-1)/2-k}}{(q^2/a;q^2)_{(n-1)/2-k}}\equiv(-a)^{(n-1)/2-2k}\frac{(aq;q^2)_k}{(q^2/a;q^2)_k}q^{(n-1)^2/4+k} \pmod{\Phi_n(q)}.$$

Meanwhile, we will utilize a powerful transformation of Andrews (see [1, Theorem 4]), which can be stated as follows:

$$
\begin{aligned}
&\sum_{k\geqslant 0}\frac{(a,q\sqrt{a},-q\sqrt{a},b_1,c_1,\dots,b_m,c_m,q^{-N};q)_k}{(q,\sqrt{a},-\sqrt{a},aq/b_1,aq/c_1,\dots,aq/b_m,aq/c_m,aq^{N+1};q)_k}\left(\frac{a^mq^{m+N}}{b_1c_1\cdots b_mc_m}\right)^k\\
&\quad=\frac{(aq,aq/b_mc_m;q)_N}{(aq/b_m,aq/c_m;q)_N}\sum_{j_1,\dots,j_{m-1}\geqslant 0}\frac{(aq/b_1c_1;q)_{j_1}\cdots(aq/b_{m-1}c_{m-1};q)_{j_{m-1}}}{(q;q)_{j_1}\cdots(q;q)_{j_{m-1}}}\\
&\qquad\times\frac{(b_2,c_2;q)_{j_1}\dots(b_m,c_m;q)_{j_1+\cdots+j_{m-1}}}{(aq/b_1,aq/c_1;q)_{j_1}\dots(aq/b_{m-1},aq/c_{m-1};q)_{j_1+\cdots+j_{m-1}}}\\
&\qquad\times\frac{(q^{-N};q)_{j_1+\cdots+j_{m-1}}}{(b_mc_mq^{-N}/a;q)_{j_1+\cdots+j_{m-1}}}\frac{(aq)^{j_{m-2}+\cdots+(m-2)j_1}q^{j_1+\cdots+j_{m-1}}}{(b_2c_2)^{j_1}\cdots(b_{m-1}c_{m-1})^{j_1+\cdots+j_{m-2}}}.
\end{aligned}\tag{2.1}
$$

It should be pointed out that Andrews' transformation is a multiseries generalization of Watson's ${}_8\phi_7$ transformation:

$$
\begin{aligned}
&{}_8\phi_7\left[\begin{matrix}a, & qa^{\frac12}, & -qa^{\frac12}, & b, & c, & d, & e, & q^{-n}\\ & a^{\frac12}, & -a^{\frac12}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1}\end{matrix};q,\ \frac{a^2q^{n+2}}{bcde}\right]\\
&\quad=\frac{(aq,aq/de;q)_n}{(aq/d,aq/e;q)_n}\,{}_4\phi_3\left[\begin{matrix}aq/bc,\ d,\ e,\ q^{-n}\\ aq/b,\ aq/c,\ deq^{-n}/a\end{matrix};q,\ q\right]
\end{aligned}
$$

(see [3, Appendix (III.18)]), where the *basic hypergeometric series* ${}_{r+1}\phi_r$ is defined as

$$
{}_{r+1}\phi_r\left[\begin{matrix}a_1,a_2,\dots,a_{r+1}\\ b_1,b_2,\dots,b_r\end{matrix};q,\ z\right]=\sum_{k=0}^{\infty}\frac{(a_1,a_2,\dots,a_{r+1};q)_k}{(q,b_1,\dots,b_r;q)_k}z^k.
$$

We shall also use Andrews' terminating q-analogue of Watson's ${}_3F_2$ summation (see [2] or [3, (II.17)]):

$$
{}_4\phi_3\left[\begin{matrix}q^{-n},\ a^2q^{n+1},\ c,\ -c\\ aq,\ -aq,\ c^2\end{matrix};q,q\right]=\begin{cases}0, & \text{if } n \text{ is odd},\\ \dfrac{c^n(q,a^2q^2/c^2;q^2)_{n/2}}{(a^2q^2,\ c^2q;q^2)_{n/2}}, & \text{if } n \text{ is even},\end{cases}\tag{2.2}
$$

and the following q-Watson-type summation due to Wei et al. [19, Corollary 5]:

$$
{}_4\phi_3\left[\begin{matrix}q^{-n},\ a^2q^{n+1},\ c,\ -cq\\ aq,\ -aq,\ c^2q\end{matrix};q,q\right]=\begin{cases}\dfrac{c^n(q;q^2)_{(n+1)/2}(a^2q^2/c^2;q^2)_{(n-1)/2}}{(a^2q^2;q^2)_{(n-1)/2}(c^2q;q^2)_{(n+1)/2}}, & \text{if } n \text{ is odd},\\ \dfrac{c^n(q,a^2q^2/c^2;q^2)_{n/2}}{(a^2q^2,\ c^2q;q^2)_{n/2}}, & \text{if } n \text{ is even}.\end{cases}\tag{2.3}
$$

We first give the following parametric version of Theorem 1.1.

