PROOF OF A BASIC HYPERGEOMETRIC SUPERCONGRUENCE MODULO THE FIFTH POWER OF A CYCLOTOMIC POLYNOMIAL

VICTOR J. W. GUO AND MICHAEL J. SCHLOSSER

ABSTRACT. By means of the $q$-Zeilberger algorithm, we prove a basic hypergeometric supercongruence modulo the fifth power of the cyclotomic polynomial $\Phi_n(q)$. This result appears to be quite unique, as in the existing literature so far no basic hypergeometric supercongruences modulo a power greater than the fourth of a cyclotomic polynomial have been proved. We also establish a couple of related results, including a parametric supercongruence.

1. INTRODUCTION

In 1997, Van Hamme [27] conjectured that 13 Ramanujan-type series including

$$\sum_{k=0}^{\infty} (-1)^k(4k + 1) \frac{(1/2)^3}{k!^3} = \frac{2}{\pi},$$

admit nice $p$-adic analogues, such as

$$\sum_{k=0}^{p-1} (-1)^k(4k + 1) \frac{(1/2)^3}{k!^3} \equiv p(-1)^{p-1} \pmod{p^3},$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ denotes the Pochhammer symbol and $p$ is an odd prime. Up to present, all of the 13 supercongruences have been confirmed. See [21,24] for historic remarks on these supercongruences. Recently, $q$-analogues of congruences and supercongruences have caught the interests of many authors (see, for example, [1–20,25,26,29]). In particular, the first author and Zudilin [16] devised a method, called ‘creative microscoping’, to prove quite a few $q$-supercongruences by introducing an additional parameter $a$.

In [13], the authors of the present paper proved many additional $q$-supercongruences by the creative microscoping method. Supercongruences modulo a higher integer power of a prime, or, in the $q$-case, of a cyclotomic polynomial, are very special and usually difficult to prove. As far as we know, until now the result

$$\sum_{k=0}^{n-1} [4k + 1] \frac{(q;q^2)_k^4}{(q^2; q^2)_k^4} \equiv q^{\frac{1-n}{2}} [n] + \frac{(n^2 - 1)(1 - q)^2}{24} q^{\frac{1-n}{2}} [n]^3 \pmod{[n] \Phi_n(q)^3}, \quad (1)$$

2010 Mathematics Subject Classification. Primary 33D15; Secondary 11A07, 11F33.

Key words and phrases. basic hypergeometric series, $q$-series, supercongruences, identities.

The first author was partially supported by the National Natural Science Foundation of China (grant 11771175).
for an odd positive integer $n$, due to the first author and Wang [15], is the unique $q$-supercongruence modulo $[n]\Phi_n(q)^3$ in the literature that was completely proved. (Several similar conjectural $q$-supercongruences are stated in [13] and in [16].) The purpose of this paper is to establish an even higher $q$-congruence, namely modulo a fifth power of a cyclotomic polynomial. Specifically, we prove the following three theorems. (The first two together confirm a conjecture by the authors [13, Conjecture 5.4]).

**Theorem 1.1.** Let $n > 1$ be a positive odd integer. Then

$$\sum_{k=0}^{\frac{n-1}{2}} [4k - 1] \frac{(q^{-1}; q^2)^4_k}{(q^2; q^2)^4_k} q^{4k} \equiv -(1 + q + q^2)[n]^4 \pmod{[n]^4 \Phi_n(q)},$$

(2a)

and

$$\sum_{k=0}^{n-1} [4k - 1] \frac{(q^{-1}; q^2)^4_k}{(q^2; q^2)^4_k} q^{4k} \equiv -(1 + q + q^2)[n]^4 \pmod{[n]^4 \Phi_n(q)}. \quad (2b)$$

**Theorem 1.2.** Let $n > 1$ be a positive odd integer. Then

$$\sum_{k=0}^{\frac{n-1}{2}} [4k - 1] \frac{(aq^{-1}; q^2)_k(q^{-1}/a; q^2)_k(q^{-1}; q^2)^2_k}{(aq^2; q^2)_k(q^2/a; q^2)_k(q^2; q^2)^2_k} q^{4k} \equiv 0 \pmod{[n]^2(1 - aq^n)(a - q^n)},$$

and

$$\sum_{k=0}^{n-1} [4k - 1] \frac{(aq^{-1}; q^2)_k(q^{-1}/a; q^2)_k(q^{-1}; q^2)^2_k}{(aq^2; q^2)_k(q^2/a; q^2)_k(q^2; q^2)^2_k} q^{4k} \equiv 0 \pmod{[n]^2(1 - aq^n)(a - q^n)}.$$

The $a = -1$ case of Theorem 1.2 admits an even stronger $q$-congruence.