Theorem 2.2. *Let $n>1$ be an odd integer. Then, modulo $\Phi_n(q^2)(1-aq^{2n})(a-q^{2n})$,*

$$
\sum_{k=0}^{(n-1)/2}(-1)^k[4k+1]_{q^2}[4k+1]^2\frac{(aq^2,q^2/a;q^4)_k(q^2;q^4)_k^2(q^4;q^8)_k}{(aq^4,q^4/a;q^4)_k(q^4;q^4)_k^2(q^8;q^8)_k}q^{-2k}
$$

$$
\equiv\begin{cases}\left(1-\dfrac{(1+q)(1-aq^2)(1-q^2/a)}{(1-q)(1-q^4)}\right)\dfrac{(q^4;q^8)_{(n-1)/4}^2}{(q^8;q^8)_{(n-1)/4}^2}[n]_{q^2}, & \text{if } n\equiv 1 \pmod 4,\\[2mm] \dfrac{(1+q)(1-aq^2)(1-q^2/a)(q^4,q^{12};q^8)_{(n-3)/4}}{(1-q)(1-q^8)(q^8,q^{16};q^8)_{(n-3)/4}}[n]_{q^2}, & \text{if } n\equiv 3 \pmod 4.\end{cases}\tag{2.4}
$$

Proof. For $a=q^{-2n}$ or $a=q^{2n}$, the left-hand side of (2.4) may be written as

$$
\sum_{k=0}^{(n-1)/2}(-1)^k[4k+1]_{q^2}[4k+1]^2\frac{(q^{2-2n},q^{2+2n};q^4)_k(q^2;q^4)_k^2(q^4;q^8)_k}{(q^{4-2n},q^{4+2n};q^4)_k(q^4;q^4)_k^2(q^8;q^8)_k}q^{-2k}.
$$

Letting $m=3$, $q\mapsto q^4$, $a=q^2$, $b_1=c_1=q^5$, $b_2=c_2=q^2$, $b_3=-q^2$, $c_3=q^{2+2n}$, and $N=(n-1)/2$ in (2.1), we see that the above summation is equal to

$$
\begin{aligned}
&\frac{(q^6,-q^{2-2n};q^4)_{(n-1)/2}}{(-q^4,q^{4-2n};q^4)_{(n-1)/2}}\\
&\quad\times\sum_{j_1,j_2\geqslant 0}\frac{(q^{-4};q^4)_{j_1}(q^2;q^4)_{j_2}(q^2,q^2;q^4)_{j_1}(-q^2,q^{2+2n},q^{2-2n};q^4)_{j_1+j_2}}{(q^4;q^4)_{j_1}(q^4;q^4)_{j_2}(q,q;q^4)_{j_1}(q^4,q^4,-q^4;q^4)_{j_1+j_2}}q^{6j_1+4j_2}\\
&=(-1)^{(n-1)/2}q^{1-n}[n]_{q^2}\sum_{j_2=0}^{(n-1)/2}\frac{(q^2,-q^2,q^{2+2n},q^{2-2n};q^4)_{j_2}}{(q^4,q^4,q^4,-q^4;q^4)_{j_2}}q^{4j_2}\\
&\quad+(-1)^{(n+1)/2}q^{3-n}[n]_{q^2}(1+q)^2\sum_{j_2=0}^{(n-3)/2}\frac{(q^2;q^4)_{j_2}(-q^2,q^{2+2n},q^{2-2n};q^4)_{j_2+1}}{(q^4;q^4)j_2(q^4,q^4,-q^4;q^4)_{j_2+1}}q^{4j_2},
\end{aligned}\tag{2.5}
$$

where we have used the fact that $(q^{-4};q^4)_{j_1}=0$ for $j_1>1$.

Taking $q\mapsto q^4$, $a=1$, $c=q^2$, and $n\mapsto (n-1)/2$ in (2.2), we have

$$
\sum_{j_2=0}^{(n-1)/2}\frac{(q^2,-q^2,q^{2+2n},q^{2-2n};q^4)_{j_2}}{(q^4,q^4,q^4,-q^4;q^4)_{j_2}}q^{4j_2}=\begin{cases}q^{n-1}\dfrac{(q^4;q^8)_{(n-1)/4}^2}{(q^8;q^8)_{(n-1)/4}^2}, & \text{if } n\equiv 1 \pmod 4,\\ 0, & \text{if } n\equiv 3 \pmod 4.\end{cases}
$$

Similarly, taking $q\mapsto q^4$, $a=q^4$, $c=q^2$, and $n\mapsto (n-3)/2$ in (2.3), we get

$$\sum_{j_2=0}^{(n-3)/2}\frac{(q^2;q^4)_{j_2}(-q^2,q^{2+2n},q^{2-2n};q^4)_{j_2+1}}{(q^4;q^4)j_2(q^4,q^4,-q^4;q^4)_{j_2+1}}q^{4j_2}$$

$$=\frac{(1+q^2)(1-q^{2+2n})(1-q^{2-2n})}{(1-q^4)^2(1+q^4)}\sum_{j_2=0}^{(n-3)/2}\frac{(q^2,-q^6,q^{6+2n},q^{6-2n};q^4)_{j_2}}{(q^4,q^8,q^8,-q^8;q^4)_{j_2}}q^{4j_2}$$

$$=\begin{cases} q^{n-3}\dfrac{(1+q^2)(1-q^{2+2n})(1-q^{2-2n})(q^4;q^8)^2_{(n-1)/4}}{(1-q^4)^2(q^8;q^8)^2_{(n-1)/4}}, & \text{if } n\equiv 1 \pmod 4,\\ q^{n-3}\dfrac{(1+q^2)(1-q^{2+2n})(1-q^{2-2n})(q^4,q^{12};q^8)_{(n-3)/4}}{(1-q^4)^2(1+q^4)(q^8,q^{16};q^8)_{(n-3)/4}}, & \text{if } n\equiv 3 \pmod 4.\end{cases}$$

Substituting the above two identities into (2.5), we obtain

$$\sum_{k=0}^{(n-1)/2}(-1)^k[4k+1]_{q^2}[4k+1]^2\frac{(q^{2-2n},q^{2+2n};q^4)_k(q^2;q^4)_k^2(q^4;q^8)_k}{(q^{4-2n},q^{4+2n};q^4)_k(q^4;q^4)_k^2(q^8;q^8)_k}q^{-2k}$$

$$=\begin{cases}\left(1-\dfrac{(1+q)(1-q^{2+2n})(1-q^{2-2n})}{(1-q)(1-q^4)}\right)\dfrac{(q^4;q^8)^2_{(n-1)/4}}{(q^8;q^8)^2_{(n-1)/4}}[n]_{q^2}, & \text{if } n\equiv 1 \pmod 4,\\ \dfrac{(1+q)(1-q^{2+2n})(1-q^{2-2n})(q^4,q^{12};q^8)_{(n-3)/4}}{(1-q)(1-q^8)(q^8,q^{16};q^8)_{(n-3)/4}}[n]_{q^2}, & \text{if } n\equiv 3 \pmod 4.\end{cases}$$

This proves that both sides of (2.4) are equal when $a=q^{\pm 2n}$. Namely, the q-congruence (2.4) holds modulo $1-aq^{2n}$ or $a-q^{2n}$.

Moreover, in view of Lemma 2.1, we can verify that the k-th and $((n-1)/2-k)$-th summands cancel each other modulo $\Phi_n(q^2)$ for any positive odd integer n. It follows that

$$\sum_{k=0}^{(n-1)/2}(-1)^k[4k+1]_{q^2}[4k+1]^2\frac{(aq^2,q^2/a;q^4)_k(q^2;q^4)_k^2(q^4;q^8)_k}{(aq^4,q^4/a;q^4)_k(q^4;q^4)_k^2(q^8;q^8)_k}q^{-2k}\equiv 0 \pmod{\Phi_n(q^2)}. \tag{2.6}$$

Noticing that $[n]_{q^2}\equiv 0 \pmod{\Phi_n(q^2)}$ for $n>1$, we conclude that the q-congruence (2.4) also holds modulo $\Phi_n(q)$.

Since $1-aq^{2n}$, $a-q^{2n}$, and $\Phi_n(q^2)$ are pairwise relatively prime polynomials in q, we complete the proof of the theorem. □

Proof of Theorem 1.1. It is easy to see that the denominators on both sides of (2.4) when $a=1$ are relatively prime to $\Phi_n(q^2)$. On the other hand, when $a=1$, the polynomial $(1-aq^{2n})(a-q^{2n})$ contains the factor $\Phi_n(q^2)^2$. Therefore, the $a=1$ case of (2.4) implies that (1.5) is true modulo $\Phi_n(q^2)^3$ for $M=(n-1)/2$. Furthermore, since $(q^2;q^4)_k^4(q^4;q^8)_k/((q^4;q^4)_k^4(q^8;q^8)_k)\equiv 0 \pmod{\Phi_n(q^2)^5}$ for k in the range $(n-1)/2<k\leqslant n-1$, we see that (1.5) is also true modulo $\Phi_n(q^2)^3$ for $M=n-1$.