**Theorem 1.3.** Let $n > 1$ be a positive odd integer. Then

$$\sum_{k=0}^{\frac{n-1}{2}} [4k - 1] \frac{(q^{-2}; q^4)^2_k}{(q^4; q^4)^2_k} q^{4k} \equiv -q^n(1 - q + q^2)[n]_q^2 \pmod{[n]_q^2 \Phi_n(q^2)}, \quad (3a)$$

and

$$\sum_{k=0}^{n-1} [4k - 1] \frac{(q^{-2}; q^4)^2_k}{(q^4; q^4)^2_k} q^{4k} \equiv -(1 - q + q^2)[n]_q^2 \pmod{[n]_q^2 \Phi_n(q^2)}. \quad (3b)$$

In the above $q$-supercongruences and in what follows,

$$(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$$

is the $q$-shifted factorial,

$$[n] = [n]_q = 1 + q + \cdots + q^{n-1}$$

is the $q$-number,

$$\left[\frac{n}{k}\right]_q = \frac{(q; q)_n}{(q; q)_k(q; q)_{n-k}}$$
is the $q$-binomial coefficient, and $\Phi_n(q)$ is the $n$-th cyclotomic polynomial of $q$. Note that the congruences in Theorem 1.1 modulo $[n]\Phi_n(q)^2$ and the congruences in Theorem 1.2 modulo $[n](1 - aq^n)(a - q^n)$ have already been proved by the authors in \cite[eqs. (5.5) and (5.10)]{13}.

2. Proof of Theorem 1.1 by the Zeilberger Algorithm

The Zeilberger algorithm (cf. \cite{22}) can be used to find that the functions

\[
F(n, k) = (-1)^k \frac{(4n - 1)(\frac{-1}{2})^3(\frac{-1}{2})_{n+k}}{(1)_{n-k}(\frac{-1}{2})_k},
\]

\[
g(n, k) = (-1)^{k-1} \frac{4(\frac{-1}{2})^3(\frac{-1}{2})_{n+k-1}}{(1)_{n-k-1}(\frac{-1}{2})_k}
\]

satisfy the relation

\[
(2k - 3)F(n, k - 1) - (2k - 4)f(n, k) = g(n + 1, k) - g(n, k).
\]

Of course, given this relation, it is not difficult to verify by hand that it is satisfied by the above pair of doubly-indexed sequences $f(n, k)$ and $g(n, k)$.

Here we use the convention $1/(1)_m = 0$ for all negative integers $m$. We now define the $q$-analogue of $f(n, k)$ and $g(n, k)$ as follows:

\[
F(n, k) = (-1)^k q^{(k-2)(k-2n+1)} \frac{(4n - 1)(q^{-1}; q^2)_n(q^{-1}; q^2)_{n+k}}{(q^2; q^2)_n(q^2; q^2)_{n-k}(q^{-1}; q^2)_n(k)},
\]

\[
G(n, k) = (-1)^{k-1} q^{(k-2)(k-2n+3)} \frac{(q^{-1}; q^2)_n(q^{-1}; q^2)_{n+k-1}}{(1-q^2)(q^2; q^2)_{n-1}(q^2; q^2)_{n-k}(q^{-1}; q^2)_n(k)},
\]

where we have used the convention that $1/(q^2; q^2)_m = 0$ for $m = -1, -2, \ldots$ Then the functions $F(n, k)$ and $G(n, k)$ satisfy the relation

\[
[2k - 3]F(n, k - 1) - [2k - 4]F(n, k) = G(n + 1, k) - G(n, k). \tag{4}
\]

Indeed, it is straightforward to obtain the following expressions:

\[
\frac{F(n, k - 1)}{G(n, k)} = q^{2n-4k+6}(1 - q)(1 - q^{4n-1})(1 - q^{2k-3})^2
\]

\[
\frac{F(n, k)}{G(n, k)} = q^{4-2k}(1 - q)(1 - q^{4n-1})(1 - q^{2n+2k-3})
\]

\[
\frac{G(n + 1, k)}{G(n, k)} = q^{4-2k}(1 - q^{2n-1})^3(1 - q^{2n+2k-3})
\]

\[
(1 - q^{2n})^3(1 - q^{2n-2k+2})^2.
\]
It is easy to verify the identity
\[
\frac{q^{2n-4k+6}(1-q^{4n-1})(1-q^{2k-3})^3}{(1-q^{2n-2k+2})(1-q^{2n})^3} + \frac{q^{4-2k}(1-q^{2k-4})(1-q^{4n-1})(1-q^{2n+2k-3})}{(1-q^{2n})^3} \\
= \frac{q^{4-2k}(1-q^{2n-1})^3(1-q^{2n+2k-3})}{(1-q^{2n})^3(1-q^{2n-2k+2})} - 1,
\]
which is equivalent to (4). (Alternatively, we could have established (4) by only guessing \(F(n,k)\) and invoking the \(q\)-Zeilberger algorithm [28].)

Let \(m > 1\) be an odd integer. Summing (4) over \(n\) from 0 to \((m+1)/2\), we get
\[
[2k - 3 \sum_{n=0}^{m+1} F(n, k - 1) - [2k - 4 \sum_{n=0}^{m+1} F(n, k) = G \left( \frac{m+3}{2}, k \right) - G(0, k) \\
= G \left( \frac{m+3}{2}, k \right).
\]

We readily compute
\[
G \left( \frac{m+3}{2}, 1 \right) = \frac{q^{m-1}(q^{-1}; q^2)_{(m+3)/2}}{(1-q)^2(q^2; q^2)_{(m+1)/2}(1-q^{-1})^2} \\
= \frac{q^{m-3}[m]^4}{[m+1]^4(-q; q^2)_{(m-1)/2}} \left[ \frac{m-1}{(m-1)/2} \right]^4,
\]
and
\[
G \left( \frac{m+3}{2}, 2 \right) = -\frac{(q^{-1}; q^2)_{(m+3)/2}(q^{-1}; q^2)_{(m+5)/2}}{(1-q)^2(q^2; q^2)_{(m+1)/2}(q^2; q^2)_{(m-1)/2}(q^{-1}; q^2)^2} \\
= -\frac{q^{-2}[m]^4[m+2]}{[m+1]^3(-q; q^2)_{(m-1)/2}} \left[ \frac{m-1}{(m-1)/2} \right]^4.
\]
Combining (5) and (6), we have
\[
\sum_{n=0}^{m+1} F(n, 0) = \frac{-2}{[-1]} \sum_{n=0}^{m+1} F(n, 1) + \frac{1}{[-1]} G \left( \frac{m+3}{2}, 1 \right) \\
= \frac{1+q}{q} G \left( \frac{m+3}{2}, 2 \right) - qG \left( \frac{m+3}{2}, 1 \right) \\
= -\frac{(1+q)[m]^4[m+1][m+2]+q^{m+1}[m]^4}{q^3[m+1]^4(-q; q^2)_{(m-1)/2}} \left[ \frac{m-1}{(m-1)/2} \right]^4,
\]
i.e.,
\[
\sum_{n=0}^{m+1} [4n-1] \frac{(q^{-1}; q^2)_n^4}{(q^2; q^2)_n^4} q^{4n} = -\frac{(1 + q)[m]4(m + 1)[m + 2] + q^{m+1}[m]^4}{q[m+1]^4(-q; q)^8_{(m-1)/2}} \left[ \frac{m-1}{(m-1)/2} \right]^4.
\] (7)

By [4, Lemma 2.1] (or [3, Lemma 2.1]), we have \((-q; q)^2_{(m-1)/2} \equiv q^{(m^2-1)/8} \pmod{\Phi_m(q)}\). Moreover, it is easy to see that
\[
\sum_{n=0}^{m+1} [4n-1] \frac{(q^{-1}; q^2)_n^4}{(q^2; q^2)_n^4} q^{4n} \equiv -((1 + q)^2 + q)[m]^4 \pmod{[m]^4\Phi_m(q)}.
\]

Concluding, the congruence (2a) holds.