It remains to prove the following two q-congruences:

$$\sum_{k=0}^{(n-1)/2}(-1)^k[4k+1]_{q^2}[4k+1]^2\frac{(q^2;q^4)_k^4(q^4;q^8)_k}{(q^4;q^4)_k^4(q^8;q^8)_k}q^{-2k}\equiv 0 \pmod{[n]_{q^2}}, \tag{2.7}$$

$$\sum_{k=0}^{n-1}(-1)^k[4k+1]_{q^2}[4k+1]^2\frac{(q^2;q^4)_k^4(q^4;q^8)_k}{(q^4;q^4)_k^4(q^8;q^8)_k}q^{-2k}\equiv 0 \pmod{[n]_{q^2}}. \tag{2.8}$$

For $n>1$, let $\zeta\neq 1$ be an n-th root of unity, possibly not primitive. Namely, ζ is a primitive root of unity of odd degree d satisfying $d\mid n$. Let $c_q(k)$ be the k-th term on the left-hand side of the congruences (2.7) and (2.8). Namely,

$$c_q(k)=(-1)^k[4k+1]_{q^2}[4k+1]^2\frac{(q^2;q^4)_k^4(q^4;q^8)_k}{(q^4;q^4)_k^4(q^8;q^8)_k}q^{-2k}.$$

Observe that (2.6) is true for any odd $n>1$. Thus, letting $a=1$ and $n=d$ in (2.6), we obtain

$$\sum_{k=0}^{(d-1)/2}c_\zeta(k)=\sum_{k=0}^{d-1}c_\zeta(k)=0, \quad \text{and} \quad \sum_{k=0}^{(d-1)/2}c_{-\zeta}(k)=\sum_{k=0}^{d-1}c_{-\zeta}(k)=0.$$

Noticing that

$$\frac{c_\zeta(\ell d+k)}{c_\zeta(\ell d)}=\lim_{q\to\zeta}\frac{c_q(\ell d+k)}{c_q(\ell d)}=c_\zeta(k),$$

we have

$$\sum_{k=0}^{n-1}c_\zeta(k)=\sum_{\ell=0}^{n/d-1}\sum_{k=0}^{d-1}c_\zeta(\ell d+k)=\sum_{\ell=0}^{n/d-1}c_\zeta(\ell d)\sum_{k=0}^{d-1}c_\zeta(k)=0,$$

and

$$\sum_{k=0}^{(n-1)/2}c_\zeta(k)=\sum_{\ell=0}^{(n/d-3)/2}c_\zeta(\ell d)\sum_{k=0}^{d-1}c_\zeta(k)+\sum_{k=0}^{(d-1)/2}c_\zeta((n-d)/2+k)=0.$$

This means that both the sums $\sum_{k=0}^{n-1}c_q(k)$ and $\sum_{k=0}^{(n-1)/2}c_q(k)$ are divisible $\Phi_d(q)$. Similarly, we can show that they are also divisible by $\Phi_d(-q)$. Since d can be any divisor of n larger than 1, we deduce that each of them is congruent to 0 modulo

$$\prod_{d\mid n,\, d>1}\Phi_d(q)\Phi_d(-q)=[n]_{q^2},$$

thus establishing (2.7) and (2.8). □

3. Two open problems

Swisher [13] proposed many interesting conjectures on generalizations of Van Hamme's supercongruences (A.2)–(L.2). Recently, the author and Zudilin [9] have proved some conjectures of Swisher by establishing their q-analogues. Here we would like to propose a similar conjecture.

Conjecture 3.1. *Let $p \equiv 1 \pmod 4$ and let $r, s \geqslant 1$. Then*

$$\sum_{k=0}^{(p^r-1)/d} (-1)^k (4k+1)^{2s+1} \frac{(\frac{1}{2})_k^5}{k!^5} \equiv -p\Gamma_p(\tfrac{1}{4})^4 \sum_{k=0}^{(p^{r-1}-1)/d} (-1)^k (4k+1)^{2s+1} \frac{(\frac{1}{2})_k^5}{k!^5} \pmod{p^{3r-2}}, \tag{3.1}$$

where $d = 1$ or 2.

For $s = 0$, Swisher [13, (A.3)] and the author [5, Conjecture 4.1] conjectured that (3.1) holds modulo p^{5r} for $p > 5$. From (1.8) we can easily see that (3.1) is true modulo p^r for $s = 1$.

Finally, motivated by [4, Conjecture 4.5], we believe that the following generalization of Corollary 1.2 for p of the form $4k + 3$ should be true.

Conjecture 3.2. *Let $p \equiv 3 \pmod 4$ and let $r \geqslant 2$ be even. Then*

$$\sum_{k=0}^{(p^r-1)/d} (-1)^k (4k+1)(8k^2+4k+1) \frac{(\frac{1}{2})_k^5}{k!^5} \equiv 0 \pmod{p^{2r}},$$

where $d = 1$ or 2.

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