Similarly, summing (4) over \(n\) from 0 to \(m - 1\), we get
\[
[2k - 3] \sum_{n=0}^{m-1} F(n, k - 1) - [2k - 4] \sum_{n=0}^{m-1} F(n, k) = G(m, k),
\]
and so
\[
\sum_{n=0}^{m-1} [4n-1] \frac{(q^{-1}; q^2)_n^4}{(q^2; q^2)_n^4} q^{4n} = \frac{1 + q}{q} G(m, 2) - qG(m, 1)
\]
\[
= -\frac{(1 + q)[2m - 2][m - 1] + q^{2m-2}}{q(-q; q)^{8_{m-1}}} \left[ \frac{2m - 2}{m - 1} \right]^4.
\] (8)

It is easy to see that
\[
\frac{1}{[m]} \left[ \frac{2m - 2}{m - 1} \right] = \frac{1}{[m-1]} \left[ \frac{2m - 2}{m - 2} \right] \equiv (-1)^{m-2} q^{2\binom{m-1}{2}} \pmod{\Phi_m(q)},
\]
and \((-q; q)^{m-1}_{m-1} \equiv 1 \pmod{\Phi_m(q)}\) (see, for example, [4]). The proof of (2b) then follows easily from (8).

3. Proof of Theorems 1.2 and 1.3

Proof of Theorem 1.2. It is easy to see by induction on \(N\) that
\[
\sum_{k=0}^{N} [4k-1] \frac{(aq^{-1}; q^2)_k(q^{-1}/a; q^2)_k(q^{-1}; q^2)_k^2}{(aq^2; q^2)_k(q^2/a; q^2)_k(q^2; q^2)_k^2} q^{4k}
\]
\[
= \frac{(aq; q^2)_N((a - 1)^2 q^{2N+1} - a(1 + q)(1 + q^{4N+1}))}{q(a - q)(1 - aq)(aq^2; q^2)_N(q^2/a; q^2)_N(-q; q)_N^2} \left[ \frac{2N}{N} \right]^2.
\] (9)
For \( N = (n + 1)/2 \) or \( N = n - 1 \), we see that \((aq; q^2)_N(q/a; q^2)_N\) contains the factor \((1 - aq^n)(1 - q^n/a)\). Moreover,

\[
\frac{[(n+1)/2]}{[n]} \left[ \frac{n}{(n-1)/2} \right] = \left[ \frac{n-1}{(n-1)/2} \right]
\]

is a polynomial in \( q \). Since \([(n+1)/2]\) and \([n]\) are relatively prime, we conclude that \([n] = (n-1)/2\) is divisible by \([n]\). Therefore, \( [(n+1)/2]/[n] = (1 + q^{(n+1)/2}) [(n-1)/2] \) is also divisible by \([n]\). It is also well known that \( [2n-2]/n-1\) is divisible by \([n]\). Moreover, it is easy to see that \([n]\) is relatively prime to \(1 + q^m\) for any non-negative integer \(m\). The proof then follows from (9) by taking \(N = (n+1)/2\) and \(N = n - 1\).

\(\square\)

**Proof of Theorem 1.3.** For \(a = -1\), the identity (9) reduces to

\[
\sum_{k=0}^{N} \frac{[4k - 1](q^{-2}; q^4)_k^2 q^{4k}}{(q^2; q^4)_k^2} = -\frac{(-q; q^2)_N^2 (1 + q^{4N+1})}{q(1 + q)(-q^2; q^2)_N^2 (-q; q^2)_N} \left[ \frac{2N}{N} \right]^2
\]

\[
= -\frac{(1 + q^{4N+1})}{q(1 + q)(-q^2; q^2)_N} \left[ \frac{2N}{N} \right]^2
\]

(10)

Note that, in the proof of Theorem 1.2, we have proved that \( [2N]/N \) is divisible by \([n]q^2\) for both \(N = (n+1)/2\) and \(N = n - 1\). Moreover, \([n]q^2\) is relatively prime to \((-q^2; q^2)_m\) for \(m \geq 0\). Hence the right-hand side of (10) is congruent to 0 modulo \([n]q^2\) for \(N = (n+1)/2\) or \(N = n - 1\). To further determine the right-hand side of (10) modulo \([n]q^2\), we need only to use the same congruences (with \(q \mapsto q^2\)) used in the proof of Theorem 1.1.

\(\square\)

**4. Immediate consequences**

Notice that for \(n = p^r\) being an odd prime power, \(\Phi^r(p) = [p]_{q^{p^r-1}}\) holds. This observation was used in [15] to extend (1) to a supercongruence modulo \([p^r][p]^3_{q^{p^r-1}}\). In the same vein we immediately deduce from Theorem 1.1 the following result:

**Corollary 4.1.** Let \(p\) be an odd prime and \(r\) a positive integer. Then

\[
\sum_{k=0}^{p^{r-1}} [4k - 1] \frac{(q^{-1}; q^2)_k q^{4k}}{(q^2; q^2)_k} \equiv -(1 + 3q + q^2)[p^r]^4 (\text{mod } [p^r]^4[p]_{q^{p^r-1}}),
\]

(11a)

and

\[
\sum_{k=0}^{p^{r-1}} [4k - 1] \frac{(q^{-1}; q^2)_k q^{4k}}{(q^2; q^2)_k} \equiv -(1 + 3q + q^2)[p^r]^4 (\text{mod } [p^r]^4[p]_{q^{p^r-1}}).
\]

(11b)

The \(q \rightarrow 1\) limiting cases of these two identities yield the following supercongruences:
Corollary 4.2. Let $p$ be an odd prime and $r$ a positive integer. Then
\[ \sum_{k=0}^{p^r-1} \frac{4k + 3}{16(k+1)^4 256^k} \left( \begin{array}{c} 2k \\ k \end{array} \right)^4 \equiv 1 - 5p^{4r} \pmod{p^{4r+1}}, \tag{12a} \]
and
\[ \sum_{k=0}^{p^r-2} \frac{4k + 3}{16(k+1)^4 256^k} \left( \begin{array}{c} 2k \\ k \end{array} \right)^4 \equiv 1 - 5p^{4r} \pmod{p^{4r+1}}. \tag{12b} \]

Similarly, we deduce from Theorem 1.3 the following result:

Corollary 4.3. Let $p$ be an odd prime and $r$ a positive integer. Then
\[ \sum_{k=0}^{p^r+1} [4k - 1] \frac{(q^2 - q^4)^2}{(q^4 - q^4)^2} q^{4k} \equiv -q^{p^r}(1 - q + q^2)[p^r]_q^{2} \pmod{[p^r]_q^2 [p]_{q^{2p^r-1}}}, \tag{13a} \]
and
\[ \sum_{k=0}^{p^r-1} [4k - 1] \frac{(q^2 - q^4)^2}{(q^4 - q^4)^2} q^{4k} \equiv -(1 - q + q^2)[p^r]_q^{2} \pmod{[p^r]_q^2 [p]_{q^{2p^r-1}}}. \tag{13b} \]

The $q \to 1$ limiting cases of these two identities yield the following supercongruences:

Corollary 4.4. Let $p$ be an odd prime and $r$ a positive integer. Then
\[ \sum_{k=0}^{p^r-1} \frac{4k + 3}{4(k+1)^2 16^k} \left( \begin{array}{c} 2k \\ k \end{array} \right)^2 \equiv 1 - p^{2r} \pmod{p^{2r+1}}, \tag{14a} \]
and
\[ \sum_{k=0}^{p^r-2} \frac{4k + 3}{4(k+1)^2 16^k} \left( \begin{array}{c} 2k \\ k \end{array} \right)^2 \equiv 1 - p^{2r} \pmod{p^{2r+1}}. \tag{14b} \]

The supercongruences in Corollaries 4.2 and 4.4 are remarkable since they are valid for arbitrarily high prime powers. Swisher [24] had empirically observed several similar but different hypergeometric supercongruences and stated them without proof.

References

8 VICTOR J. W. GUO AND MICHAEL J. SCHLOSSER


School of Mathematical Sciences, Huaiyin Normal University, Huai’an 223300, Jiangsu, People’s Republic of China
E-mail address: jwguo@hytc.edu.cn

Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria
E-mail address: michael.schlosser@univie.ac.at